GCMP

GREATER CLEVELAND MATHEMATICS PROGRAM

TEACHER'S GUIDE FOR
SEQUENTIAL WRITE-IN TEXTS
LEVELS E THROUGH H

4 e f g h

SRA

SCOPE AND SEQUENCE

GREATER CLEVELAND MATHEMATICS PROGRAM

	K	1	2	3	4	5	6
Whole Numbers							
Development & Use of Models	✓	✓	✓	✓	✓	✓	
Investigation & Exploration	✓	✓	✓	✓	✓	✓	✓
Development of Computational Skill		✓	✓	✓	✓	✓	✓
Application		✓	✓	✓	✓	✓	✓
Fractional Numbers							
Development & Use of Models		✓	✓	✓	✓	✓	✓
Investigation & Exploration		✓	✓	✓	✓	✓	✓
Development of Computational Skill					✓	✓	✓
Application			✓	✓	✓	✓	✓
Integers							
Development & Use of Models					✓	✓	✓
Investigation & Exploration					✓	✓	✓
Application					✓	✓	✓
Rationals							
Development & Use of Models							✓
Investigation & Exploration							✓
Numeration							
Measurement							
Investigation & Exploration	✓	✓	✓	✓	✓	✓	✓
Development of Skills		✓	✓	✓	✓	✓	✓
Application			✓	✓	✓	✓	✓
Geometry							
Development & Use of Models	✓	✓	✓	✓	✓	✓	✓
Investigation & Exploration	✓	✓	✓	✓	✓	✓	✓
Problem Solving							

GCMP

GREATER CLEVELAND MATHEMATICS PROGRAM

Teacher's Guide for Sequential Write-In Texts Levels E through H

4

Prepared by the staff of the Educational Research Council of America, assisted by teachers from the participating Council Schools.

George S. Cunningham	Consultant	Charles Bastis	Research Assistant
Ronald P. Fisher	Elementary Coordinator	Ruth Humiston	Research Assistant
Rae Marie Parsons Creps	Research Associate	Judy Klein	Research Assistant
Lucille McCraith	Research Associate	Barrett Robinson	Research Assistant
David Raskin	Research Associate	William T. Hale	Assistant Director
Margaret Russell	Research Associate	John F. Mehegan	Director

SRA Science Research Associates, Inc., 259 East Erie Street, Chicago, Illinois 60611
A Subsidiary of IBM

THE EDUCATIONAL RESEARCH COUNCIL OF AMERICA
ACKNOWLEDGES THE GENEROUS SUPPORT OF
THE CLEVELAND FOUNDATION
WHICH ORIGINALLY MADE THE GREATER CLEVELAND
MATHEMATICS PROGRAM POSSIBLE.

CONTENTS

Pupil Pages 65–80
Guide Pages 75–91

UNIT 6 • QUOTIENT AND REMAINDER

Using counters to explore solutions to the quotient-remainder equation, $b = aq + r$
Using remainders in building a sequence of equations that lead to the least remainder and the greatest partial quotient
Associating the b of $b = aq + r$ with distances from 0 on the number line
Using multiples of 10 and 100 to solve quotient-remainder equations
Applying equations of the form $b = aq + r$ to story exercises

Pupil Pages 81–92
Guide Pages 93–104

UNIT 7 • THE DIVISION ALGORISM

Recognizing that the greatest quotient is the sum of partial quotients
Relating quotient-remainder equations to a division algorism
Using multiples of 10 and 100 as partial quotients in a division algorism
Using a division algorism to find the quotient and remainder
Using multiplication as a check for division
Computing the greatest quotient and least remainder in application situations

Pupil Pages 93–106
Guide Pages 105–118

UNIT 8 • MULTIPLES AND THE DIVISION ALGORISM

Recognizing and interpreting multiples of 10, 100, and 1000
Computing products and missing factors that are multiples of 10, 100, and 1000
Using multiples of 10, 100, and 1000 as partial quotients in a division algorism
Recognizing the importance of subtraction in a division algorism
Using subtraction and division in application situations

Pupil Pages 107–120
Guide Pages 119–128

UNIT 9 • GEOMETRY: POINTS

Developing a concept of a point by exploring locations
Using three references to describe the precise location of a point
Using pairs of numbers to locate points on a grid
Perceiving the purpose of lines of longitude and latitude on a map
Recognizing that the prime meridian and the equator are reference lines on a map
Using latitude and longitude to locate points on a map
Exploring triangular numbers
Investigating number of distances determined by two or more points

Pupil Pages 121–136
Guide Pages 129–138

UNIT 10 • ROUNDING NUMBERS

Strengthening understanding of place value
Associating numbers with multiples of 10 on the number line
Interpreting numerals as a whole number of tens
Rounding numbers from 0 through 9999 to the nearest multiple of 10, the next higher 10, and the next lower 10
Computing estimated sums and products using round numbers
Estimating sums and products in application situations

Pupil Pages 137–148
Guide Pages 139–145

UNIT 11 • THE DOZENAL SYSTEM

Exploring a dozenal system of numeration
Expressing whole numbers in a dozenal system; practice with division
Writing standard decimal numerals for numbers expressed in a dozenal system
Applying a dozenal system to story exercises

Pupil Pages 149–164
Guide Pages 147–156

UNIT 12 • MEASUREMENT

Perceiving measurement as comparison
Using a balance scale to compare the weight of objects
Comparing the length of objects to the length of a given object
Recognizing that assigning a number to one of two lengths (or weights) determines a number for the other length (or weight)
Recognizing the need for standard units of measure
Identifying the standard units used for measuring length (inch, foot, yard, mile), weight (ounce, pound, ton), and liquids (cup, pint, quart, gallon)
Using a ruler to measure the length of objects
Computing sums, differences, and products of quantities that are measured in standard units
Reading various types of scales that are used to measure weight
Computing sums, differences, and products that involve standard units used for measuring liquids
Reading and solving story exercises that involve standard units of measure
Comparing one standard unit to another standard unit

Pupil Pages 165–184
Guide Pages 157–171

UNIT 13 • THE SET OF FRACTIONAL NUMBERS

Recognizing that numbers may be assigned to regions
Using partitioned regions to investigate quotients of whole numbers
Developing understanding of a fractional number as a quotient of two whole numbers
Interpreting fractional numbers as quotients, products, and sums
Associating fractional numbers with distances from 0 on the number line
Recognizing that some distances on the number line are both a whole number and a fractional number
Applying concepts of fractional numbers to comparison of weights
Using fractional numbers in application situations

Pupil Pages 185–196
Guide Pages 173–180

UNIT 14 • FRACTIONAL NUMBERS: ADDITION AND SUBTRACTION

Using regions as models for addition and subtraction of fractional numbers with the same denominators
Computing sums and differences of fractional numbers with the same denominators
Associating sums and differences of fractional numbers with distances from 0 on the number line
Supplying the missing addend
Strengthening understanding of the concept of difference as a missing addend

Pupil Pages 197–208
Guide Pages 181–188

UNIT 15 • GEOMETRY: LINES

Developing a perception of line
Using a folded paper model of a line to determine which points are on one line
Recognizing that two points determine a line
Counting the number of lines determined by a given number of points
Investigating the intersection of lines in space
Determining the greatest number of points of intersection of a given number of lines

Pupil Pages 209–232
Guide Pages 189–206

UNIT 16 • EXACT DIVISION—FRACTIONS AND MIXED FRACTIONS

Recognizing that some fractional numbers are whole numbers and some are not
Using regions and the number line to locate fractional numbers that are not whole numbers between two consecutive whole numbers
Using regions to illustrate a fraction for a mixed fraction and a mixed fraction for a fraction
Using the division algorism to compute mixed fractions
Writing fractions for mixed fractions and writing mixed fractions for fractions

Pupil Pages 233–266
Guide Pages 207–224

UNIT 17 • DIVISION AND MULTIPLICATION

Using multiplication as a check for division
Reviewing the computation of fractions for mixed fractions and mixed fractions for fractions
Using guessing and testing as a means of discovery
Using the multiplication algorism to compute the product of a whole number and a mixed fraction
Using multiples of 10 and 100 for partial quotients in a division algorism
Using division as a check for multiplication
Investigating short multiplication algorisms
Using division in application situations

Pupil Pages 267–276
Guide Pages 225–231

UNIT 18 • BAR GRAPHS

Recognizing that a bar graph is a way to display data
Making and interpreting bar graphs
Developing the concepts of range and arithmetic mean
Computing the mean and range

Pupil Pages 277–288
Guide Pages 233–240

UNIT 19 • STORY EXERCISES

Identifying the information in a story that is relevant to the number structure
Recognizing that the number structure of a story exercise may be represented by more than one equation
Translating the number structure of a story to an appropriate equation
Using a □ or a letter to represent a number that is not specified
Solving equations for story exercises

Pupil Pages 289–300
Guide Pages 241–250

UNIT 20 • OPEN SENTENCES

Distinguishing between true and false sentences
Recognizing that mathematical sentences may be equations or inequalities
Reviewing the symbols for greater than and less than (>, <)
Developing open sentences as sentences that may be used to generate other sentences that are either true or false
Identifying sentences as being true, false, or open
Replacing the variable in an open sentence with a number and deciding whether the resulting sentence is true or false
Recognizing that open sentences may have one solution, many solutions, or no solution
Using a trial-and-success method to find solutions to open sentences

Pupil Pages 301–320
Guide Pages 251–266

UNIT 21 • DIFFERENCES: THE SET OF INTEGERS

Using arrows to illustrate missing-addend equations on a scale
Using differences of whole numbers as solutions to missing-addend equations
Recognizing that differences of whole numbers are whole numbers or opposites of whole numbers
Recognizing that arrows with the same length and same direction represent the same number; arrows with the same length and opposite direction represent opposites
Recognizing that if a and b are whole numbers, the equation $a + \square = b$ always has a solution in the Set of Integers
Using integers to solve missing-addend problems
Using the concepts of length and direction in application situations

Pupil Pages 321–332
Guide Pages 267–275

UNIT 22 • GEOMETRY: POINTS, LINES, AND PLANES

Recognizing that a plane is determined by three points not on a line
Understanding that a plane is unbounded
Observing that four points either lie in one plane or determine four planes
Testing to show whether a fourth point is in the same plane as three given points
Perceiving that when two planes intersect, they intersect in a line
Perceiving that when two planes do not intersect, they are parallel

Pupil Pages 333–340
Guide Pages 277–282

UNIT 23 • USING THE RULER

Developing skill in the use of a ruler
Comparing lengths
Measuring lengths to the nearest inch, $\frac{1}{2}$ inch, $\frac{1}{4}$ inch, and $\frac{1}{8}$ inch
Perceiving a number of inches as a length rather than as a mark on a ruler
Interpreting distances between points on a ruler

Pupil Pages 341–368
Guide Pages 283–294

UNIT 24 • DIVISION REVIEWS COMPUTATION

Using division and multiplication algorisms to practice the computation of sums, quotients, differences, and products
Computing mixed fractions for fractions
Computing sums and differences of fractional numbers with the same denominators
Writing and solving equations for story exercises
Recognizing patterns
Determining whether one quotient is greater than or less than another quotient

Guide Pages 295–299 *Diffo I Cards*

INTRODUCTION

The child has a natural curiosity about his world. He eagerly explores his environment and is able to make simple generalizations about the objects that surround him. As he progresses through the grades, this enthusiasm for learning tends to diminish unless it is carefully nurtured.

The elementary teacher is especially sensitive to the child's eagerness to understand the world around him, and makes every effort to create a learning environment that will stimulate and encourage this active curiosity. She knows that the pupil is always at his best when he is in the process of discovery. She realizes that adhering rigidly to narrowly prescribed bounds and learning facts in neat categories may seem efficient, but this very efficiency may destroy enthusiasm for learning and stifle intellectual curiosity. She is acutely aware that her own intellectual curiosity has far-reaching effects in the learning environment. The zest for learning which the teacher evidences will be copied by the child. The teacher's use of questions will encourage the pupil to raise questions and investigate ideas.

Elementary school mathematics programs are beginning to take advantage of the child's curiosity by encouraging him to put a greater emphasis on the search for patterns, basic principles, and a logical structure of the mathematics he learns. There is a growing enthusiasm for skillful teaching procedures, well-planned activities, and the inclusion of new topics in the elementary school curriculum. The authors of the Greater Cleveland Mathematics Program are experienced classroom teachers, child psychologists, mathematics educators, curriculum specialists, and professional mathematicians. Each is aware of the trends in mathematics education and has utilized this awareness in the preparation of materials designed to provide variety and flexibility in meeting the needs of the child. The program is designed to help the child develop his power to reason, and security of understanding is provided in every phase.

GCMP GUIDELINES

The first concern of education is the learner. A good learning program provides an efficient and workable sequence of learning experiences to meet the challenge of providing better mathematics instruction for all pupils.

The concepts in the program may be as new to the teacher as to the pupil. Since instruction is the responsibility of the teacher, teacher-training films and KEY TOPICS IN MATHEMATICS FOR THE INTERMEDIATE TEACHER have been prepared to develop the understanding needed to teach the concepts effectively. This Teacher's Guide offers a rich variety of suggestions for effective teaching of the concepts and skills.

The following is a brief summary of the main features of GCMP.

The program meets two parallel objectives for mathematics education for the elementary school:

1. Development of a clear understanding of the number relationships needed in gaining proficiency in solving problems and in applying concepts.
   - Questions are emphasized more than answers. The effective application of mathematics requires the ability to define questions with precision.
   - Reasoning and understanding, rather than memorizing rules, are stressed.
   - Pupils are taught to look for powerful patterns and concepts that answer many questions of detail.
   - Every effort is made to develop an idea prior to the use of words that name the idea. Language is useless without ideas; the language of mathematics is learned as the pupil participates in developmental activities.
   - Review is built into the development and application of new concepts instead of being emphasized as an end in itself.
   - A variety of teaching techniques, ranging from information-giving to guided discovery, is recommended in the activities used to develop concepts.
2. Development of computational skills.
   - All basic facts are learned by referring to a model or picture.
   - Models or pictures are used to develop algorisms for each operation.
   - Computational skills are developed and used in application situations.
   - More practice exercises than necessary for most classes are provided in the activities and on the pupil pages.
   - A variety of teaching techniques is recommended in the activities used to develop computational skills.

The program is articulated and sequential.
- No pupil's progress is handicapped.
- No important understanding is omitted from any pupil's experience.
- No pupil is burdened with busywork of dubious value.

The program is field-tested.
- While the program demands serious and sustained effort from teachers, field testing has shown that this teacher effort results in better education for pupils.

The most important part of the program is the development of concepts and skills through a variety of experiences, many of which are described in detail in this Teacher's Guide.
- Using the pupil page is only one of many activities designed to introduce and develop basic concepts and skills.
- The pupil page may be used for discussion of ideas, practice, and to give a chance for the pupil to test what he has perceived.

Teaching the Program for Sequential Write-In Texts E, F, G, and H

Sequential Write-In Texts are textbooks rather than workbooks. They should be used by themselves rather than in conjunction with other books.

The activities used prior to work on the pupil pages are planned to stress single concepts. Some activities require the teacher to lead a group discussion, to ask carefully worded questions, and to guide the pupils to discover the underlying concept. Others indicate a direct, straightforward presentation. It seems to be true that the interaction of the pupils in a group is a necessary factor in the development of the pupil's reasoning ability.

The pupil pages marked *For Class Discussion* are used to develop significant ideas in the program. The discussion for these pages should include both pupil-pupil and teacher-pupil interaction. The pupil pages marked *reference page* contain important developmental ideas, discussion questions, generalizations, or basic facts. The teacher should plan to have pupils refer to these pages whenever they want to.

As the pupil participates in discovery situations, he perceives mathematics as a system of generalizations drawn from his experiences and begins to appreciate its simplicity. The skillful teacher realizes that it is better to gain insight into a principle than to memorize a great number of unrelated facts and procedures. The principle, when grasped, enables the pupil to discover the facts for himself and devise procedures whenever he needs them.

The teacher will notice the following features in the program contained in Sequential Write-In Texts E, F, G, and H.

1. Visual models are used extensively in deepening perception of number structure.
2. Many of the activities require the pupil to manipulate objects and make generalizations.
3. Set pictures are translated to addition equations and subtraction equations.
4. Arrays are translated to multiplication equations and division equations.
5. Place-value concepts are extended with the help of activities using wooden cubes, the Countingman, and the number line.
6. Addition, subtraction, multiplication, and division equations are used to show the number structure in given situations.
7. Algorisms for computing sums, differences, and products are reviewed.
8. Arrays are rearranged to illustrate quotient-remainder problems.
9. Algorisms for computing quotients are developed and used.
10. Measurement concepts are developed around the question "How much?" This question leads to the concept of a fractional number as a quotient of whole numbers.
11. Lengths, areas, weights, and amounts of liquid are compared to standard units. Comparison is used to develop an understanding of numbers as a measure of quantity.
12. Regions and segments are divided into equal parts to illustrate fractional numbers and sums and differences of fractional numbers.
13. A region model is used to interpret a mixed fraction for a fraction and fraction for a mixed fraction.
14. Arrow diagrams are used to investigate differences of whole numbers—integers.
15. The geometric concepts of point, line, and plane are explored. Grids and maps are used to locate points in space.
16. Pictures and trial-and-success techniques are used to solve problems.
17. Computation is used to answer questions in application situations.
18. Games, experiments, and solving interesting problems are used to help develop concepts and skills.

As the teacher guides the pupils in the program, she will see them making discoveries. As they make generalizations about their discoveries, she will want to lead them to new situations where their generalizations may be checked. She will encourage the pupils to search for new patterns, new ideas, and new relationships.

Pacing the Program

The teacher must assume responsibility for pacing the program to meet the needs of the individual pupil and of the group. The amount of time required for each unit depends on the purpose of the unit and on the ability of the group. Write-In Texts E, F, G, and H should be used in that sequence. No books should be omitted.

Sequential Write-In Text E

Unit 1 reviews addition and subtraction combinations as sets, and set pictures are translated to addition equations and subtraction equations. Units 2 and 3 review products and introduce the concept of quotient. Arrays are translated to multiplication equations and division equations reviewing basic combinations. These units offer the opportunity to move easily into the fourth-grade program by using manipulative materials and models to build new concepts and extend concepts with which pupils are already somewhat familiar. Units 5 and 6 use arrays to investigate the missing-factor problem and to introduce quotient-remainder solutions when a whole-number missing factor does not exist. Pacing throughout these five units should sustain interest but allow time for a feeling of security while materials are used to develop and review basic mathematical concepts.

In Unit 4, the multiplication algorism and place-value concepts are reviewed. Unit 7 introduces and develops an algorism for division. Presentation of these two units should proceed at a pace that allows for building computational skill and a feeling of confidence while sustaining interest.

Sequential Write-In Text F

Unit 8 continues the development of an algorism for division. Presentation of this unit should proceed at a pace that allows for building computational skill

and a feeling of confidence while sustaining interest.

Unit 9 provides exploration of some geometric concepts. In Unit 10, numeration is investigated as numbers are rounded to 10's; the rounded numbers are used to estimate sums and products. In Unit 11, a dozenal system of numeration is explored and an opportunity to apply knowledge of division is presented. Since major emphasis in these three units is on exploration rather than mastery, pacing should proceed at a rather rapid rate.

In Units 12 and 13, interesting problems and experiments in measuring weight, length, and area lead to the need for fractional numbers and develop concepts of fractional numbers. Pacing throughout these two units should sustain interest but allow time for a feeling of security while materials are used to develop and review basic mathematical concepts.

Sequential Write-In Text G

Unit 14 investigates and develops algorisms for addition and for subtraction of fractional numbers having the same denominators. Unit 16 introduces and develops an algorism for exact division. Unit 17 provides further practice in computing products, quotients, and differences of whole numbers. Presentation of these three units should proceed at a pace that allows for building computational skill and a feeling of confidence while sustaining interest.

Unit 15 provides exploration of some geometric concepts. In Unit 18, bar graphs are constructed and interpreted. Since major emphasis in these two units is on exploration rather than mastery, pacing should proceed at a rather rapid rate.

Sequential Write-In Text H

Unit 19 provides an opportunity to translate the number structure of story exercises to mathematical sentences. In Unit 21, a model is used to investigate differences of whole numbers and introduce integers. Pacing throughout these two units should sustain interest but allow time for a feeling of security while materials are used to develop and review basic mathematical concepts.

Unit 20 provides an interesting investigation of mathematical sentences. Unit 22 provides exploration of some geometric concepts. Unit 23 provides experience in using the ruler. Since major emphasis in these three units is on exploration rather than mastery, pacing should proceed at a rather rapid rate.

Unit 24 provides further practice in computing products, quotients, and differences of whole numbers. Presentation of this unit should proceed at a pace that allows for building computational skill and a feeling of confidence while sustaining interest.

Organization of the Teacher's Guide

Each unit has sections devoted to *Objectives, Key Ideas, Concepts, Scope, Fundamentals, Readiness for Understanding,* and *Developmental Experiences.* In most units there are instructions for *Pupil Work Pages* and a *Supplemental Experiences* section. To help the teacher use the guide effectively, each of these sections is discussed here.

- The *Objectives* section includes brief statements of the major goals of a unit; the *Key Ideas* and *Concepts* that are developed in the unit; and a reference to the chapter in KEY TOPICS IN MATHEMATICS FOR THE INTERMEDIATE TEACHER that presents the relevant mathematical concepts in greater detail.
- The *Key Ideas* are usually brief phrases that describe one or more of the major ideas or important insights for understanding.
- The *Scope* section indicates specific concepts or skills to be developed in the activities that follow.
- The *Fundamentals* section presents to the teacher a concise overview of the basic mathematical concepts contained in the material presented to the pupils and some pedagogical hints.
- In the *Developmental Experiences* section, the teacher will find a variety of activities used to introduce concepts, provide practice, and extend the pupil's perception. It also includes the materials list that indicates the devices and materials needed for the developmental experiences.
- The suggestions for *Pupil Work Pages* extend and implement the concepts presented on each page, and suggest additional activities and questions. Note that reduced pupil pages in the Teacher's Guide do not contain elements that appear in a second color in the Write-In Texts. In units dealing with geometry or fractional numbers, the teacher will find it helpful to refer to use of color in the pupil book to clarify some of the illustrations and exercises.
- The *Supplemental Experiences* sections may be used for review work or to reinforce a pupil's understanding of a concept. No attempt has been made to classify a supplemental activity as "enrichment" or "remedial," since what serves as review for one pupil may bring initial understanding to another.

Different types of activities are marked as follows:

▶ Developmental activity

■ Supplemental activity

● Pupil Work Page suggestions

There is a materials list at the beginning of each *Developmental Experiences* section. Some teachers ask pupils to help prepare the materials. Teachers have also found it advisable for each pupil to have a small box to hold his own supplemental materials.

UNIT 1
SETS AND PROBLEM SOLVING
Pages 1 Through 14

OBJECTIVE

To explore the relationships among the numbers in story problems.

The pupil observes the union of two disjoint sets and learns that a model of the union is any of eight equivalent equations, four addition and four subtraction. He learns to express the relationship among the numbers in a story problem by a set-picture and subsequently with an appropriate equation.

See Key Topics in Mathematics for the Intermediate Teacher: Sets and Numbers; Problem Solving.

KEY IDEAS

Subtraction is a way of looking at addition.
$a = b$ says that $b = a$.
$a + b = b + a$.

CONCEPTS

addend
difference
equation
number property
sum

KEY IDEA

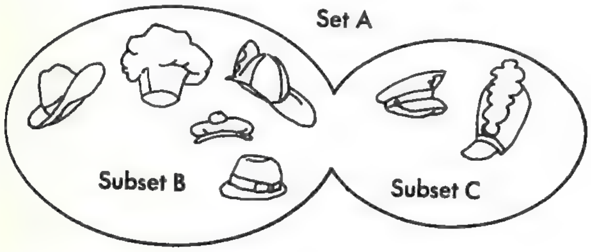
Subtraction is a way of looking at addition.

Scope

To use addition and subtraction equations to describe the relationship among numbers of a partitioned set.

Fundamentals

The number of objects in a set is the number prop-erty of the set. For example, the number property of Set A is 7, and the number properties of Subsets B and C are 5 and 2 respectively.



A partitioned set is a model for addition. Set A is artitioned into Subsets B and C, so the sum of the umber properties of Subsets B and C is the number roperty of Set A. Therefore, the number property of et A is $5 + 2$. The numbers 5 and 2 are addends; e number $5 + 2$ is the sum. Since the number prop-rty of Set A is 7, as well as $5 + 2$, we write these uations:

$$5 + 2 = 7 \qquad 7 = 5 + 2$$

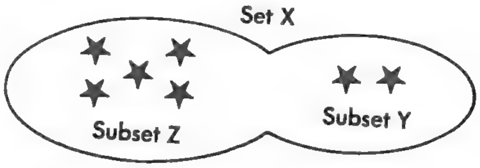
$5 + 2 = 7$ means the number $5 + 2$ is the number 7. $7 = 5 + 2$ means the number 7 is the number $5 + 2$.

The partitioned set shows that addition is commutative—in the example, $2 + 5$ is $5 + 2$.

Any of these equations express number relationships of this partitioned set.

$$5 + 2 = 7 \qquad 2 + 5 = 7$$
$$7 = 5 + 2 \qquad 7 = 2 + 5$$

The difference $b - a$ is the number y such that $a + y = b$. We write: $a + (b - a) = b$. The number relationships of this equation are illustrated by a partitioned set. Let Set X be partitioned into Subsets Y and Z, so that each of its members is placed in one or the other of Subset Y and Subset Z.



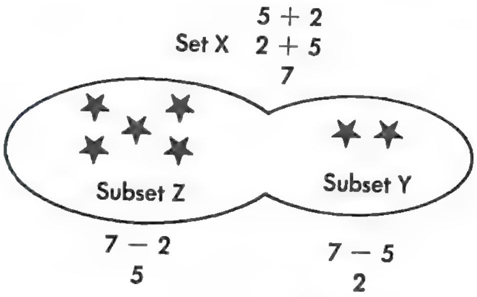
If, for example, Set X has 7 members and Subset Y has 2 members, the number property of Subset Z is the difference, $7 - 2$. We can express the number relationships of the partitioned set with this equation:

$$2 + (7 - 2) = 7$$

Since the number property of Subset Z (by counting) is 5, as well as $7 - 2$, we write the equation:

$$5 = 7 - 2$$

A summary of the relationships among the numbers in this example is shown in the following diagram.



The number structure of this partitioned set is expressed by each of the following eight equations:

$$5 + 2 = 7 \qquad 7 = 5 + 2$$
$$2 + 5 = 7 \qquad 7 = 2 + 5$$
$$7 - 2 = 5 \qquad 5 = 7 - 2$$
$$7 - 5 = 2 \qquad 2 = 7 - 5$$

Many different story exercises can have the same number structure. For example, each of these eight equations (four addition and four subtraction equations) expresses the number structure of stories like these:

John's uncle chopped down 7 trees. Five of them were oak and 2 of them were maple.

Harry's father had 7 peach trees. He chopped down 2 of them. Now he has 5 peach trees.

When one of the numbers in a story is missing,

a placeholder is used in the equation. (At this stage, a placeholder is a symbol we use when we want to say "a number" without specifying the number.)

> Kenneth climbed 5 trees yesterday and 2 other trees today. How many trees has he climbed during these two days?

If the total number of trees climbed is m, any of the following equations expresses the number structure of the story.

$5 + 2 = m \quad 2 + 5 = m \quad m - 5 = 2 \quad m - 2 = 5$
$m = 5 + 2 \quad m = 2 + 5 \quad 2 = m - 5 \quad 5 = m - 2$

The placeholder in each of these equations names 7. The answer to the story question is, "He climbed 7 trees during these two days."

If the children need more developmental experiences in addition or subtraction, the teacher may refer to Unit 6 and Unit 7 of the Third-Grade Teacher's Guide and adapt the activities presented there. It may be helpful to provide the children with practice and a chance to review addition and subtraction.

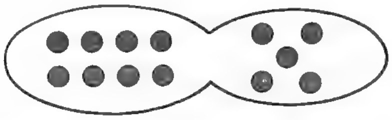
Readiness for Understanding

Ability to count.
Knowledge of sum and difference.

Developmental Experiences

for each child
20 counters
tagboard strip ($\frac{1}{4}'' \times 12''$)

► Draw the following partitioned set on the chalkboard.



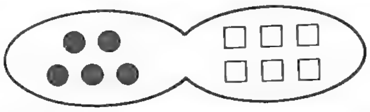
Write the following equations on the chalkboard.

$8 + 5 = 13 \qquad 13 = 8 + 5$
$5 + 8 = 13 \qquad 13 = 5 + 8$

Point out that 8 and 5 are the numbers of things in the parts of the set, and $8 + 5$ is the number of things in the set. Also, when you count the number of things in the whole set, you get 13.

By turns, ask several children to match each number in the equation $8 + 5 = 13$ with the corresponding set (8 and 5 with the parts; $8 + 5$, or 13, with the whole set). Do this with some of the other equations. Then ask the children what "equals" (=) means. They should understand that $13 = 8 + 5$ means that the number $8 + 5$ is the number 13.

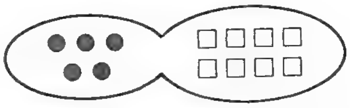
Now draw this partitioned set on the board.



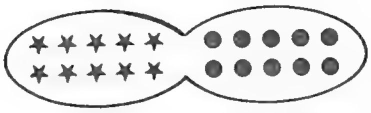
Ask a child to come to the chalkboard and write an addition equation that represents the set. Remind the class that four different addition equations can be written for this set. Ask for volunteers to write the other three equations on the board.

$5 + 6 = 11 \qquad 11 = 5 + 6$
$6 + 5 = 11 \qquad 11 = 6 + 5$

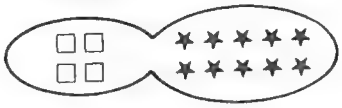
Use the following sets to continue this activity.



5 + 8



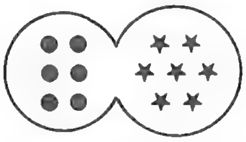
10 + 10



4 + 10

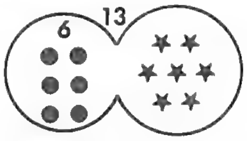
Four addition equations should be written for each of these partitioned sets.

► Draw the following partitioned set on the chalkboard, and ask a child to write the four addition equations that the set-picture suggests.

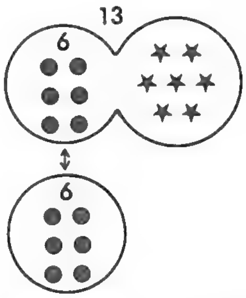


$6 + 7 = 13 \qquad 13 = 6 + 7$
$7 + 6 = 13 \qquad 13 = 7 + 6$

Write the number properties of two of the sets:

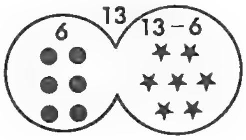


Tell the class that the number of things in the remaining set is the difference $13 - 6$. Show this by comparing the whole set to a set of 6.



The number property of the unmatched set is the difference of the number property of the set of 13 and the number property of the set of 6.

Write the number $13 - 6$ in the unlabeled set:



Ask a pupil to count this set; he will find that it has 7 members. Ask him to describe the relationship between $13 - 6$ and 7; he should say that they are the same number. Emphasize that the number $13 - 6$ is the number 7 by writing the equation $13 - 6 = 7$.

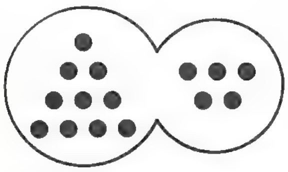
Ask another pupil to tell the difference of the number of elements in the set of 13 compared to a set of 7 elements; he should say $13 - 7$. This set has 6 elements. Ask if there is an equation that states this ($13 - 7 = 6$). Ask a volunteer to write it below the equation $13 - 6 = 7$ already on the board. Ask if there are other equations that state that $13 - 6 = 7$ or $13 - 7 = 6$. Then have a pupil write these equations on the chalkboard.

$$13 - 6 = 7 \qquad 7 = 13 - 6$$
$$13 - 7 = 6 \qquad 6 = 13 - 7$$

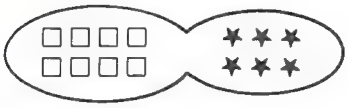
Explain how the set-picture shows the difference represented in each equation.

Have each child read his equation to the class. The first equation is read, "Thirteen minus six equals seven." Ask the meaning of this statement. The children should understand that the equal sign means that the difference $13 - 6$ and 7 are the same number.

Continue the activity with these sets.



$10 + 5$



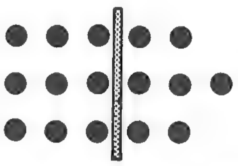
$8 + 6$

Four addition equations and four subtraction equations should be written for each set.

Give each child a set of 20 counters (washers, bottle caps, cardboard disks, or something similar) and a $\frac{1}{4}$-inch by 12-inch strip of tagboard. Write on the chalkboard an addition equation that uses upper-decade facts.

$$16 = 9 + 7$$

Direct the children to use their counters and tagboard strips to show a partitioned set that is represented by the given equation. The children will show a set with 16 members that has parts with 9 and 7 members respectively.



After the pupils have done this, discuss the meaning of each number in the equation. (16, or $9 + 7$, is the number of things in the whole set and 9 and 7 are the numbers of things in the parts.)

Ask someone to write on the chalkboard a subtraction equation that shows the relationship among the numbers 16, 9, and 7. The pupil may write:

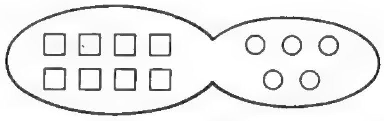
$$16 - 9 = 7$$

After the child has done this, discuss the meaning of each number in the equation. ($16 - 9$ and 7 are the number of things in one part, 9 is the number of things in the other part, and 16 is the number of things in the whole set.)

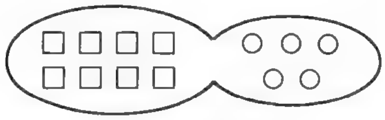
Next, choose some other child to write another subtraction equation that shows the relationship among the same three numbers. Again discuss the numbers in the equation with the class.

Continue in this way until the eight addition and subtraction equations have all been examined. Adapt this procedure to other equations involving upper-decade addition and subtraction facts.

Draw on the chalkboard a set of 8 squares and 5 circles.



Ask the children to count the things in the whole set and in each of its parts. On the chalkboard below the set, write the equation $8 + 5 = 13$.



$8 + 5 = 13$

Ask a child to write another equation that has the addends in the same order.

$$8 + 5 = 13 \qquad 13 = 8 + 5$$

The class should observe that $8 + 5$ is 13, and 13 is $8 + 5$. Call on a child to write an equation below $8 + 5 = 13$; in this equation he is to change the order of the addends. Then ask another child to do the same with the equation $13 = 8 + 5$.

$$8 + 5 = 13 \qquad 13 = 8 + 5$$
$$5 + 8 = 13 \qquad 13 = 5 + 8$$

Have the children again identify the number of things in the whole set and the number of things in each part.

On the chalkboard beside the equation $8 + 5 = 13$, write the sentence $\_\_\_ = 13 - 8$. Have a pupil complete this sentence.

$$\underline{5} = 13 - 8 \qquad 8 + 5 = 13 \qquad 13 = 8 + 5$$
$$5 + 8 = 13 \qquad 13 = 5 + 8$$

Discuss this definition: The difference $13 - 8$ is the number that, when added to 8, gives 13.

$$8 + (13 - 8) = 13$$
$$8 + \quad 5 \quad = 13$$

Now ask a child to write another equation by interchanging the difference $13 - 8$ with 5.

$13 - 8 = 5 \quad \underline{5} = 13 - 8 \quad 8 + 5 = 13 \quad 13 = 8 + 5$
$5 + 8 = 13 \quad 13 = 5 + 8$

The class should observe that 5 is 13 − 8, and 13 − 8 is 5.

Beside the equation 5 + 8 = 13, write the following incomplete sentence: ___ = 13 − 5. Have a child complete this sentence.

$13 - 8 = 5 \quad \underline{5} = 13 - 8 \quad 8 + 5 = 13 \quad 13 = 8 + 5$
$\underline{8} = 13 - 5 \quad 5 + 8 = 13 \quad 13 = 5 + 8$

Then discuss with the class the fact that the difference 13 − 5 is the number that, when added to 5, gives 13. Ask a child to write another equation by interchanging the difference 13 − 5 with 8.

$13 - 8 = 5 \quad \underline{5} = 13 - 8 \quad 8 + 5 = 13 \quad 13 = 8 + 5$
$13 - 5 = 8 \quad \underline{8} = 13 - 5 \quad 5 + 8 = 13 \quad 13 = 5 + 8$

The class should observe that if 8 is 13 − 5, then 13 − 5 is 8.

Ask someone to point out in each of the new equations the number property of the whole set (13), the number property of the subset of circles (5), and the number property of the subset of squares (13 − 5, which is the same as 8).

Pages 1 through 4

● Use page 1 for class discussion. After the children have studied the illustration at the top of the page, discuss the numbers they think of as they look at the pictures. There are 8 children. The children can also be seen as 3 boys and 5 girls (3 + 5), or as 5 girls and 3 boys (5 + 3). The number of girls is 5; the number of children who are not boys is the same, 8 − 3. The number of boys is 3; the number of children who are not girls is the same, 8 − 5.

Next ask the class to discuss how each equation describes the set of boys and girls. One child might suggest that each of the addition equations describes the entire set of children. Another child may suggest that the equations 8 − 3 = 5 and 5 = 8 − 3 describe the subset of girls. Finally, a third child may state that the equations 8 − 5 = 3 and 3 = 8 − 5 describe the subset of boys. Every equation shows the relationship among the numbers 5, 3, and 8.

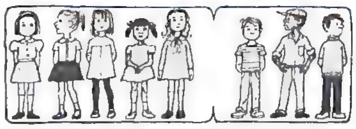
Complete the exercises on the page with the class. Discuss the equations with the children.

● Use pages 2 and 3 to give the children an opportunity to write addition and subtraction equations for partitioned sets. Assign the exercises for independent work.

Name

UNIT 1 SETS AND PROBLEM SOLVING

For Class Discussion *See pupil page suggestions.*

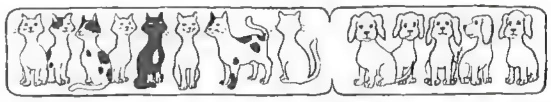


We can use many equations to describe a set. How does each equation describe this set of boys and girls?

3 + 5 = 8	8 = 3 + 5	8 − 3 = 5	5 = 8 − 3
5 + 3 = 8	8 = 5 + 3	8 − 5 = 3	3 = 8 − 5

Write four addition equations and four subtraction equations to describe each set.

9 + 7 = 16	16 = 7 + 9	16 − 9 = 7	9 = 16 − 7
7 + 9 = 16	16 = 9 + 7	16 − 7 = 9	7 = 16 − 9



8 + 5 = 13	13 = 5 + 8	13 − 8 = 5	8 = 13 − 5
5 + 8 = 13	13 = 8 + 5	13 − 5 = 8	5 = 13 − 8

reference page

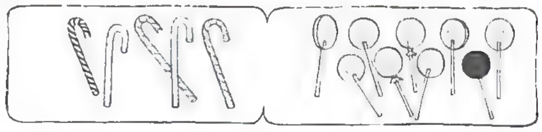
E-1

Write four addition equations and four subtraction equations for each set.

1.

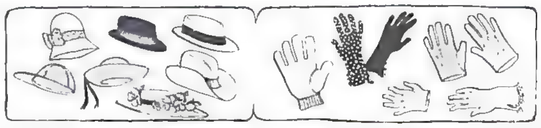
9 + 4 = 13	13 = 9 + 4	13 − 9 = 4	4 = 13 − 9
4 + 9 = 13	13 = 4 + 9	13 − 4 = 9	9 = 13 − 4

2.



5 + 9 = 14	14 = 5 + 9	14 − 5 = 9	9 = 14 − 5
9 + 5 = 14	14 = 9 + 5	14 − 9 = 5	5 = 14 − 9

3.

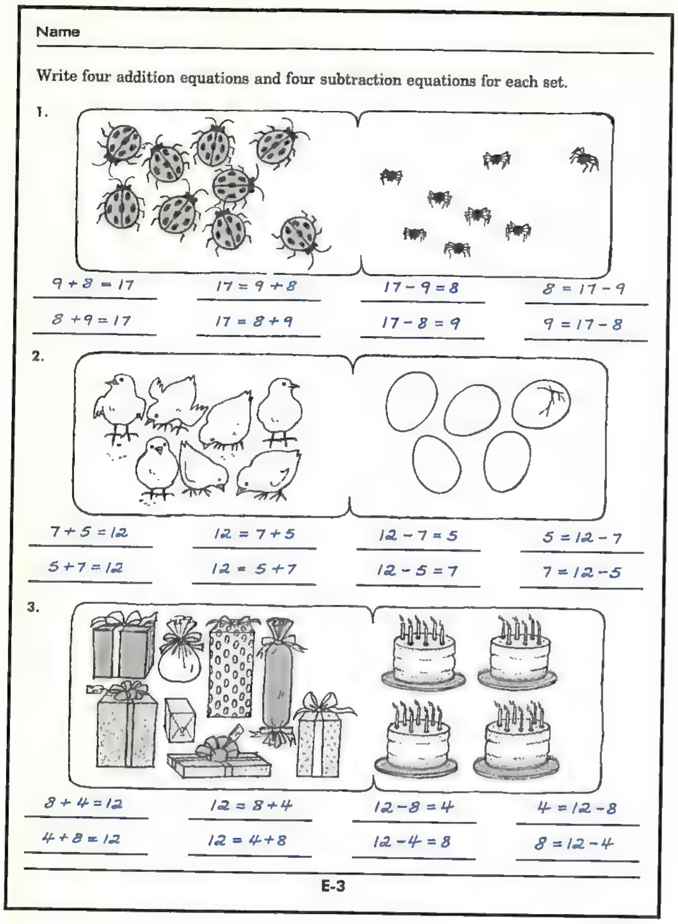


7 + 7 = 14	14 = 7 + 7	14 − 7 = 7	7 = 14 − 7

4.

6 + 7 = 13	13 = 6 + 7	13 − 6 = 7	7 = 13 − 6
7 + 6 = 13	13 = 7 + 6	13 − 7 = 6	6 = 13 − 7

E-2


Name

Write four addition equations and four subtraction equations for each set.

1.

$9+8=17$ $17=9+8$ $17-9=8$ $8=17-9$

$8+9=17$ $17=8+9$ $17-8=9$ $9=17-8$

2.

$7+5=12$ $12=7+5$ $12-7=5$ $5=12-7$

$5+7=12$ $12=5+7$ $12-5=7$ $7=12-5$

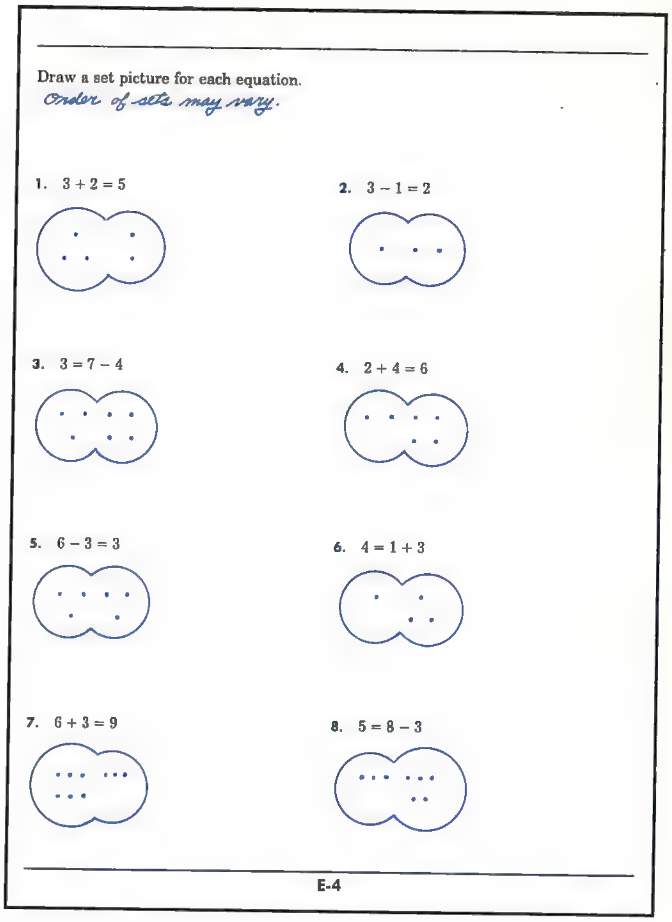
3.

$8+4=12$ $12=8+4$ $12-8=4$ $4=12-8$

$4+8=12$ $12=4+8$ $12-4=8$ $8=12-4$

E-3

● Page 4 gives children the opportunity to draw set-pictures and represent given equations. With the class, discuss and complete one or two exercises; then tell the children to complete the other exercises independently. Be sure the children understand that, in this case, it is the *number* of objects in a set that is important, not the kind of objects in the set.


Draw a set picture for each equation.

Order of sets may vary.

1. $3+2=5$

2. $3-1=2$

3. $3=7-4$

4. $2+4=6$

5. $6-3=3$

6. $4=1+3$

7. $6+3=9$

8. $5=8-3$

E-4

Supplemental Experiences

■ Play an arithmetic game. Write some of the basic addition and subtraction combinations on a set of cards. Emphasize those combinations involving sums from 10 through 18.

Let the children form two teams and stand on opposite sides of the room. Show them the cards one at a time. Each time you show a card, call on the first child who thinks he knows the computed sum or difference. If he answers correctly, he should return to his seat. If his answer is incorrect, he must remain standing, and a member of the opposing team is given

the opportunity to answer. Those pupils who are seated may serve as judges and scorekeepers. The team that first has all of its members seated wins the game.

■ An interesting challenge for the class is to ask the children to write addition equations that represent every possible partitioning of a particular set into two subsets. For example, display a set of 10. Ask the children to tell the various ways in which the set can be partitioned into two subsets. As each suggestion is made, write on the chalkboard the appropriate addition equation. These equations show the structure of any possible partitioning of a set of 10 into two subsets.

$$\begin{aligned} 0 + 10 &= 10 \\ 1 + 9 &= 10 \\ 2 + 8 &= 10 \\ 3 + 7 &= 10 \\ 4 + 6 &= 10 \\ 5 + 5 &= 10 \end{aligned}$$

Continue the activity using other sets that contain from 11 through 18 objects.

■ Make duplicate copies of a list of equations similar to these:

$$\begin{array}{l} 5 + (8 - \underline{\quad}) = 8 \\ (14 - 8) + \underline{\quad} = 14 \\ 12 = \underline{\quad} + (12 - 9) \\ 6 + (11 - 6) = \underline{\quad} \\ 9 + (\underline{\quad} - 9) = 13 \\ 9 + 5 = (\underline{\quad} - 5) + (14 - \underline{\quad}) \\ 16 = 9 + (16 - \underline{\quad}) \\ 7 + 6 = (13 - \underline{\quad}) + (13 - \underline{\quad}) \\ (15 - \underline{\quad}) + 7 = 15 \\ 17 = (17 - \underline{\quad}) + 9 \end{array}$$

Give each child a copy, and have the children complete the equations. If they understand the inverse relationship between addition and subtraction, they will be able to complete the equations without computing. Some children, however, will not recognize these as applications of the inverse relationship. Permit them to compute.

After the exercises have been completed, ask individuals to explain how they completed specific equations. One pupil may suggest that he knew the missing number was 5 in the equation $5 + (8 - \underline{\quad}) = 8$ because the difference $8 - 5$ is the number that, when added to 5, gives 8. Another pupil may explain his solution to the equation $7 + 6 = (13 - \underline{\quad}) + (13 - \underline{\quad})$ in terms of differences:

> The difference $13 - 6$ is the number that gives 13 when it is added to 6; this is the 7 in the sum $7 + 6$. The difference $13 - 7$ is the number that gives 13 when it is added to 7; this is the 6 in the sum $7 + 6$.

Allow the children to explain in their own words. The important thing is that they tell their classmates the meaning of difference, and that addition and subtraction are inverse operations. They may use words like "opposite," "backwards," or "undoing" to say these things; do not insist upon precision. An activity such as this helps the children see the advantage of getting the whole picture—looking at the entire equation before analyzing the parts.

Some of the children can make up their own equations for their classmates to complete. Perhaps you will want to encourage this. The child who can complete the equation $13 - 7 = \underline{\quad}$ is also capable of completing the equation $130 - 70 = \underline{\quad}$ (13 tens minus 7 tens is how many tens). You may want to work this idea into the activity too.

KEY IDEA

$a = b$ says that $b = a$.

Scope

To develop the relationship among numbers in a story problem.

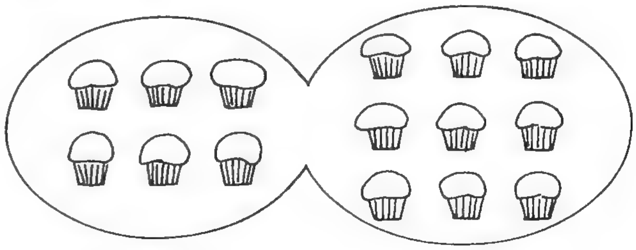
To review basic addition and subtraction facts.

Fundamentals

The third-grade child used partitioned sets and arrays as physical models of the operations of addition, subtraction, and multiplication. The union of disjoint sets shows addition. The comparison of two disjoint sets shows subtraction. The array shows multiplication. The ability to abstract number relationships from a situation expressed in a story is the next goal. The child reads a story and perceives the situation in terms of the sets involved in the story. The relationships of the cardinalities of the sets are then expressed in equations.

Subtraction is the inverse of addition, and addition is the inverse of subtraction; one operation undoes the other. Therefore, if a story situation can be represented by an addition equation, it can also be represented by a subtraction equation. For example:

> Mother baked 15 cupcakes. She gave the children 6. There were □ left.



6	□
$15 - \square$	$15 - 6$

The set-picture of the cupcakes is an aid to understanding the story. Note from the set-picture that the

sum is 15, the addends are $\square$ and 6. Each addend is a difference; 6 is $15 - \square$ and $\square$ is $15 - 6$. The equations of the set-picture are:

$$\begin{array}{ll} \square + 6 = 15 & 15 - 6 = \square \\ 6 + \square = 15 & 15 - \square = 6 \\ 15 = \square + 6 & \square = 15 - 6 \\ 15 = 6 + \square & 6 = 15 - \square \end{array}$$

There are eight different equations, each of which properly describes the mathematical structure of the story.

Readiness for Understanding

Knowledge of sum and difference.
Understanding of inverse.

Developmental Experiences

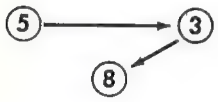
gummed circle stickers
tagboard cards (8″ × 10″)
felt-tip pen

► Place the following diagram on the chalkboard.

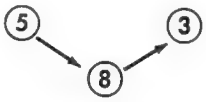
Tell the pupils that this is a picture of three "Number Towns"—"Five," "Three," and "Eight." Ask them to show as many ways as possible to travel through all three towns. Explain that in order to take a trip it will be necessary to write an equation that shows a relationship among the three numbers. For example, to begin at (5) and move through (3) to (8), the equation $5 + 3 = 8$ could be given. Ask for volunteers to tell what trip they would like to take and have them give an equation for their tickets. Continue until all possibilities have been suggested.

Beginning at (5), there are two routes:

$5 + 3 = 8$

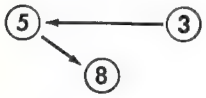


$5 = 8 - 3$



Beginning at (3), there are also two routes:

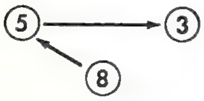
$3 + 5 = 8$



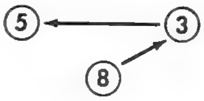
$3 = 8 - 5$

Beginning at (8), there are two routes:

$8 = 5 + 3$
$8 - 5 = 3$



$8 = 3 + 5$
$8 - 3 = 5$



When all eight equations have been written, write the following on the chalkboard.

Four ways of describing the sum-addend relationship
addend + addend = sum
sum = addend + addend
sum − addend = addend
addend = sum − addend

Ask the pupils to tell which of the equations have the sum-addend relationship *addend + addend = sum* ($3 + 5 = 8$, $5 + 3 = 8$). Write these as indicated in the chart below. Continue in this way until all 8 equations have been listed.

4 ways of describing the sum-addend relationship	8 equations for the sum-addend relationship
addend + addend = sum	$3 + 5 = 8$ $5 + 3 = 8$
sum = addend + addend	$8 = 3 + 5$ $8 = 5 + 3$
sum − addend = addend	$8 - 3 = 5$ $8 - 5 = 3$
addend = sum − addend	$5 = 8 - 3$ $3 = 8 - 5$

Then have the class observe that the sum 8 and its addends, 5 and 3, appear in each equation. Each of the eight equations describes the sum-addend relationship of the numbers 8, 5, and 3.

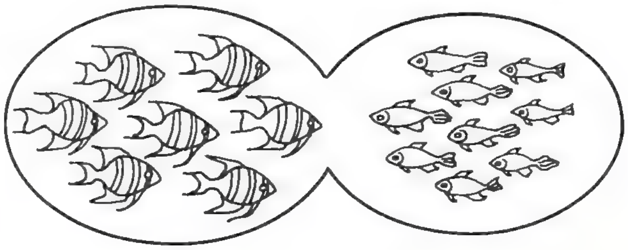
Repeat this activity with other numbers, such as the following.

6, 9, 15
8, 3, 11
5, 7, 12
4, 9, 13
6, 8, 14

► Write a story like this on the chalkboard:

Max has 16 fish in his aquarium. Of the 16 fish, 7 are baby angelfish and 9 are guppies.

Ask one child to read the story to the class, and another child to draw a partitioned set to illustrate the story situation.



Then have the class give eight equations that express the number structure of the story. Record each equation on the chalkboard as it is given.

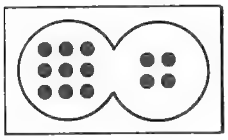
$16 = 7 + 9$	$7 + 9 = 16$
$16 = 9 + 7$	$9 + 7 = 16$
$9 = 16 - 7$	$16 - 7 = 9$
$7 = 16 - 9$	$16 - 9 = 7$

Tell each of eight children to select a different equation and discuss the sum-addend relationship among the numbers given in the equation and in the story. Repeat the activity using other stories.

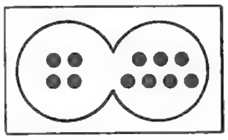
▶ Arrange gummed circle stickers on tagboard cards to show sets that are partitioned into two subsets. The number properties of the whole sets should be between 11 and 18. Make one card for each child in the class.

Have the children form two teams. Then ask a member of each team to come forward. Tell each of these two children to choose a card and to place it on the chalktray. Direct each child to write, on the chalkboard above the card, one addition equation and one subtraction equation that expresses the sum-addend relationship shown by his set.

$13 - 9 = 4$
$4 + 9 = 13$



$7 + 4 = 11$
$4 = 11 - 7$



Ask the class to tell who was first to complete the assignment correctly. Give one point to each child who completed the assignment correctly; give one additional point to the child who correctly completed his assignment first.

Continue in this way until all the children have had an opportunity to participate in the activity. Then total each team's points and declare a winner.

Pages 5 through 8

● On page 5, stories illustrated by set-pictures are introduced. There is no information missing from these stories, so there is no need for the pupil to answer questions as yet. The pupil, however, is asked to write four addition and four subtraction equations to show the number structure of each story. The teacher may want to read the stories either with or to those pupils who need this help.

Work through one of the exercises with the class. Help the pupils understand that the set-pictures are illustrations for the stories. After the procedure for work has been established, assign the remaining exercises for independent work. Then discuss the equations the pupils have chosen for each story and set-picture. Some pupils may become concerned with the monetary value of the coins in exercise 3. If the question of value arises, discuss it with the class. Help them see that the concern is with how many coins rather than how much money.

● Page 6 provides practice in matching set-pictures and equations. Once the children understand the procedure, direct them to complete the exercises independently. When they are finished, discuss the equations they selected for each set. While doing this, you may wish to have them give other equations that describe these sets.

● Page 7 provides further practice for the children in writing the eight equations that show the number structure of a story illustrated by a set-picture. Assign the exercises for independent work. Then discuss the equations the pupils have chosen for each story and set-picture.

● The set-pictures are omitted on page 8. The child is asked to draw a partitioned set and to write one addition and one subtraction equation for each story. Work exercise 1 with the class, and then assign the remaining exercises for independent work.

Some children may enjoy making up stories similar to those in exercise 2. Let these children tell their stories to the class and call on volunteers to give an addition and subtraction equation for each story.

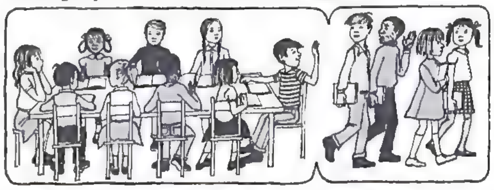
Name ______________________

Write four addition equations and four subtraction equations for each story.
See pupil page suggestions

1. Jim delivers 10 newspapers each day. He delivers 7 papers on Elm Street and 3 papers on Maple Lane.

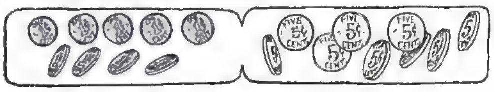
$3 + 7 = 10$ $10 = 7 + 3$ $10 - 7 = 3$ $7 = 10 - 3$
$7 + 3 = 10$ $10 = 3 + 7$ $10 - 3 = 7$ $3 = 10 - 7$

2. There were 13 pupils in a reading group. Four pupils left the group. Now there are 9 pupils in the group.



$4 + 9 = 13$ $13 = 4 + 9$ $13 - 4 = 9$ $4 = 13 - 9$
$9 + 4 = 13$ $13 = 9 + 4$ $13 - 9 = 4$ $9 = 13 - 4$

3. Charles has 18 coins in his piggy bank. Nine of the coins are pennies and 9 are nickels.



$18 = 9 + 9$ $9 + 9 = 18$ $18 - 9 = 9$ $9 = 18 - 9$

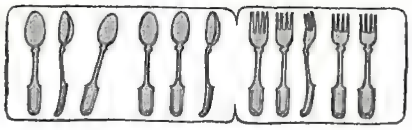
reference page

E-5

Match each equation with a set picture.

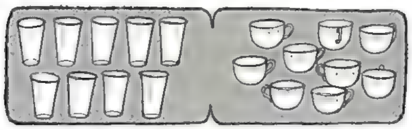
1.	$7 + 4 = 11$	D
2.	$3 = 11 - 8$	B
3.	$11 = 6 + 5$	A
4.	$15 - 7 = 8$	E
5.	$11 - 6 = 5$	A
6.	$9 = 18 - 9$	C
7.	$4 = 11 - 7$	D
8.	$3 + 8 = 11$	B
9.	$18 = 9 + 9$	C
10.	$9 + 8 = 17$	None
11.	$8 = 11 - 3$	B
12.	$15 = 8 + 7$	E

A

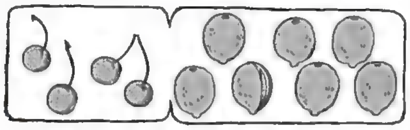


B

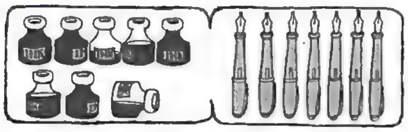
C



D



E



E-6

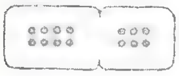
Name

Write four addition equations and four subtraction equations for each story.

1. Roberta had 8 records. She bought 5 more records. Now she has 13 records.

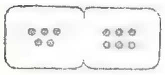
$8+5=13$	$13-5=8$
$5+8=13$	$13-8=5$
$13=5+8$	$5=13-8$
$13=8+5$	$8=13-5$

2. George had 14 pencils. He gave 8 of them to his brother. Now he has 6 pencils.



$8+6=14$	$14-8=6$
$6+8=14$	$14-6=8$
$14=6+8$	$6=14-8$
$14=8+6$	$8=14-6$

3. There were 11 birds on a branch of a tree. Five birds flew away. Six birds stayed on the branch.



$5+6=11$	$11-5=6$
$6+5=11$	$11-6=5$
$11=5+6$	$6=11-5$
$11=6+5$	$5=11-6$

4. Keith had 16 tickets to sell for the school play. He sold 9 of the tickets. He has 7 tickets left to sell.

$9+7=16$	$16-9=7$
$7+9=16$	$16-7=9$
$16=7+9$	$7=16-9$
$16=9+7$	$9=16-7$

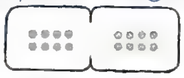
E-7

See pupil page suggestions.

Draw a set picture for each story. Write one addition equation and one subtraction equation for each story.

Order of sets may vary. Equations may vary.

1. Harry began a game with 16 marbles. He lost 8 marbles. He has 8 marbles left.

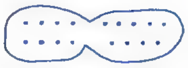


$8+8=16$
$16-8=8$

2. There were 12 plonks in a murple. Five moved away. There were 7 plonks left.

$7+5=12$
$12-5=7$

3. There were 8 peas in one pod and 9 peas in another. There were 17 peas in the two pods.



$8+9=17$
$17-8=9$

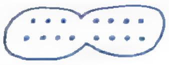
4. Bob walked 6 miles on Tuesday and 6 miles on Wednesday. He walked 12 miles in the two days.

$6+6=12$
$12-6=6$

5. Bridget had 15 cents. After she bought a pen for 9 cents, she had 6 cents left.

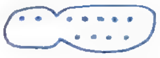
$9+6=15$
$15-9=6$

6. Ray has 7 dollars and Jim has 8 dollars. The two boys have 15 dollars.



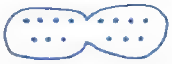
$7+8=15$
$15-7=8$

7. John had 11 marbles. He lost 2 of them. Now he has 9.



$2+9=11$
$11-2=9$

8. Evelyn had 14 cents. She gave 7 cents to her sister and 7 cents to her brother.



$7+7=14$
$14-7=7$

9. Al's dog buried 6 bones on Monday and 7 bones on Tuesday. He buried 13 bones.

$6+7=13$
$13-6=7$

E-8

Supplemental Experiences

■ To help the children become proficient in computation, read several exercises for them to compute without using pencil and paper. For example, say, "Seven plus six," pause, "minus eight," pause, "minus one," pause, "plus nine." Then call on someone to give the result (thirteen). Give another exercise, "Eighteen minus nine," pause, "plus five," pause, "minus five," pause, "minus six." Again call on someone to give the result (three).
Continue the activity by using similar oral exercises.

YOU READ	RESULT
$3 + 4 \quad + 8 \quad - 6 \quad + 2$	11
$16 - 8 \quad + 3 \quad - 4 \quad + 0$	7
$18 - 9 \quad + 6 \quad - 7 \quad - 0$	8
$7 + 8 \quad - 6 \quad + 4 \quad - 6$	7
$6 + 2 \quad + 3 \quad - 4 \quad + 3 \quad - 2 \quad + 9$	17
$8 + 5 \quad - 6 \quad + 2 \quad + 4 \quad - 7 \quad + 9$	15
$16 - 8 \quad - 3 \quad + 6 \quad - 2 \quad + 7 \quad - 8$	8

■ Provide additional opportunities for the children to make up stories and draw set-pictures that illustrate the stories. Write on the chalkboard the equations $13 = 8 + 5$ and $8 = 13 - 5$. Call on several children to tell stories related to these two equations. After the stories have been told, have the class pick the one they liked best. Write the story on the chalkboard, and ask one of the children to draw a set-picture that illustrates the story.
Now write on the chalkboard other pairs of equivalent equations.

$6 = 10 - \ 4$ $\quad$ $15 - 7 = 8$ $\quad$ $9 + 7 = 16$
$4 + \ 6 = 10$ $\quad$ $15 = 7 + 8$ $\quad$ $16 - 9 = \ 7$

Have the children each choose a pair of equations and write a story and draw a set-picture that illustrates it. Encourage the pupils to write imaginative stories. When they have finished, select several stories to share with the class. Base your selection on both the mathematical accuracy and the creativity of the story. If you want to emphasize the mechanics of writing, examine the stories in language class and have improvements suggested at that time.

■ For those children who need additional review of the upper-decade addition combinations, make a table like this.

+	9	8	6	7	5
7	7 + 9 16				
5	5 + 9 14	5 + 8 13			
9					
8			8 + 6 14		
6					

Have the pupils complete the table. The completed table will look like this:

+	9	8	6	7	5
7	7 + 9 16	7 + 8 15	7 + 6 13	7 + 7 14	7 + 5 12
5	5 + 9 14	5 + 8 13	5 + 6 11	5 + 7 12	5 + 5 10
9	9 + 9 18	9 + 8 17	9 + 6 15	9 + 7 16	9 + 5 14
8	8 + 9 17	8 + 8 16	8 + 6 14	8 + 7 15	8 + 5 13
6	6 + 9 15	6 + 8 14	6 + 6 12	6 + 7 13	6 + 5 11

The table may be varied by using different numbers.

KEY IDEA

$a + b = b + a$.

Scope

To explore some mathematical models.
To review basic addition and subtraction facts.

Fundamentals

$3 + 6 = 9$ $\quad$ $9 - 6 = 3$
$6 + 3 = 9$ $\quad$ $9 - 3 = 6$
$9 = 3 + 6$ $\quad$ $3 = 9 - 6$
$9 = 6 + 3$ $\quad$ $6 = 9 - 3$

Each of these equations expresses the sum-addend relationship of the numbers 3, 6, and 9.

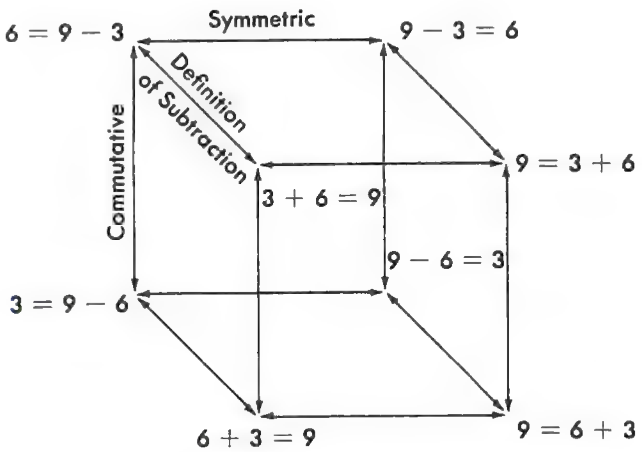
The equation $3 + 6 = 9$ expresses the same sum-addend relationship as the equation $6 + 3 = 9$. This change of addends reflects the commutative property of addition—the addends may be interchanged without affecting the sum.

The equation $6 = 9 - 3$ is related to the equation $3 + 6 = 9$, by the definition of subtraction $(3 + (9 - 3) = 9)$.

The equation $6 = 9 - 3$ is related to the equation $9 - 3 = 6$. This exchange of position reflects the symmetric property of equality.

The equation $6 = 9 - 3$ is related to the equation $3 = 9 - 6$. This change of position of the addends reflects the commutative property of addition.

The eight equivalent equations can be analyzed by using the following three-dimensional diagram. Each equation is related to three others.



As you go horizontally from one equation to the other, you are showing the symmetric property of equality.

As you go vertically from one equation to the other, you are showing the commutative property of addition.

As you go obliquely from one equation to the other, you are showing the definition of subtraction.

Readiness for Understanding

Knowledge of sum and difference.

Developmental Experiences

tagboard cards (3″ × 9″)
felt-tip pen
pocket chart

► Write on the chalkboard several stories similar to the following.

Seventeen pupils made projects for the science fair. Eight of these pupils were girls. The others were boys.

Alice has some fruit in a basket. She has 7 apples and 5 pears.

Charles and George collect unusual stones. Charles has 6 stones in his collection, and George has 8 stones.

David had 13 dollars. He spent 7 dollars of his money. He put the money he did not spend into his bank.

Tell the class that often in a story, numbers are not specifically named. This number can be expressed by a placeholder, □, and its relationship to the other numbers in the story can be expressed in an equation.

Ask the children what numbers are stated in the first story (the story states that the number of pupils is 17 and the number of girls is 8). Ask the class what number is not specifically named (the story does not specifically name the number of boys). Select one child to write an equation that represents this story situation. Then call on seven other pupils to come forward in turn and express the relationship among the numbers 17, 8, and □, by writing equations equivalent to the equation written by the first pupil.

$17 = 8 + \square$	$17 - 8 = \square$
$17 = \square + 8$	$17 - \square = 8$
$8 + \square = 17$	$8 = 17 - \square$
$\square + 8 = 17$	$\square = 17 - 8$

Repeat this procedure for the other stories on the chalkboard.

► Write this list on the chalkboard.

9 lemons and 6 oranges
13 boys and 7 girls
2 pages and 11 pages
14 birds and 6 birds
8 women and 17 men
4 eggs and 12 eggs

Help the class make up a story for each pair of items in the list. (None of the sums in these exercises should be greater than 18.) Record each story as it is constructed by the class. The following, for example, might be made up for "9 lemons and 6 oranges."

Katy made fruit punch. She used 9 lemons and 6 oranges. How many pieces of fruit did Katy use?

For each story have one child write an addition equation on the chalkboard and another child write a subtraction equation. Ask them to use a □ for any number that is not specifically named. For example, in the story about Katy's fruit punch, the number of pieces of fruit that Katy used is not specifically named. The children may write the following equations for this story.

$$\square = 9 + 6 \qquad \square - 9 = 6$$

After a pair of related equations has been written for each story, ask the children to describe how they would compute the number which □ represents. A child may say that he would add 6 and 9 and get 15. Write this solution below each equation.

$\square = 9 + 6$	$\square - 9 = 6$
$\square = 15$	$\square = 15$

Have the children discuss why both solutions are the same. They should explain that the relationship among the three numbers is:

Sum = addend + addend.

This is the same as:

Sum − addend = addend.

► Write the following four equations on the chalkboard.

$9 - 8 = \square$	$8 + \square = 9$
$\square - 8 = 9$	$9 + 8 = \square$

Next to these equations write this story:

Just before the last bell rang, the policeman helped 9 girls and 8 boys to cross the street safely. How many children were helped across the street?

Ask the children which of the four equations shows the relationship among the numbers in the story ($\square - 8 = 9$ and $9 + 8 = \square$). Have them explain why the equations they selected are appropriate. The pupils may say:

The number of girls is 9. The number of boys is 8. The number of children, □, is 9 + 8. The equation $9 + 8 = \square$ expresses this idea.

The number of children, □, minus the number of boys, 8, equals the number of girls, 9. The equation $\square - 8 = 9$ expresses this idea.

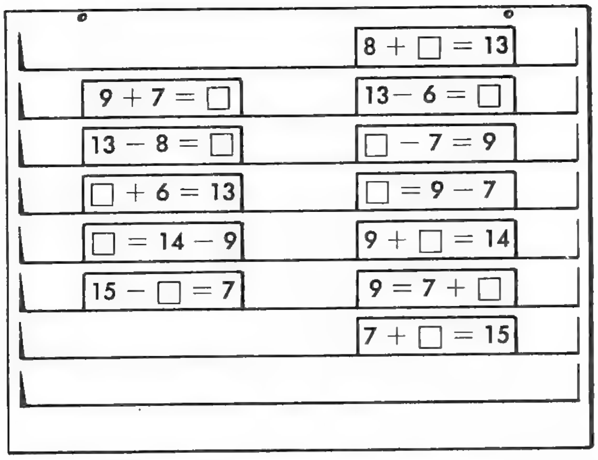
Adapt this procedure to several other stories. Remember to write four equations for each story.

► Make 100 cards like these: Each card should have an equation with a placeholder.

$3 + \square = 11$	$12 = \square + 5$	$7 + 6 = \square$
$\square = 11 - 3$	$12 - \square = 5$	$\square - 7 = 6$

Make sure that for each addition equation you have an equivalent subtraction equation.

Place five of the cards on one side of the pocket chart. Place seven cards on the other side of the chart; five of these cards should show equations that are equivalent to the first five equations. The other two cards should show equations that are not.



Call five pupils forward in turn; ask each to match an equation on the left side of the chart with one equivalent to it on the right side. Tell each that he may do this by placing the two equations side-by-side in the chart. After the assignment has been carried out, have the class decide whether or not the equations have been matched correctly. Have any errors corrected.

Continue in a similar way with other sets of 12 cards until all the pupils have been given at least one opportunity to match a pair of equivalent equations.

▶ Write on the chalkboard several addition and subtraction equations in which a placeholder is used. Have the pupils tell whether $\square$ is the sum or one of the addends in each equation.

$3 + 9 = \square$	(sum)	$6 - \square = 4$	(addend)
$4 + \square = 10$	(addend)	$\square - 7 = 2$	(sum)
$\square + 7 = 13$	(addend)	$14 - 6 = \square$	(addend)
$\square = 5 + 6$	(sum)	$\square = 18 - 9$	(addend)
$12 = \square + 3$	(addend)	$5 = \square - 7$	(sum)
$16 = 8 + \square$	(addend)	$2 = 11 - \square$	(addend)

Next, write on the chalkboard the equation $5 + 4 = \square$. Then call on a child to write an equivalent subtraction equation just below it. He may write any of the following:

$$\square - 5 = 4 \qquad 5 = \square - 4$$
$$\square - 4 = 5 \qquad 4 = \square - 5$$

For both the original equation and the equation that was chosen, have the class tell whether $\square$ is the sum or one of the addends.

Follow a similar procedure with the subtraction equation $8 - 3 = \square$. This time, after you write the equation on the chalkboard, ask a child to write an equivalent addition equation. He may write any one of the following equations.

$$\square + 3 = 8 \qquad 8 = \square + 3$$
$$3 + \square = 8 \qquad 8 = 3 + \square$$

Again ask the class whether $\square$ is the sum or one of the addends.

Continue with other addition and subtraction equations that contain a placeholder. Such equations as the following may be used.

$$2 + 8 = \square \qquad \square = 9 + 1 \qquad 8 - \square = 4$$
$$4 + \square = 13 \qquad 12 = \square + 0 \qquad \square - 5 = 6$$
$$\square + 3 = 5 \qquad 15 = 7 + \square \qquad 14 - 7 = \square$$

In each instance where an addition equation is used, have an equivalent subtraction equation written; in each instance where subtraction is used, have an equivalent addition equation written. As the children observe whether $\square$ is the sum or one of the addends in the pair of equations, they will have the opportunity to reinforce their understanding of the relationship between addition and subtraction.

▶ On the chalkboard, write a story in which no numbers are specified. For example:

Paul collected some baseball cards in March and some baseball cards in April. How many baseball cards did he collect in these months?

Explain to the pupils that they can write equations which express the relationship of the numbers in this story by using placeholders. For instance:

Let $\square$ be the number of cards Paul collected in March.
Let $\triangle$ be the number of cards Paul collected in April.
Let $\bigcirc$ be the number of cards Paul collected in March and April.

Since the number of cards collected in March and April, $\bigcirc$, is $\square + \triangle$, we may write $\square + \triangle = \bigcirc$ to express the number relationship in this story.

Choose volunteers. Have each one write an addition or subtraction equation that expresses the same number relationship.

$$\square + \triangle = \bigcirc \qquad \bigcirc = \square + \triangle$$
$$\triangle + \square = \bigcirc \qquad \bigcirc = \triangle + \square$$
$$\bigcirc - \square = \triangle \qquad \triangle = \bigcirc - \square$$
$$\bigcirc - \triangle = \square \qquad \square = \bigcirc - \triangle$$

When you have finished, adapt this procedure to the following stories. Allow the pupils to choose placeholders for the numbers in the story.

George has several books at home. He borrowed some of the books from the library. The others belong to him.

Linda's cat had a litter of kittens. Linda kept some of these kittens. Some she gave away.

Mrs. Hall used some eggs in her salad. She got them from a carton of eggs. After Mrs. Hall took the eggs for her salad, some eggs were left in the carton.

The ability to solve story exercises will vary with the ability of each pupil. Understanding the number structure of a story and realizing that the number structure may stay the same when the story situation is changed, should increase a pupil's comprehension. The improvement of each pupil in this difficult area must be taken into consideration. Small gains are important; they should be accepted, settled for, and given credit.

After help has been given in needed areas, some pupils will not live up to expectations. At this stage, additional work often results in a waste of time. Pupils need time to grow, so give them this time. One or two story exercises placed on the chalkboard occasionally while working in other units may be all that is needed to maintain the ability to understand the number struc-

ture of story exercises. The spiral effect of the fourth-grade program also provides for review, and this will clear up deficiencies for many pupils. It is most important to remember that individual differences in the class will not allow for complete mastery in all areas.

Pages 9 through 14

● Since an understanding of number structure is a key to problem solving, the pupil is gradually introduced to stories in which some of the numbers are not specifically named. Up to this time, the pupil has written equations for stories in which all numbers are specifically named. On pages 9 through 11, the pupil is asked to write one addition and one subtraction equation for each of several stories. A placeholder is used in the story and in the equations to represent the number that is not specifically named. For the time being, no attempt is made to solve the equations.

Discuss the example at the top of page 9 with the class. Questions such as these may help the pupils to better understand the number structure of the story.

What information is given? (10 pieces of fruit; 7 were apples; □ were pears)

What number does □ represent? (the number of pears)

Ask pupils what other addition and subtraction equations could have been given for this story.

$$\square + 7 = 10 \qquad 10 - 7 = \square$$
$$10 = \square + 7 \qquad \square = 10 - 7$$
$$10 = 7 + \square \qquad 7 = 10 - \square$$

Then work through exercises 1 and 2 with the class. After the procedure for work has been established, assign exercises 3 through 7 for independent work. Upon completion of these exercises, discuss the equations that the pupils have written. For each equation, ask whether □ is a sum or a missing addend. Help the pupils see that the sum and addends do not always occur in the same position in an equation. For example, in the equation, $\square = 7 + 8$, the sum is in the first position; in the equation, $7 = 15 - \square$, the sum is in the middle position; and in $7 + 8 = \square$, the sum is in the last position.

The children have had practice in writing addition and subtraction equations for set-pictures and stories. At the bottom of page 9, they are given an addition equation and a related subtraction equation and are asked to supply a story. Most children at this grade level have good imaginations and are quite creative. Do not let the actual writing of a story, or struggles with grammar or spelling, slow down the learning process; some children may prefer to tell their stories. Stories can also be composed by the class as a whole. An activity such as this helps the pupils realize that the number structure of one story may be identical to the number structure of a different story. They begin to conclude that the number structure is not dependent on the people, places, objects, or color of the objects described in the story. Remind the pupils to use a placeholder to represent a number that is not specifically named.

Name ____________

Terry had 10 pieces of fruit in a basket. Seven were apples and □ were pears.

See pupil page suggestions.

$7 + \square = 10$ $\qquad$ $10 - \square = 7$

Write one addition equation and one subtraction equation for each story.

Equations may vary.

1. Jenny had 12 pages to read to finish a story. She read 3 of them before dinner. She has □ more pages to read. — $3 + \square = 12$; $12 - 3 = \square$
2. A classroom library had 8 books one year. The next year the class bought 5 more books. There are now □ books in the classroom library. — $5 + 8 = \square$; $\square - 8 = 5$
3. Joanne could work 9 of 11 math exercises. She needed help with □ of the exercises. — $9 + \square = 11$; $11 - 9 = \square$
4. Jack bought □ zeeks. On his way to school he dropped and broke 7 of them. He had 4 zeeks left. — $4 + 7 = \square$; $\square - 7 = 4$
5. Tom had 8 pieces of bubble gum. He gave 5 pieces to his friends. Tom kept □ pieces of gum. — $5 + \square = 8$; $8 - \square = 5$
6. Jan jumped rope 7 times without missing. Lois jumped 16 times without missing. Lois jumped □ more times without missing than Jan. — $\square + 7 = 16$; $16 - 7 = \square$
7. The school supply room is 18 feet long and 9 feet wide. It is □ feet longer than it is wide. — $9 + \square = 18$; $18 - 9 = \square$
8. Write a story for $5 + \square = 7$ and $7 - 5 = \square$.

Any story that uses a correct relationship for 7, 5, and 2 (7 − 5 = 2, 2 + 5 = 7, and so forth) is acceptable.

reference page

E-9

● Pages 10 and 11 contain story exercises for which the pupils are to write one addition and one subtraction equation. Work through the example and exercise 1 on page 10 with the class. Assign the remaining exercises on page 10 and the exercises on page 11 for independent work. An effort has been made to keep the vocabulary relatively simple so that most pupils can complete one or both of these pages independently. Those pupils who find the set-picture helpful in seeing the number structure of the stories should be permitted to draw their own pictures. In discussing the equations that are selected for each exercise, ask whether □ is the sum or one of the addends. After correcting just one of these pages, it should be possible for the teacher to identify those pupils who are having difficulty.

● The story exercises on pages 12 through 14 are designed to help the pupils see that usually a question asks that the unspecified number be computed. They are then asked to write and solve one equation for each story and to express the answer to the question in the form of a sentence.

Discuss the story at the top of page 12. Remind the children that certain numbers may not be specifically named. Have someone read the example story to the class. Ask what number is not specifically named (the number of home runs hit by the other players). Point out that □ has been used to represent this number. Ask what other addition equations might be given for this story ($10 = \square + 4$, $\square + 4 = 10$, $4 + \square = 10$). Ask what subtraction equations could have been given ($10 - 4 = \square$, $10 - \square = 4$, $\square = 10 - 4$, $4 = 10 - \square$). Ask whether □ in each of these equations is the sum or one of the addends (one of the addends). Show the children where the computed number of home runs hit by the other players has been recorded ($\square = 6$). This number can be used in the sentence that answers the question.

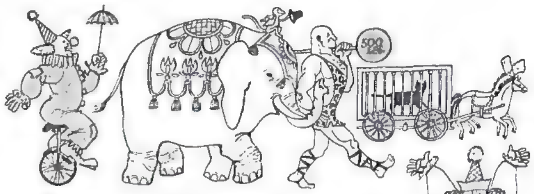
To help the children make the transition from incomplete sentences such as, "She lost □ games." to questions such as, "How many games did she lose?" have them express the incomplete sentences given on previous pages as questions. Until now, the children have had merely to write one addition and one subtraction equation for each story. Now they are asked to compose their own sentences to answer the question.

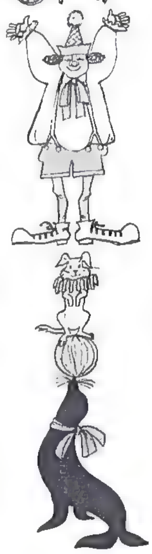
Complete the exercises on page 12. Encourage the children to discuss the equations that describe each story and to compose sentences that answer the story questions. After the page has been completed, have them make up other story problems for the same equations. For example, if someone wrote the equation $7 + 4 = \square$ for exercise 2, he may make up this story problem for the same equation:

> Charles has 7 stones and George has 4. How many stones do these boys have?

Making up several story problems for the same equation may help the children see the number structure in any story.

● Work one or two exercises on pages 13 and 14 with the children so that they understand the procedure to be followed. Then assign the rest of each page for independent work. Since the children are aware that the additive number structure of a particular story may be expressed either by an addition equation or by a subtraction equation, they should be encouraged to select an equation that seems easiest to them. No attempt should be made to seek uniformity of choice or to lead the pupils to write particular kinds of equations for particular types of stories. While checking the exercises, have each child who gives an answer state a subtraction or addition equation equivalent to the one he used.





See pupil page suggestions.

In the spring the circus came to Clearfield. It had 9 lions and 8 tigers. There were □ lions and tigers.

$\square = 8 + 9$ $\qquad$ $\square - 9 = 8$

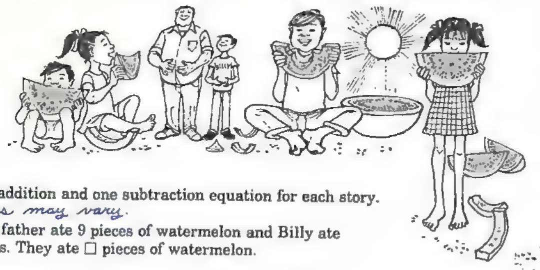
Write one addition and one subtraction equation for each story.

Equations may vary.

1. The men spent 9 hours taking the animals off the train. They spent 5 hours setting up the tents. The men worked for □ hours.
   $9 + 5 = \square$ $\qquad$ $\square - 5 = 9$
2. Ray watched the men unload the elephants and the horses. He counted 14 animals. There were 6 elephants and □ horses.
   $6 + \square = 14$ $\qquad$ $14 - \square = 6$
3. Ray had 17 peanuts. He fed 8 peanuts to the elephants and he ate the rest. Ray ate □ peanuts.
   $8 + \square = 17$ $\qquad$ $17 - \square = 8$
4. There were 9 lions in one act. Four lions stood up on their hind legs. The other □ lions jumped through a flaming hoop.
   $\square + 4 = 9$ $\qquad$ $9 - 4 = \square$
5. There were 16 clowns. There were 7 happy clowns and □ sad clowns.
   $7 + \square = 16$ $\qquad$ $16 - \square = 7$

E-10

Name



Write one addition and one subtraction equation for each story. *Equations may vary.*

1. Billy's father ate 9 pieces of watermelon and Billy ate 3 pieces. They ate □ pieces of watermelon.

   9 + 3 = □ □ − 3 = 9

2. Mrs. Smith's rose bush had 15 roses on it. She cut off 6 roses. There were □ roses left on the bush.

   6 + □ = 15 15 − □ = 6

3. Mrs. Wood canned 6 jars of tomatoes and 4 jars of beans. She canned □ jars in all.

   6 + 4 = □ □ − 4 = 6

4. Tony and Fran ate 8 pieces of toast. Tony ate 4 pieces. Fran ate □ pieces of toast.

   4 + □ = 8 8 − □ = 4

5. In a ball game Joe's team made 6 runs. There were 13 runs in the entire game. The other team made □ runs.

   6 + □ = 13 13 − □ = 6

6. Eight blicks and 2 snads were choozling together. There were □ blicks and snads that were choozling.

   8 + 2 = □ □ − 8 = 2

See pupil page suggestions.

There were 10 home runs hit in the ball game. Jim hit 4. How many did the other players hit?

10 = 4 + □ □ = 6

The other players hit 6 home runs.

Write an equation for each story. Solve the equation. Write your answer in a sentence. *Equations may vary.*

1. Alice wants to buy a notebook that costs 16 cents. She has 8 cents. How much more money does she need?

   8 + □ = 16

   □ = 8 *She needs 8¢ more.*

2. On Saturday Betty put 7 stamps in her album. On Sunday she put in 4 more stamps. How many stamps did she put in her album on those days?

   7 + 4 = □

   □ = 11 *She puts 11 stamps in her album.*

3. Al will be 15 years old in 6 years. How old is Al now?

   6 + □ = 15

   □ = 9 *He is 9 years old now.*

4. Tom had 9 baskets of apples. He sold 7 baskets. How many does he have left?

   9 − 7 = □

   □ = 2 *He has 2 baskets left.*

5. Cindy, Betty, and Ruth joined a sewing club. There are now 12 girls in the club. How many girls were in the sewing club before these girls joined?

   □ + 3 = 12

   □ = 9 *9 girls were in the club.*

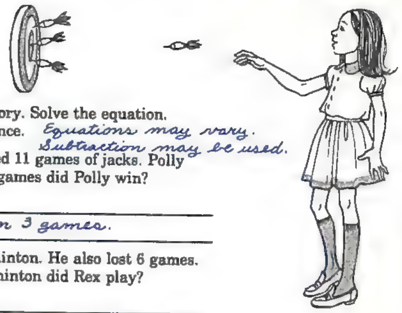
6. Kay blew up 11 balloons for a relay race. Her friends broke 6 of them in the race. How many were not broken?

   6 + □ = 11

   □ = 5 *5 balloons were not broken.*

reference page

Name



Write an equation for each story. Solve the equation. Write your answer in a sentence. *Equations may vary. Subtraction may be used.*

1. Polly and her sister played 11 games of jacks. Polly lost 8 games. How many games did Polly win?

   8 + □ = 11

   □ = 3 *Polly won 3 games.*

2. Rex won 4 games of badminton. He also lost 6 games. How many games of badminton did Rex play?

   4 + 6 = □

   □ = 10 *Rex played 10 games.*

3. Nan threw 16 darts at a target. She missed the bull's-eye 9 times. How many times did she hit the bull's-eye?

   9 + □ = 16

   □ = 7 *She hit it 7 times.*

4. Ken scored 9 points in a game. Sue scored the same number of points. How many points did they score together?

   9 + 9 = □

   □ = 18 *They scored 18 points.*

5. Tim scored 9 points in a basketball game. Bill scored 2 more points than Tim. How many points did Bill score?

   9 + 2 = □

   □ = 11 *Bill scored 11 points.*

6. Five boys were flying kites. Eight girls were flying kites. How many children were flying kites?

   5 + 8 = □

   □ = 13 *13 children were flying kites.*

reference page

Write an equation for each story. Solve the equation. Write your answer in a sentence. *Equations may vary.*

1. Cathy and Judy found 5 leaves. Dick and John found 5 leaves. How many leaves did the children find?

   5 + 5 = □

   □ = 10 *The children found 10 leaves.*

2. Eight girls were playing ball. Two girls stopped playing. How many girls were still playing ball?

   8 − 2 = □

   □ = 6 *6 girls were still playing ball.*

3. Connie cleaned 7 chalkboard erasers. There were 5 more that needed cleaning. How many erasers were there?

   7 + 5 = □

   □ = 12 *There were 12 erasers.*

4. Dan's family hiked 6 miles from home and 6 miles back. How far did they hike?

   6 + 6 = □

   □ = 12 *They hiked 12 miles.*

5. A baseball team played 10 games. They lost 3 games and won the rest. How many games did they win?

   10 − 3 = □

   □ = 7 *They won 7 games.*

6. Carol bought 1 new hair ribbon. She had 9 ribbons. How many ribbons does she have?

   1 + 9 = □

   □ = 10 *She has 10 ribbons.*

Supplemental Experiences

■ Divide the class into two teams. Call out a number from 10 through 18 and let the children take turns naming the given number as a sum. As the sums are given, list them on the chalkboard. Give a point for each correct response.

The pupils should not repeat a sum that has been given previously, nor give it in commuted form. For example, if you name 14, the first member of Team A may give 7 + 7, and the first member of Team B may give 8 + 6. Continue alternating between Team A and Team B until all of the possible sums for 14 have been listed. Then name another number and follow the same procedure. Several possibilities are listed here.

14	10	17	15
0 + 14	0 + 10	0 + 17	0 + 15
1 + 13	1 + 9	1 + 16	1 + 14
2 + 12	2 + 8	2 + 15	2 + 13
3 + 11	3 + 7	3 + 14	3 + 12
4 + 10	4 + 6	4 + 13	4 + 11
5 + 9	5 + 5	5 + 12	5 + 10
6 + 8		6 + 11	6 + 9
7 + 7		7 + 10	7 + 8
		8 + 9	

The team with the greatest number of points is declared the winner.

■ Vary the activity by telling the class a number from 0 through 9 and having the team members name the given number as a difference. Several possibilities are listed here.

9	8	7
18 − 9	17 − 9	16 − 9
17 − 8	16 − 8	15 − 8
16 − 7	15 − 7	14 − 7
15 − 6	14 − 6	13 − 6
14 − 5	13 − 5	12 − 5
13 − 4	12 − 4	11 − 4
12 − 3	11 − 3	10 − 3
11 − 2	10 − 2	9 − 2
10 − 1	9 − 1	8 − 1
9 − 0	8 − 0	7 − 0

■ Write on the chalkboard two equations that lack operation signs.

$$5 \triangle 4 = 9 \qquad 13 \triangle 4 = 9$$

Ask the children what operation sign belongs in the first equation (addition). Let someone write the plus sign in the placeholder. Ask another child to complete the second equation and read it to the class. Continue with other equations like these:

$$13 \triangle 9 = 4 \qquad 4 = 13 \bigtriangledown 9$$
$$13 = 9 \square 4 \qquad 16 = 9 \square 7$$
$$9 = 13 \bigtriangledown 4 \qquad 9 \triangle 7 = 16$$

UNIT 2
DIVISION—INVERSE MULTIPLICATION

Pages 15 Through 22

OBJECTIVE

To introduce the quotient as a factor of a product.

The pupil learns that a factor, for example, $\square$ in $a \times \square = b$, is a quotient. He learns that such a quotient may be named $b \div a$ or $\frac{b}{a}$. He uses his knowledge of multiplication to compute the quotient.

See Key Topics in Mathematics for the Intermediate Teacher: Multiplication and Division of Whole Numbers.

KEY IDEA

Division is the inverse of multiplication.

CONCEPTS

quotient

KEY IDEA

Division is the inverse of multiplication.

Scope

To use arrays to show inverse multiplication.

Fundamentals

Division is the operation that is the inverse of multiplication and is defined in terms of multiplication. The basic multiplication equation is:

Factor times factor equals product.

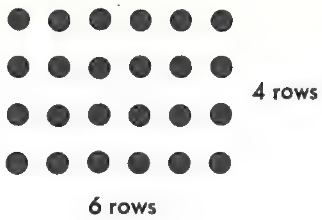
The basic division equation is:

Dividend divided by divisor equals quotient.

The dividend is the product of the divisor and the quotient, so the divisor and quotient are factors of the dividend.

The array is the basic model for multiplication in the set of whole numbers. The array therefore provides a basic model for division in the set of whole numbers.

For example, consider the product 4×6 which is the number of members of a 4 by 6 array.



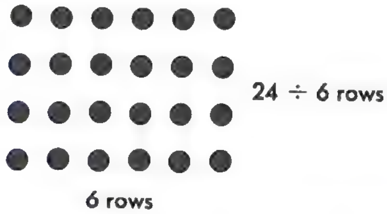
The factored product 4×6 is the same number as the sum $20 + 4$; this can be determined by counting or by using other processes. Thus the computed product, 24, is the same number as the factored product, 4×6.

$$4 \times 6 = 24$$

Each factor can be described as the quotient of the product divided by the other factor.

$$4 = 24 \div 6 \text{ or } 4 = \tfrac{24}{6}$$

$$6 = 24 \div 4 \text{ or } 6 = \tfrac{24}{4}$$



When the array is used as a model for multiplication, a product (factor $\times$ factor) is the number of objects in the array.

$$\underbrace{\overset{\text{factor}}{4} \times \overset{\text{factor}}{6}}_{\text{product}} = \underset{\text{product}}{24}$$

When the array is used as a model for division, a quotient (product $\div$ factor) is a number of rows.

$$\begin{array}{ccccc} 4 & = & 24 & \div & 6 \\ | & & | & & | \\ \text{factor} & & \text{product} & & \text{factor} \\ | & & | & & | \\ 6 & = & 24 & \div & 4 \end{array}$$

$$\begin{array}{ccc} & & \text{dividend} \quad \text{divisor} \\ 4 & = & 24 \div 6 \\ | & & \\ \text{quotient} & & \\ | & & \\ 6 & = & 24 \div 4 \\ & & \text{dividend} \quad \text{divisor} \end{array}$$

In this unit, the pupil identifies the dividend as a product and the divisor and quotient as factors. He learns that:

factor = product ÷ factor

quotient = product ÷ factor

He also learns that the same idea can be written:

quotient = dividend ÷ divisor

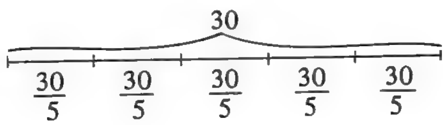
He learns to express quotients using the division sign (÷) as well as the fraction bar (—). He learns that the quotient 24 *divided by* 6 can be expressed in each of the following ways:

$4, \frac{24}{6}, 24 \div 6$

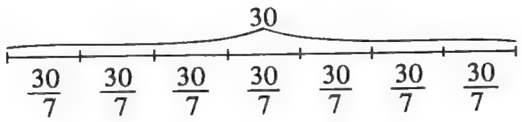
Since the fraction bar is a division sign, it is acceptable for him to read both the fraction bar and the division sign in the same way. We read "$\frac{24}{6}$" both as "twenty-four sixths" and as "twenty-four divided by 6." We read "24 ÷ 6" both as "twenty-four sixths" and as "twenty-four divided by 6." Exactly the same concept underlies both symbols.

When we use a model to represent multiplication and division in the set of whole numbers, we should be able to use the model to demonstrate the properties of multiplication and division in the set of whole numbers. We can do this with the array. For example, the model should show that the product of two whole numbers is a whole number (it does). It should also show that some quotients of two whole numbers are not whole numbers. For example, an array of 30 objects in 5 rows does exist but an array of 30 objects in 7 rows does not. Thus, 30 ÷ 5 is a whole number but 30 ÷ 7 is not. We say 30 is divisible by 5 and 30 is not divisible by 7.

In later units, the pupil is introduced to a model for division that does not have the limitations of the array. This model, in which a measure is divided into equal parts, serves as the basic model for division in the set of fractional numbers. Using a line segment, the model for 30 *divided by* 5 is as follows:



Each of the 5 equal parts represents the fractional number 30 ÷ 5, or $\frac{30}{5}$. The model for 30 *divided by* 7 is as follows:



Each of the 7 equal parts represents the fractional number 30 ÷ 7, or $\frac{30}{7}$. This model works in the set of whole numbers and in the set of fractional numbers, and shows that whole numbers are fractional numbers as well.

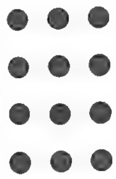
Readiness for Understanding

Knowledge of product.

Developmental Experiences

for each child
40 counters

► Give each pupil 40 counters—washers, bottle caps, cardboard squares, or something similar. First have them form an array with 3 rows of counters, 4 counters in each row. Ask them how many counters are in the array. Draw the following illustration on the chalkboard:



3 × 4
4 × 3
12

Point out that the product for this array may be named as 3 × 4, 4 × 3, or 12.

Have the class form other arrays, such as these, and name the product for each array in 3 ways.

2 by 8,
3 by 6,
4 by 7,
6 by 6,
9 by 3.

Write 3 × 4 =___on the chalkboard. Ask a pupil to write the standard numeral for the product. Review the idea that 3 × 4 and 12 are the same number. In this case 3 and 4 are factors of the product.

factor × factor = product

3 × 4 = 12

Continue with products for the other arrays: 2 × 8 =___, 3 × 6 =___, 4 × 7 =___, 7 × 8 =___, 9 × 6 =___, and 8 × 9 =___. In each example, have the pupils compute the product and identify the factors.

► Write the following equation on the chalkboard.

6 × □ = 18

Have the class arrange 18 counters in rows of 6 to form an array. Tell the class that the "missing" number of rows is 18 *divided by* 6. Write this on the chalkboard:

□ is 18 divided by 6.

Have 30 counters arranged in rows of 5 to form an array. Write the following on the chalkboard.

5 × □ = 30
□ is 30 divided by___

Ask a pupil to tell the number that goes in the blank. Then ask for the "missing" number of rows. A child may respond, 6. Tell the class that 30 *divided by* 5 is also the missing factor.

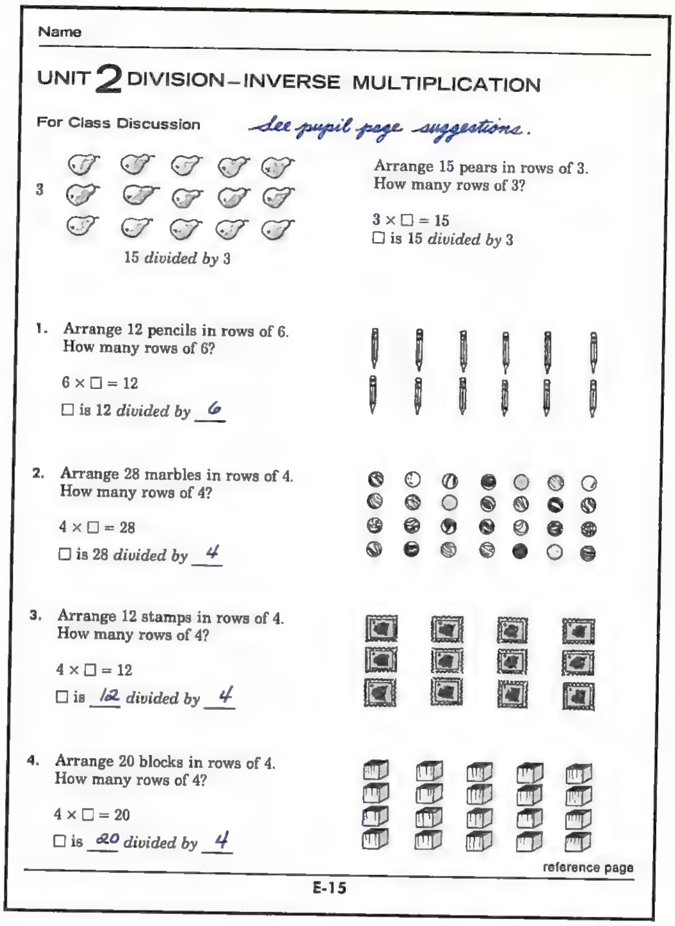
Write examples such as the following on the chalkboard. Have the pupils make an appropriate array for each example. Then ask them to fill in the blanks to complete the sentences.

7 × □ = 28
□ is __ *divided by* __
2 × □ = 18
□ is __ *divided by* __
3 × □ = 21
□ is __ *divided by* __
6 × □ = 24
□ is __ *divided by* __
5 × □ = 40
□ is __ *divided by* __
4 × □ = 16
□ is __ *divided by* __

Pages 15 through 19

● Pages 15 and 16 provide experience for the children in recognizing quotients. As page 15 is discussed, tell the pupils that the unstated number of rows in each case is a quotient. By using quotients to answer the question, "How many rows?", it should be clear to the pupils that quotients are numbers. For example, when 15 pears are arranged in 3 rows one way, there are 15 ÷ 3 rows the other way and a 3 by 15 ÷ 3 array is formed.

As page 16 is discussed, introduce the division sign (÷). Tell the class that 18 ÷ 6 may be read, "eighteen divided by six." For each exercise, have the pupils use the division sign (÷) in writing each quotient on the chalkboard.



Name

UNIT 2 DIVISION–INVERSE MULTIPLICATION

For Class Discussion *see pupil page suggestions.*

3

15 *divided by* 3

Arrange 15 pears in rows of 3. How many rows of 3?

3 × □ = 15
□ is 15 *divided by* 3

1. Arrange 12 pencils in rows of 6. How many rows of 6?
6 × □ = 12
□ is 12 *divided by* 6

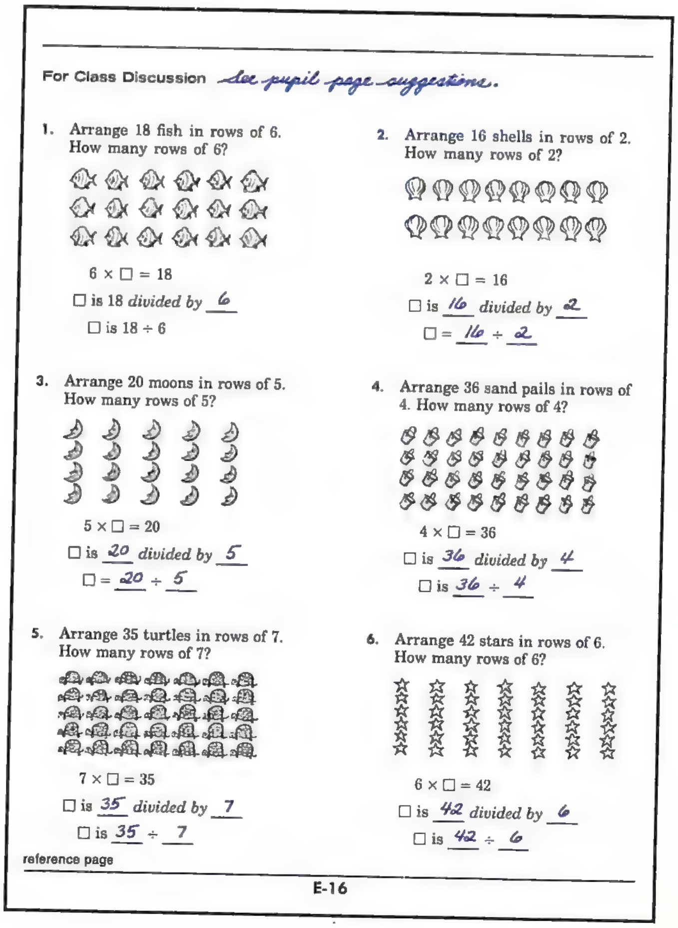
2. Arrange 28 marbles in rows of 4. How many rows of 4?
4 × □ = 28
□ is 28 *divided by* 4

3. Arrange 12 stamps in rows of 4. How many rows of 4?
4 × □ = 12
□ is 12 *divided by* 4

4. Arrange 20 blocks in rows of 4. How many rows of 4?
4 × □ = 20
□ is 20 *divided by* 4

reference page

E-15



For Class Discussion *see pupil page suggestions.*

1. Arrange 18 fish in rows of 6. How many rows of 6?
6 × □ = 18
□ is 18 *divided by* 6
□ is 18 ÷ 6

2. Arrange 16 shells in rows of 2. How many rows of 2?
2 × □ = 16
□ is 16 *divided by* 2
□ = 16 ÷ 2

3. Arrange 20 moons in rows of 5. How many rows of 5?
5 × □ = 20
□ is 20 *divided by* 5
□ = 20 ÷ 5

4. Arrange 36 sand pails in rows of 4. How many rows of 4?
4 × □ = 36
□ is 36 *divided by* 4
□ is 36 ÷ 4

5. Arrange 35 turtles in rows of 7. How many rows of 7?
7 × □ = 35
□ is 35 *divided by* 7
□ is 35 ÷ 7

6. Arrange 42 stars in rows of 6. How many rows of 6?
6 × □ = 42
□ is 42 *divided by* 6
□ is 42 ÷ 6

reference page

E-16

● Page 17 provides further experience in recognizing and using quotients. Another division sign, the fraction bar, is introduced.

As the illustration at the top of page 17 is discussed, the class should understand that the quotient is $54 \div 9$ or $\frac{54}{9}$. Each may be read, "fifty-four divided by nine." Write the following on the chalkboard and direct the pupils to read each quotient.

$$\frac{6}{3} \quad \frac{12}{4} \quad \frac{15}{5} \qquad 8 \div 2 \qquad 16 \div 8$$

The children should then rewrite these quotients using $\div$ instead of the fraction bar and vice versa.

As exercise 1 is discussed, have the children describe the array which corresponds to the equation $8 \times \square = 32$. (Thirty-two dots are arranged in rows of 8. The quotient, $\square$, is the number of rows.) Have the pupils write the quotient using both $\div$ and the fraction bar. Continue with the other exercises in a similar manner.

● Page 18 provides experience in computing quotients. Write an equation such as $8 \times \square = 16$ on the chalkboard. Ask a pupil to draw the array. The class should realize that the array can be made by drawing rows of 8 until there are 16 in the array. Point out that the number of rows is a quotient.

$\frac{16}{8}$ or $16 \div 8$

8

Then ask the class to count the rows (2). Tell the pupils that 2 is the quotient. Two is the same number as $\frac{16}{8}$ and $16 \div 8$. Tell the pupils that, if $8 \times \square = 16$, then $16 \div 8$ is $\square$. Since they already know that 8×2 is 16, $\square$ is also 2. Tell the class that, when $16 \div 8$ is computed, the result is 2.

$$8 \times \square = 16$$
$$\square = 16 \div 8$$
$$\square = 2$$

Discuss the example on the top of the page. Ask the children how they could make an array to illustrate $9 \times \square = 72$ (arrange 72 objects in an array with rows of 9). Then ask the pupils to describe the array. The quotient, $\square$, is 72 divided by 9. Ask them to compute 72 divided by 9. The result is 8 because 9×8 is 72; there are 8 rows of 9 in the array. Assign the exercises for independent work. If the pupils do not know how to compute a quotient, they can make an array and examine it. Discuss the results.

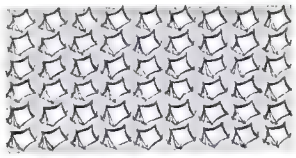
● Page 19 provides further practice in recognizing and computing quotients. Discuss the example. Assign the exercises for independent work. Discuss the results. Then write the following on the chalkboard.

$6 \times \square = 48$	$4 \times \square = 36$
$7 \times \square = 35$	$6 \times \square = 30$
$9 \times \square = 36$	$7 \times \square = 56$
$9 \times \square = 45$	$9 \times \square = 54$

Let the pupils write and compute the quotient for $\square$.

Name

Arrange 54 tents in rows of 9. How many rows of 9? *See pupil page suggestions.*



$9 \times \square = 54$

$\square = 54 \div 9$

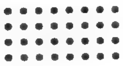
$\square = \frac{54}{9}$

The fraction bar and $\div$ are division signs.

$54 \div 9$ and $\frac{54}{9}$ are read *Fifty-four divided by nine.*

What quotient is $\square$?

1.

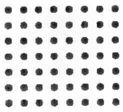


$8 \times \square = 32$

$\square = 32 \div$ 8

$\square = \frac{32}{8}$

2.

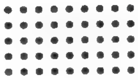


$7 \times \square = 56$

$\square =$ 56 $\div$ 7

$\square = \frac{56}{7}$

3.



$9 \times \square = 45$

$\square =$ 45 $\div$ 9

$\square = \frac{45}{9}$

4.

$4 \times \square = 24$

$\square =$ 24 $\div$ 4

$\square = \frac{24}{4}$

reference page

E-17

What quotient is $\square$? *See pupil page suggestions.*

$9 \times \square = 72$	$9 \times \square = 72$
$\square = 72 \div 9$	$\square = \frac{72}{9}$
$\square = 8$	$\square = 8$

What quotient is $\square$? Compute $\square$.

1. $8 \times \square = 56$
   $\square = 56 \div$ 8
   $\square =$ 7
2. $7 \times \square = 63$
   $\square = \frac{63}{7}$
   $\square =$ 9
3. $3 \times \square = 27$
   $\square =$ 27 $\div$ 3
   $\square =$ 9
4. $4 \times \square = 28$
   $\square = \frac{28}{4}$
   $\square =$ 7
5. $6 \times \square = 54$
   $\square =$ 54 $\div$ 6
   $\square =$ 9
6. $9 \times \square = 81$
   $\square = \frac{81}{9}$
   $\square =$ 9
7. $7 \times \square = 56$
   $\square =$ 56 $\div$ 7
   $\square =$ 8
8. $5 \times \square = 40$
   $\square = \frac{40}{5}$
   $\square =$ 8

reference page

E-18

Name

What quotient is □? Compute □. *See pupil page suggestions.*

$9 \times \square = 54$	$9 \times \square = 54$
$\square = 54 \div 9$	$\square = \frac{54}{9}$
$\square = \underline{6}$	$\square = \underline{6}$

Complete.

1. $6 \times \square = 18$	2. $9 \times \square = 63$
$\square = \underline{18} \div \underline{6}$	$\square = \frac{63}{9}$
$\square = \underline{3}$	$\square = \underline{7}$
3. $7 \times \square = 49$	4. $8 \times \square = 72$
$\square = \underline{49} \div \underline{7}$	$\square = \frac{72}{8}$
$\square = \underline{7}$	$\square = \underline{9}$
5. $6 \times \square = 42$	6. $3 \times \square = 21$
$\square = \underline{42} \div \underline{6}$	$\square = \frac{21}{3}$
$\square = \underline{7}$	$\square = \underline{7}$
7. $4 \times \square = 32$	8. $5 \times \square = 45$
$\square = \underline{32} \div \underline{4}$	$\square = \frac{45}{5}$
$\square = \underline{8}$	$\square = \underline{9}$

E-19

Developmental Experiences

► Put the following array on the chalkboard.

```
  □□□□□□□□□
3 □□□□□□□□□
  □□□□□□□□□
```

$$3 \times \underline{?} = 27$$
$$27 = 3 \times \underline{?}$$

Ask the pupils to complete the sentences. Tell them that the number of rows, 9, is a missing factor. Then draw another array beside the previous one.

```
  □□□□□□□□□     □□□□□□□□□
3 □□□□□□□□□   3 □□□□□□□□□
  □□□□□□□□□     □□□□□□□□□
      9           27 ÷ 3
```

Point out that the number of rows is the quotient $27 \div 3$. Let the pupils complete the following sentences using this quotient.

$$3 \times \underline{? \div ?} = 27$$
$$27 = 3 \times \underline{? \div ?}$$

The pupils should see that 9 and $27 \div 3$ are the same number. We can write:

$$9 = 27 \div 3$$
$$27 \div 3 = 9$$

Be sure the pupils understand the meaning of the equal sign.

Put the following two arrays side by side on the chalkboard.

```
  ✱✱✱✱✱✱     ✱✱✱✱✱✱
  ✱✱✱✱✱✱     ✱✱✱✱✱✱
5 ✱✱✱✱✱✱   5 ✱✱✱✱✱✱
  ✱✱✱✱✱✱     ✱✱✱✱✱✱
  ✱✱✱✱✱✱     ✱✱✱✱✱✱
     □          ? ÷ ?
```

Point out that the two arrays are the same. Then ask the pupils to complete the following sentences.

$5 \times \square = 30$	$5 \times \underline{? \div ?} = 30$	$\square = \underline{? \div ?}$
$30 = 5 \times \square$	$30 = 5 \times \underline{? \div ?}$	$\underline{? \div ?} = \square$

The pupils may now use arrays to illustrate and complete the following sets of equations.

(1)	$8 \times \square = 32$ $32 = 8 \times \square$	$8 \times \underline{? \div ?} = 32$ $32 = 8 \times \underline{? \div ?}$	$\square = \underline{? \div ?}$ $\underline{? \div ?} = \square$
(2)	$4 \times \square = 20$ $20 = 4 \times \square$	$4 \times \underline{? \div ?} = 20$ $20 = 4 \times \underline{? \div ?}$	$\square = \underline{? \div ?}$ $\underline{? \div ?} = \square$
(3)	$5 \times \square = 45$ $45 = 5 \times \square$	$5 \times \underline{? \div ?} = 45$ $45 = 5 \times \underline{? \div ?}$	$\square = \underline{? \div ?}$ $\underline{? \div ?} = \square$
(4)	$9 \times \square = 63$ $63 = 9 \times \square$	$9 \times \underline{? \div ?} = 63$ $63 = 9 \times \underline{? \div ?}$	$\square = \underline{? \div ?}$ $\underline{? \div ?} = \square$

Explain that each equation in the right-hand column corresponds in a special way to the equation in the left-hand column. It uses the inverse operation, division, to show how the same three numbers are related; it shows that the missing factor (□) is a quotient.

► The children are now ready to play the Inverse Operation Game. Write these equations in a column on one side of the chalkboard.

$$4 \times \square = 24$$
$$24 = 6 \times \square$$

Ask the pupils to write a corresponding equation which uses the inverse operation.

$$4 \times \square = 24 \longleftrightarrow \square = 24 \div 4 \text{ (or } 24 \div 4 = \square)$$
$$24 = 6 \times \square \longleftrightarrow 24 \div 6 = \square \text{ (or } \square = 24 \div 6)$$

Continue the game using other equations. It will also be appropriate to write division equations first and ask the pupils to supply the corresponding multiplication equations.

$$\square = 35 \div 7 \longleftrightarrow 7 \times \square = 35 \text{ (or } 35 = 7 \times \square)$$
$$35 \div \square = 5 \longleftrightarrow 35 = 5 \times \square \text{ (or } 5 \times \square = 35)$$

This will help the pupils to understand that the relationship in question works in both directions, from multiplication to division and vice versa. The teacher may draw two-headed arrows between the pairs of corresponding equations to show this.

▶ Have the class arrange 21 counters in 3 rows. Ask a pupil to give the quotient for the missing factor. He may answer either 7 or 21 ÷ 3. If he answers 21 ÷ 3, have another pupil compute the quotient. Put the following on the chalkboard.

○ ○ ○ ○ ○ ○ ○
3 ○ ○ ○ ○ ○ ○ ○
○ ○ ○ ○ ○ ○ ○
21 ÷ 3

$$3 \times \square = 21$$
$$\square \times 3 = 21$$

Let the pupils complete the sentences. They should see that $\square$ is 21 ÷ 3:

$$3 \times 21 \div 3 = 21$$
$$21 \div 3 \times 3 = 21$$

Point out that another way to write 21 ÷ 3 is $\frac{21}{3}$.

$$3 \times \frac{21}{3} = 21$$
$$\frac{21}{3} \times 3 = 21$$

Continue the activity by having the class arrange 30 counters in 6 rows. Draw the array on the chalkboard and label the parts 6 and $\frac{30}{6}$. Have the pupils complete the following sentences.

$$6 \times \frac{30}{6} = 30$$
$$\frac{30}{6} \times 6 = 30$$

The class may then use arrays to illustrate and complete these sentences.

$$8 \times \underline{? \div ?} = 32$$
$$5 \times \underline{? \div ?} = 45$$
$$9 \times \underline{\tfrac{?}{?}} = 63$$

$$\underline{? \div ?} \times 8 = 32$$
$$\underline{? \div ?} \times 5 = 45$$
$$\underline{\tfrac{?}{?}} \times 9 = 63$$

Write the following sentences on the chalkboard.

$$8 \times 64 \div 8 = \underline{?}$$
$$9 \times 54 \div 9 = \underline{?}$$
$$\frac{48}{6} \times 6 = \underline{?}$$
$$\frac{14}{3} \times 3 = \underline{?}$$
$$18 \div 3 = \underline{?}$$
$$24 \div 6 = \underline{?}$$
$$\frac{36}{4} = \underline{?}$$

When the class has completed the sentences, ask several pupils how they found the missing numbers. Let the class compare their answers.

Pages 20 through 22

● Pages 20 and 21 provide practice in recognizing and computing quotients and products. Discuss the example at the top of page 20. Ask the pupils how they can tell that 24 ÷ 6 is 4 and that 6 × 24 ÷ 6 is 24. Work the first exercise on page 20 with the class. Assign the other exercises for independent work. Discuss the results. Pages 21 and 22 provide more exercises of the same type.

24 ÷ 6  6

$24 \div 6 = 4$	$\frac{24}{6} = 4$
$6 \times (24 \div 6) = 24$	$6 \times \frac{24}{6} = 24$
$6 \times 4 = \underline{24}$	$6 \times 4 = \underline{24}$

Compute.

1. $42 \div 7 = \underline{6}$	2. $\frac{21}{7} = \underline{3}$	3. $48 \div 6 = \underline{8}$
$7 \times (42 \div 7) = \underline{42}$	$7 \times \frac{21}{7} = \underline{21}$	$6 \times (48 \div 6) = \underline{48}$
$7 \times 6 = \underline{42}$	$7 \times 3 = \underline{21}$	$6 \times 8 = \underline{48}$
4. $\frac{27}{9} = \underline{3}$	5. $24 \div 4 = \underline{6}$	6. $\frac{18}{2} = \underline{9}$
$9 \times \frac{27}{9} = \underline{27}$	$4 \times (24 \div 4) = \underline{24}$	$2 \times \frac{18}{2} = \underline{18}$
$9 \times 3 = \underline{27}$	$4 \times 6 = \underline{24}$	$2 \times 9 = \underline{18}$
7. $24 \div 3 = \underline{8}$	8. $\frac{36}{6} = \underline{6}$	9. $\frac{30}{5} = \underline{6}$
$3 \times (24 \div 3) = \underline{24}$	$6 \times \frac{36}{6} = \underline{36}$	$5 \times \frac{30}{5} = \underline{30}$
$3 \times 8 = \underline{24}$	$6 \times 6 = \underline{36}$	$5 \times 6 = \underline{30}$

reference page

E-20

Name

Compute.

1. $36 \div 4 = 9$
   $4 \times (36 \div 4) = 36$
   $4 \times 9 = 36$

2. $\frac{28}{7} = 4$
   $7 \times \frac{28}{7} = 28$
   $7 \times 4 = 28$

3. $35 \div 5 = 7$
   $5 \times (35 \div 5) = 35$
   $5 \times 7 = 35$

4. $\frac{40}{8} = 5$
   $8 \times \frac{40}{8} = 40$
   $8 \times 5 = 40$

5. $81 \div 9 = 9$
   $9 \times (81 \div 9) = 81$
   $9 \times 9 = 81$

6. $\frac{12}{4} = 3$
   $4 \times \frac{12}{4} = 12$
   $4 \times 3 = 12$

7. $72 \div 9 = 8$
   $(9 \times 72) \div 9 = 72$
   $9 \times 8 = 72$

8. $\frac{64}{8} = 8$
   $8 \times \frac{64}{8} = 64$
   $8 \times 8 = 64$

9. $8 \div 8 = 1$
   $8 \times 8 \div 8 = 8$
   $8 \times 1 = 8$

10. $\frac{56}{8} = 7$
    $8 \times \frac{56}{8} = 56$
    $8 \times 7 = 56$

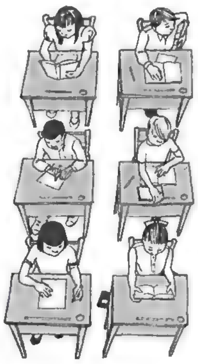
11. $48 \div 8 = 6$
    $8 \times 48 \div 8 = 48$
    $8 \times 6 = 48$

12. $\frac{63}{9} = 7$
    $9 \times \frac{63}{9} = 63$
    $9 \times 7 = 63$

E-21

$16 \div 4 = 4$
$4 \times (16 \div 4) = 16$
$(16 \div 4) \times 4 = 16$

$\frac{10}{2} = 5$
$2 \times \frac{10}{2} = 10$
$\frac{10}{2} \times 2 = 10$



Compute.

1. $6 \div 3 = 2$
   $3 \times (6 \div 3) = 6$
   $(6 \div 3) \times 3 = 6$

2. $\frac{42}{7} = 6$
   $7 \times \frac{42}{7} = 42$
   $\frac{42}{7} \times 7 = 42$

3. $81 \div 9 = 9$
   $9 \times (81 \div 9) = 81$
   $(81 \div 9) \times 9 = 81$

4. $\frac{48}{8} = 6$
   $8 \times \frac{48}{8} = 48$
   $\frac{48}{8} \times 8 = 48$

5. $27 \div 3 = 9$
   $(3 \times 27) \div 3 = 27$
   $(27 \div 3) \times 3 = 27$

6. $\frac{72}{8} = 9$
   $8 \times \frac{72}{8} = 72$
   $\frac{72}{8} \times 8 = 72$

7. $49 \div 7 = 7$
   $7 \times (49 \div 7) = 49$
   $(49 \div 7) \times 7 = 49$

8. $\frac{28}{7} = 4$
   $7 \times \frac{28}{7} = 28$
   $\frac{28}{7} \times 7 = 28$

9. $\frac{63}{7} = 9$
   $7 \times \frac{63}{7} = 63$
   $\frac{63}{7} \times 7 = 63$

10. $54 \div 6 = 9$
    $(6 \times 54) \div 6 = 54$
    $(54 \div 6) \times 6 = 54$

E-22

Supplemental Experience

■ Play a factor game to review basic multiplication facts. Write the following on the chalkboard.

Factor Game
$24 \longleftrightarrow 6 \times 4$
$24 \longleftrightarrow 3 \times 8$
$15 \longleftrightarrow$
60

Encourage the pupils to discover the rules of the game by observing the examples. Then ask them to follow these rules with the number 15. Suggestions such as 3×5, 5×3, 15×1, and 1×15 are all correct. Do not insist that all possibilities be given. Write another product such as 54. Some pupils may suggest that a factored form is 2×27. This is correct, but it is more important that the pupils recognize 54 as 6×9 or 9×6.

Have the children supply products for the first column as the game continues.

UNIT 3
ARRAYS AND PROBLEM SOLVING
Pages 23 Through 38

OBJECTIVE

To explore the relationships among the numbers in story problems.

The pupil observes an array and learns that a model of the array is any of eight equivalent equations, four multiplication and four division. He learns to express the relationship among the numbers in a story problem with an array and subsequently with an appropriate equation.

See Key Topics in Mathematics for the Intermediate Teacher: Sets and Numbers; Multiplication and Division of Whole Numbers.

KEY IDEAS

Division is a way of looking at multiplication.
$6a = 6b$ says that $6b = 6a$.
$ab = ba$.

KEY IDEA

Division is a way of looking at multiplication.

Scope

To use both multiplication and division equations to describe the relationship among numbers of an array.
To review basic multiplication facts.

Fundamentals

Products may be shown by arrays. The product, factor times factor, is the number of elements in the array. In this array, the factors are 6 and 3 and the product is 6×3 or 3×6:

	6
	✱ ✱ ✱ ✱ ✱ ✱
3	✱ ✱ ✱ ✱ ✱ ✱
	✱ ✱ ✱ ✱ ✱ ✱

Since the number of elements in the array is 18, as well as 6×3 and 3×6, we write these equations:

$$18 = 6 \times 3 \qquad 18 = 3 \times 6$$
$$6 \times 3 = 18 \qquad 3 \times 6 = 18$$

These equations state that the number 6×3 is the number 18; the number 18 is the number 6×3, and so forth.

The same array also shows related quotients. Since division undoes multiplication ($6 \times 3 \div 3 = 6$), an equation in terms of multiplication may be expressed in terms of division.

$$6 \times 3 = 18$$

Factor $\times$ Factor = Product

$$6 = 18 \div 3$$

Factor = Product $\div$ Factor

The quotients $18 \div 3$ and $18 \div 6$ are the numbers of rows in the array.

	6 18 ÷ 3
	✱ ✱ ✱ ✱ ✱ ✱
3 18 ÷ 6	✱ ✱ ✱ ✱ ✱ ✱
	✱ ✱ ✱ ✱ ✱ ✱

The quotient $18 \div 3$, or 6, is a number of rows since there are 3 objects in each vertical row. We may write these equations:

$$6 = 18 \div 3 \text{ and } 18 \div 3 = 6$$

And $18 \div 6$, or 3, is also a number of rows since there are 6 in each horizontal row. We may write these equations:

$$3 = 18 \div 6 \text{ and } 18 \div 6 = 3$$

Each of the following 8 equations expresses the relationship among the numbers 6, 3, and 18 illustrated by the array.

$$\begin{array}{ll} 3 \times 6 = 18 & 6 = 18 \div 3 \\ 6 \times 3 = 18 & 3 = 18 \div 6 \\ 18 = 3 \times 6 & 18 \div 3 = 6 \\ 18 = 6 \times 3 & 18 \div 6 = 3 \end{array}$$

To help the children understand the relationship among the numbers in a story, the story situation is first illustrated by an array. For example, a 3 by 6 array illustrates the number structure of stories such as these.

Three boys each ate 6 apples. Together they ate 18 apples.

Mr. Smith gave 18 ribbons to his 3 daughters. Each girl received 6 ribbons.

Any one of the eight equations associated with the array will express the relationship among the numbers in the stories.

Story problems are introduced after pupils are able to write equations that express the number structure of an array. The pupil names the missing number with

a placeholder and writes equations to express the relationship among the numbers in the story. Consider this story.

Jim had 15 baseball cards in 3 packs. How many cards were in each pack?

The pupil may express the relationship among the numbers in this story by writing any one of these equations.

$3 \times \square = 15$	$15 = 3 \times \square$
$\square \times 3 = 15$	$15 = \square \times 3$
$15 \div 3 = \square$	$\square = 15 \div 3$
$15 \div \square = 3$	$3 = 15 \div \square$

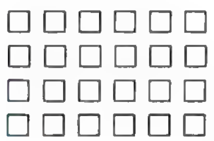
Readiness for Understanding

Understanding of product and array.

Developmental Experiences

tagboard cards (3″ × 9″)	*for each child*
felt-tip pen	40 counters
pocket chart	

▶ Draw a 4 by 6 array on the chalkboard.



Have the class observe that in a 4 by 6 array there are 4 rows viewed one way and 6 rows viewed the other way. Tell the class that an array is a model for multiplication. Write the product 4×6 under the array. Have the class note that in the product 4×6 there are 2 factors. The factor 4 is the number of rows in the array viewed one way; the factor 6 is the number of rows in the array viewed the other way. The product 4×6 is the number of members belonging to the whole array.

Ask a child to count the objects in this array (24). Write this equation on the chalkboard:

$$4 \times 6 = 24$$

Review what the equal sign means. In the equation $4 \times 6 = 24$, the equal sign means that the number 4×6 is the same number as 24; each is the number of objects in the array.

Ask another child to write an equation that interchanges 4×6 and 24.

$$24 = 4 \times 6$$

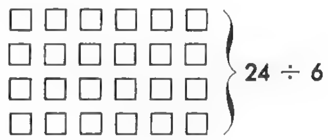
The class should observe that, if $4 \times 6 = 24$, then it must be true that $24 = 4 \times 6$.

Call on a pupil to write an equation in which the factors in $4 \times 6 = 24$ are commuted. Ask another pupil to examine the equation $24 = 4 \times 6$, and then write it with its factors in commuted form.

$4 \times 6 = 24$	$24 = 4 \times 6$
$6 \times 4 = 24$	$24 = 6 \times 4$

Ask the children to point to the parts of each equation that show the number of members in the array, the number of rows viewed one way, and the number of rows viewed the other way.

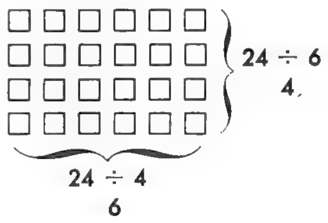
Direct the children's attention to the 4 rows in the 4 by 6 array. This number of rows is also the quotient $24 \div 6$. Write this quotient on the chalkboard so that the class can observe the relationship of this number to the 4 rows in the array.



Explain to the children that an array that has 24 objects arranged in 6 rows one way, has $24 \div 6$ rows the other way.

Now point to the 6 rows in the 4 by 6 array. This number of rows is $24 \div 4$. If an array of 24 objects has 4 rows one way, then there are $24 \div 4$ rows the other way. Write $24 \div 4$ on the chalkboard so that the class can note the relationship of the quotient to the 6 rows in the array.

Ask some child to count the number of rows viewed each way (4 and 6). Write these numbers on the chalkboard next to the array.



Now write the equation $6 = 24 \div 4$ beside the multiplication equations for the array:

$6 = 24 \div 4$	$4 \times 6 = 24$	$24 = 4 \times 6$
	$6 \times 4 = 24$	$24 = 6 \times 4$

The children should note that the number $24 \div 4$ is the number 6. Ask another child to write an equation in which the position of the quotient $24 \div 4$ and the 6 are reversed.

$24 \div 4 = 6$	$6 = 24 \div 4$	$4 \times 6 = 24$	$24 = 4 \times 6$
		$6 \times 4 = 24$	$24 = 6 \times 4$

Point out to the class that $24 \div 4$ and 6 are the same number.

Ask a child to point out, in each of the two division equations, the number of members in the array (24). Let some other child point out the number of rows viewed one way (4). Then ask someone to give the number of rows viewed the other way ($24 \div 4$, which is the same number as 6).

Also, do this with $4 = 24 \div 6$ and $24 \div 6 = 4$.

$24 \div 4 = 6$	$6 = 24 \div 4$	$4 \times 6 = 24$	$24 = 4 \times 6$
$24 \div 6 = 4$	$4 = 24 \div 6$	$6 \times 4 = 24$	$24 = 6 \times 4$

Be sure the class realizes that both equations state that $24 \div 6$ is the same number as 4; and that if $4 = 24 \div 6$, then $24 \div 6 = 4$.

Have someone identify, in each of the two new equations, the number of members in the array (24). Ask someone to point out the number of rows viewed one way (6) and the number of rows viewed the other way ($24 \div 6$, which is the same number as 4).

Remind the children that these numbers (the product, 24 and two factors, 4 and 6) have been given in each of the eight equations on the chalkboard.

Continue in this same way to let the children write and discuss eight equations that express the relationship of the numbers in a 3 by 5 array, a 2 by 8 array, a 7 by 3 array, and a 9 by 4 array.

► Give each child a set of 40 counters (washers, bottle caps, cardboard disks, or something similar). Write on the chalkboard the equation $15 \div 3 = 5$. Ask the children to use their counters at their desks to show an array for this equation.

After each child has completed his array, ask the class to give the number of objects in the array (15). Then have a child tell the number of rows viewed one way ($15 \div 3$, which is the same number as 5) and the number of rows viewed the other way (3). They should point out these numbers in the equations.

Select someone to write beside the division equation a multiplication equation that shows the relationship among the three numbers. For example, a child might write $3 \times 5 = 15$. Continue in this manner until the eight equivalent equations that show the relationship among 15, 3, and 5 have been discussed.

$15 \div 3 = 5$	$3 \times 5 = 15$
$15 \div 5 = 3$	$5 \times 3 = 15$
$5 = 15 \div 3$	$15 = 3 \times 5$
$3 = 15 \div 5$	$15 = 5 \times 3$

(3, 5, $15 \div 5$, and $15 \div 3$ are the numbers of rows and 3×5 and 15 are the number of members in the array.)

Adapt this procedure to other multiplication and division equations for any basic multiplication facts through 40. You may want to begin with the equations $18 = 3 \times 6$, $7 = 28 \div 4$, and $4 \times 8 = 32$. In each instance, have the other seven related equations written on the chalkboard.

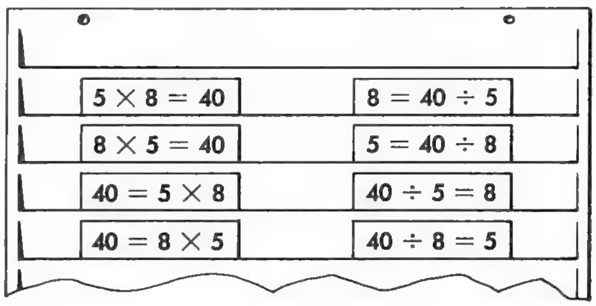
► Prepare several sets of 3 by 9 inch tagboard cards such as these.

$40 = 8 \times 5$	$8 \times 5 = 40$	$40 \div 5 = 8$
$8 = 40 \div 5$	$40 = 5 \times 8$	$5 = 40 \div 8$
$40 \div 8 = 5$		$5 \times 8 = 40$

Each set of cards should contain eight equations which express the product-factor relationship among three numbers. Four cards show multiplication equations and four show division equations. Shuffle several sets together to make a stack.

Form two teams. Ask a member of Team A to select any card from the shuffled stack and place it in the pocket chart. Then ask this player to draw an array on the chalkboard for the product-factor relationship expressed on the card.

Next, ask a member of Team B to select a card from the shuffled stack which shows the same product-factor relationship. Continue, alternating play, until the eight equations for this relationship are in the pocket chart.



A team earns 1 point for each card it places in the chart. If a player does not place an appropriate card in the chart, the play goes to the other team. The class should decide if a card is appropriate.

► You can use the same pocket chart and sets of eight cards to play a Commutativity Game. Place the following cards on the left side of the chart.

$5 \times 8 = 40$	
$40 = 5 \times 8$	
$8 = 40 \div 5$	
$40 \div 5 = 8$	

Ask the pupils to identify the factors and product in each of these equations. Then have them arrange the remaining cards to form pairs of equations in which the factors have been interchanged.

$5 \times 8 = 40$	$8 \times 5 = 40$
$40 = 5 \times 8$	$40 = 8 \times 5$
$8 = 40 \div 5$	$5 = 40 \div 8$
$40 \div 5 = 8$	$40 \div 8 = 5$

Continue the game using other sets of eight equations.

▶ Use the same materials to play a Symmetry Game. In this game, the sets of eight equations are to be arranged in symmetrical pairs. (Two equations are symmetrical if their right-hand and left-hand members are interchanged.) A completed Symmetry Game is shown here.

$5 \times 8 = 40$	$40 = 5 \times 8$
$8 \times 5 = 40$	$40 = 8 \times 5$
$40 \div 5 = 8$	$8 = 40 \div 5$
$40 \div 8 = 5$	$5 = 40 \div 8$

Play this game using other sets of eight equations.

Pages 23 through 28

● Pages 23 through 28 provide opportunity to consider division as the inverse of multiplication. These pages provide practice in writing and identifying equations that describe arrays.

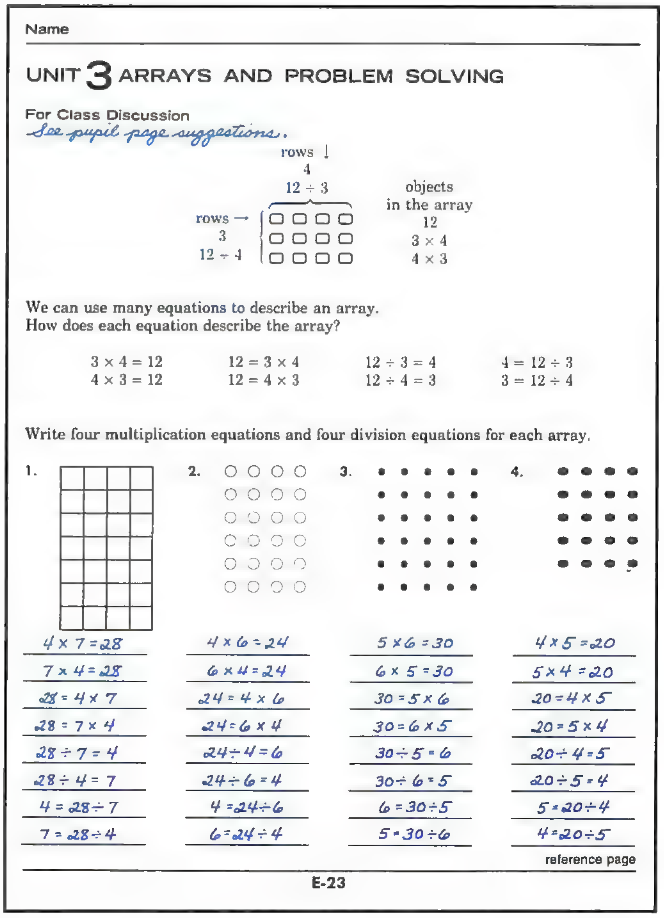
Use page 23 as a discussion page. After the pupils have had an opportunity to study the illustration at the top of the page, discuss the numbers that describe the array. The number of objects in the array is 12, 3×4, or 4×3. The number of rows one way is 3 or $12 \div 4$. The number of rows the other way is 4 or $12 \div 3$. Again, eight equivalent equations can be written to describe the array. Have the pupils discuss the numbers in each equation. $12 \div 4$ and 3 are the number of rows viewed one way. $12 \div 3$ and 4 are the number of rows viewed the other way. 4×3, 3×4, and 12 are the number of members in the array. Help the pupils to see that, in this way, each equation expresses the multiplicative relationship of the numbers 3, 4, and 12.

Now have the children complete the exercises on the page. They should understand that each of the 8 equations shows the product-factor relationship of the numbers illustrated by each array. Ask several children to read to the class the equations they have written that describe specific arrays; they may need practice in reading division equations.

● Use pages 24 through 26 to give the children a chance to develop their ability to write multiplication and division equations for given arrays. Assign the exercises for independent work.

Let the children write on the chalkboard the eight equations for each exercise. As you discuss the equations, use the terms *factor* and *product* to help the children learn to use these names.

● Page 27 provides practice in drawing arrays that illustrate given equations. Work at least one exercise with the children, and have them complete the other exercises. Ask them not to be concerned with exact reproductions of the objects of the array. If the children need further practice, have them write the other equations that describe the arrays they have drawn.


Name

UNIT 3 ARRAYS AND PROBLEM SOLVING

For Class Discussion

See pupil page suggestions.

We can use many equations to describe an array.
How does each equation describe the array?

$3 \times 4 = 12$	$12 = 3 \times 4$	$12 \div 3 = 4$	$4 = 12 \div 3$
$4 \times 3 = 12$	$12 = 4 \times 3$	$12 \div 4 = 3$	$3 = 12 \div 4$

Write four multiplication equations and four division equations for each array.

1.	2.	3.	4.
$4 \times 7 = 28$	$4 \times 6 = 24$	$5 \times 6 = 30$	$4 \times 5 = 20$
$7 \times 4 = 28$	$6 \times 4 = 24$	$6 \times 5 = 30$	$5 \times 4 = 20$
$28 = 4 \times 7$	$24 = 4 \times 6$	$30 = 5 \times 6$	$20 = 4 \times 5$
$28 = 7 \times 4$	$24 = 6 \times 4$	$30 = 6 \times 5$	$20 = 5 \times 4$
$28 \div 7 = 4$	$24 \div 4 = 6$	$30 \div 5 = 6$	$20 \div 4 = 5$
$28 \div 4 = 7$	$24 \div 6 = 4$	$30 \div 6 = 5$	$20 \div 5 = 4$
$4 = 28 \div 7$	$4 = 24 \div 6$	$6 = 30 \div 5$	$5 = 20 \div 4$
$7 = 28 \div 4$	$6 = 24 \div 4$	$5 = 30 \div 6$	$4 = 20 \div 5$

reference page

E-23

Write four multiplication equations and four division equations to describe each array.

1.

● ● ● ● ● ●
● ● ● ● ● ●
● ● ● ● ● ●

$3 \times 6 = 18$	$18 \div 6 = 3$
$6 \times 3 = 18$	$18 \div 3 = 6$
$18 = 6 \times 3$	$3 = 18 \div 6$
$18 = 3 \times 6$	$6 = 18 \div 3$

2.

● ● ● ● ●
● ● ● ● ●
● ● ● ● ●
● ● ● ● ●

$5 \times 4 = 20$	$20 \div 5 = 4$
$4 \times 5 = 20$	$20 \div 4 = 5$
$20 = 5 \times 4$	$4 = 20 \div 5$
$20 = 4 \times 5$	$5 = 20 \div 4$

3.

● ● ● ● ● ● ●
● ● ● ● ● ● ●

$2 \times 7 = 14$	$14 \div 2 = 7$
$7 \times 2 = 14$	$14 \div 7 = 2$
$14 = 2 \times 7$	$7 = 14 \div 2$
$14 = 7 \times 2$	$2 = 14 \div 7$

4.

● ● ● ● ● ● ●
● ● ● ● ● ● ●
● ● ● ● ● ● ●

$3 \times 7 = 21$	$21 \div 3 = 7$
$7 \times 3 = 21$	$21 \div 7 = 3$
$21 = 3 \times 7$	$7 = 21 \div 3$
$21 = 7 \times 3$	$3 = 21 \div 7$

5.

● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ●

$3 \times 8 = 24$	$24 \div 3 = 8$
$8 \times 3 = 24$	$24 \div 8 = 3$
$24 = 8 \times 3$	$8 = 24 \div 3$
$24 = 3 \times 8$	$3 = 24 \div 8$

6.

● ● ● ● ●
● ● ● ● ●
● ● ● ● ●

$5 \times 3 = 15$	$15 \div 5 = 3$
$3 \times 5 = 15$	$15 \div 3 = 5$
$15 = 5 \times 3$	$3 = 15 \div 5$
$15 = 3 \times 5$	$5 = 15 \div 3$

E-24

Write four multiplication equations and four division equations to describe each array.

1.

● ● ● ●
● ● ● ●
● ● ● ●

2.

● ● ● ● ●
● ● ● ● ●

3.

● ● ● ● ● ● ●
● ● ● ● ● ● ●
● ● ● ● ● ● ●
● ● ● ● ● ● ●
● ● ● ● ● ● ●

1.	2.	3.
$4 \times 3 = 12$	$5 \times 2 = 10$	$7 \times 5 = 35$
$3 \times 4 = 12$	$2 \times 5 = 10$	$5 \times 7 = 35$
$12 = 4 \times 3$	$10 = 5 \times 2$	$35 = 5 \times 7$
$12 = 3 \times 4$	$10 = 2 \times 5$	$35 = 7 \times 5$
$12 \div 4 = 3$	$10 \div 2 = 5$	$35 \div 5 = 7$
$12 \div 3 = 4$	$10 \div 5 = 2$	$35 \div 7 = 5$
$3 = 12 \div 4$	$5 = 10 \div 2$	$5 = 35 \div 7$
$4 = 12 \div 3$	$2 = 10 \div 5$	$7 = 35 \div 5$

4.

● ● ● ● ● ●
● ● ● ● ● ●
● ● ● ● ● ●
● ● ● ● ● ●
● ● ● ● ● ●

5.

● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●

6.

● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ● ●

4.	5.	6.
$6 \times 5 = 30$	$6 \times 9 = 54$	$7 \times 9 = 63$
$5 \times 6 = 30$	$9 \times 6 = 54$	$9 \times 7 = 63$
$30 = 6 \times 5$	$54 = 6 \times 9$	$63 = 7 \times 9$
$30 = 5 \times 6$	$54 = 9 \times 6$	$63 = 9 \times 7$
$30 \div 5 = 6$	$54 \div 6 = 9$	$63 \div 7 = 9$
$30 \div 6 = 5$	$54 \div 9 = 6$	$63 \div 9 = 7$
$6 = 30 \div 5$	$9 = 54 \div 6$	$9 = 63 \div 7$
$5 = 30 \div 6$	$6 = 54 \div 9$	$7 = 63 \div 9$

E-26

Name

Write four multiplication equations and four division equations to describe each array.

1.

● ● ● ● ● ● ●
● ● ● ● ● ● ●

2.

● ● ● ● ● ● ●
● ● ● ● ● ● ●
● ● ● ● ● ● ●

3.

● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ●
● ● ● ● ● ● ● ●

1.	2.	3.
$7 \times 2 = 14$	$7 \times 3 = 21$	$8 \times 3 = 24$
$2 \times 7 = 14$	$3 \times 7 = 21$	$3 \times 8 = 24$
$14 = 7 \times 2$	$21 = 7 \times 3$	$24 = 8 \times 3$
$14 = 2 \times 7$	$21 = 3 \times 7$	$24 = 3 \times 8$
$14 \div 7 = 2$	$21 \div 7 = 3$	$24 \div 3 = 8$
$14 \div 2 = 7$	$21 \div 3 = 7$	$24 \div 8 = 3$
$2 = 14 \div 7$	$7 = 21 \div 3$	$8 = 24 = 3$
$7 = 14 \div 2$	$3 = 21 \div 7$	$3 = 24 \div 8$

4.

● ● ● ● ●
● ● ● ● ●
● ● ● ● ●

5.

● ● ● ● ●
● ● ● ● ●
● ● ● ● ●
● ● ● ● ●

6.

● ● ● ●
● ● ● ●
● ● ● ●
● ● ● ●
● ● ● ●
● ● ● ●

4.	5.	6.
$5 \times 3 = 15$	$4 \times 5 = 20$	$6 \times 4 = 24$
$3 \times 5 = 15$	$5 \times 4 = 20$	$4 \times 6 = 24$
$15 = 5 \times 3$	$20 = 4 \times 5$	$24 = 6 \times 4$
$15 = 3 \times 5$	$20 = 5 \times 4$	$24 = 4 \times 6$
$15 \div 5 = 3$	$20 \div 4 = 5$	$24 \div 4 = 6$
$15 \div 3 = 5$	$20 \div 5 = 4$	$24 \div 6 = 4$
$3 = 15 \div 5$	$5 = 20 \div 4$	$6 = 24 \div 4$
$5 = 15 \div 3$	$4 = 20 \div 5$	$4 = 24 \div 6$

E-25

Name

Draw an array for each equation. *Position of arrays may vary.*

1. $2 \times 3 = 6$

• • •
• • •

2. $1 = 6 \div 6$

• • • • • •

3. $4 \div 2 = 2$

• •
• •

4. $12 \div 6 = 2$

• • • • • •
• • • • • •

5. $9 = 3 \times 3$

• • •
• • •
• • •

6. $9 \times 1 = 9$

• • • • • • • • •

7. $8 = 2 \times 4$

• • • •
• • • •

8. $8 = 16 \div 2$

• • • • • • • •
• • • • • • • •

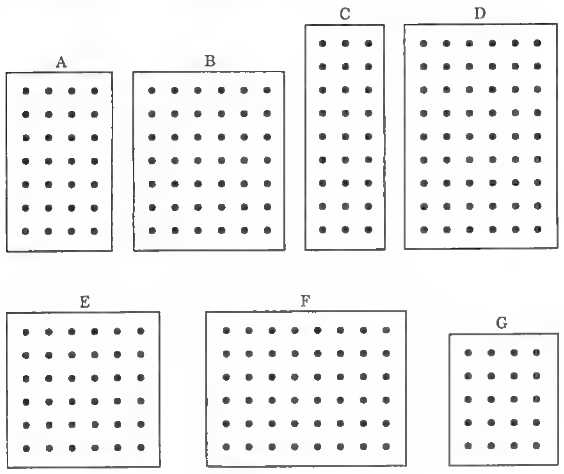
9. $10 \div 2 = 5$

• • • • •
• • • • •

E-27

● Page 28 provides practice in matching equations with given arrays. After the procedure for work has been established, assign the exercises for independent work. Examine the equations that were selected for each array and ask the children to give other equations that describe each array.

Match each equation with the array it describes.



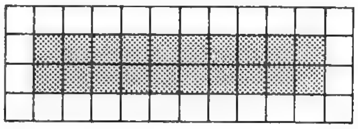
1. $3 \times 9 = 27$ C	7. $4 = 28 \div 7$ A	13. $20 \div 4 = 5$ G
2. $48 \div 8 = 6$ F	8. $42 = 7 \times 6$ B	14. $3 = 27 \div 9$ C
3. $20 = 4 \times 5$ G	9. $9 \times 6 = 54$ D	15. $21 = 3 \times 7$ None
4. $7 = 28 \div 4$ A	10. $36 \div 6 = 6$ E	16. $54 \div 6 = 9$ D
5. $42 \div 6 = 7$ B	11. $8 = 48 \div 6$ F	17. $6 = 42 \div 7$ B
6. $6 \times 6 = 36$ E	12. $28 = 7 \times 4$ A	18. $8 \times 6 = 48$ F

E-28

Supplemental Experiences

■ For additional practice, give the children sheets of graph paper and have them draw arrays for equations such as $20 \div 2 = 10$, $2 \times 9 = 18$, and $4 \div 2 = 2$. Then have them write other equations for the arrays they have drawn.

$20 \div 2$ (or 2×10) array



■ Duplicate a list of equations.

$4 \times (8 \div \_\_) = 8$
$(40 \div 5) \times \_\_ = 40$
$56 = \_\_ \times (56 \div 7)$
$9 \times (45 \div 9) = \_\_$
$4 \times 8 = (\_\_ \div 8) \times (32 \div \_\_)$
$9 \times (\_\_ \div 9) = 72$
$63 = 7 \times (63 \div \_\_)$
$7 \times 3 = (21 \div \_\_) \times (21 \div \_\_)$
$(42 \div \_\_) \times 6 = 42$
$64 = (64 \div \_\_) \times 8$

Give each child a copy and have him fill in the blanks with the correct numbers. When the children understand the inverse relationship between multiplication and division, they will be able to complete the equations without computing. Permit the children to compute if they wish.

After the exercises have been completed, encourage the children to explain in their own words how they completed specific equations. The children will help each other understand the meaning of quotient and the concept of multiplication and division as inverse operations. An activity such as this also helps the children understand the advantage of looking at the whole equation before analyzing the parts.

Some pupils may enjoy making up similar equations for their classmates to complete.

KEY IDEA

$6a = 6b$ says that $6b = 6a$.

Scope

To develop the relationship among numbers in a story problem.

To review basic multiplication facts.

Fundamentals

The ability to perceive the relationship among numbers in a story problem depends on an understanding of the story situation. Each story describes specific number relationships which may be expressed as equations. Since multiplication is the inverse of division and division is the inverse of multiplication (that is, one undoes the other), equations may be developed from either point of view.

Consider the following stories and note that both the numbers and the relationships among the numbers are the same.

> Four girls each sold 7 boxes of Girl Scout cookies. Together they sold 28 boxes.
>
> Four girls sold 28 boxes of Girl Scout cookies. They each sold 7 boxes.

Both stories describe the same situation. The relationship among the number of boxes, the number of girls, and the number of boxes each girl sold is expressed by each of the following eight equations.

$4 \times 7 = 28$	$28 \div 4 = 7$
$7 \times 4 = 28$	$28 \div 7 = 4$
$28 = 4 \times 7$	$7 = 28 \div 4$
$28 = 7 \times 4$	$4 = 28 \div 7$

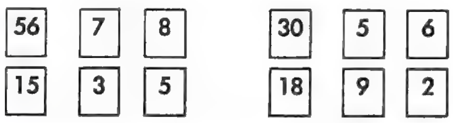
Readiness for Understanding

Knowledge of sum, difference, product, and quotient.

Developmental Experiences

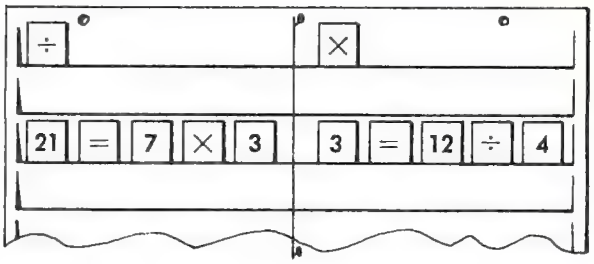
tagboard cards (8″ × 10″)	*for each child*
felt-tip pen	3 tagboard cards (2″ × 3″)
box	
pins	
paper strip	
pocket chart	
gummed circle stickers	

► Cut out three tagboard cards for use at the pocket chart for each child in the class. On one card in each set of three, write the standard numeral for a product of two whole number factors; on each of the other two cards, write a different factor of the product. Use a different basic multiplication fact for each set of three cards.



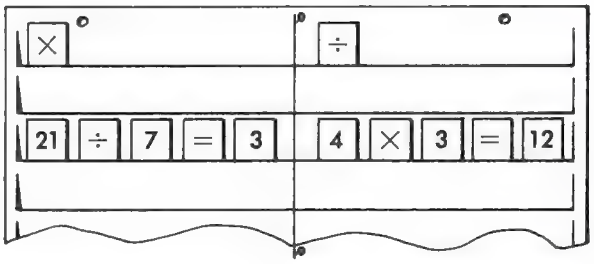
Fasten the three cards together with a paper clip and place them in a box. Finally, cut out six more cards. Write a times sign (×) on two of the cards, a division sign (÷) on two of the cards, and an equal sign (=) on the remaining two cards. Then pin a narrow strip of paper down the center of a pocket chart and place one of each type of sign in each section of the chart.

Ask the children to form two teams—one for each half of the pocket chart. A member from each team should choose a set of cards from the box. Each player will use his cards and two of the ×, ÷, or = cards to form an equation.



Score one point if the equation is correct and one more point for the team that correctly finished first.

Before a second member from each team comes to the pocket chart, explain to the children that they are going to use the numbers already in the chart. If the equation on their team's side of the chart is a multiplication equation, they should create a division equation with the given numbers. If a division equation now appears, the children should make a multiplication equation.



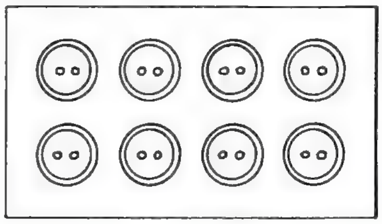
Score the points in the same way as in the first round. Then remove the number cards from the board and begin a new round.

Continue in this way to have some team members create equations and other team members create equivalent equations. After all of the children have had a chance to participate, total each team's points to find the winning team.

► Write this story on the chalkboard:

> Sally bought a card with 8 buttons on it. The buttons were stitched on the card in rows that had the same number of buttons in each. There were 2 rows if Sally looked at the buttons one way, and 4 rows if she looked at them the other way.

Ask a child to read the story to the class and to draw an array to illustrate the story.



Have the other children give eight equations that describe the relationship of the numbers in the array. Write each equation on the chalkboard as it is given.

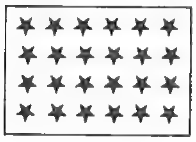
$2 \times 4 = 8$	$8 = 2 \times 4$
$8 \div 4 = 2$	$2 = 8 \div 4$
$4 \times 2 = 8$	$8 = 4 \times 2$
$8 \div 2 = 4$	$4 = 8 \div 2$

Have 8 pupils each select a different equation and tell how the numbers given in their particular equation are illustrated in the array. Repeat the activity, using other stories.

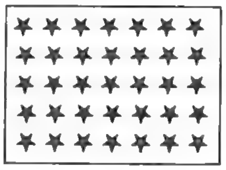
► Arrange gummed circle stickers or draw dots on tagboard cards to show arrays that represent the basic multiplication facts. Make one array card for each child.

Separate the class into two teams. Instruct a member of each team to choose an array card and place it on the chalktray. Direct each child to write on the chalkboard above his card one multiplication equation and one division equation for his array.

$24 = 6 \times 4$
$24 \div 6 = 4$



$7 \times 5 = 35$
$7 = 35 \div 5$



Have the class decide which pupil was first to finish the assignment correctly. Give one point to each child who did his work correctly; give an extra point to the child who was first to finish correctly. Try to let all the children participate.

Pages 29 through 32

● Pages 29 and 30 contain stories illustrated by arrays. Each story specifies all the numbers the children need to write four multiplication and four division equations. The children are not required to answer questions at this point in the unit. Most of the children should be able to read the stories, but if some children have reading problems, read the stories with them.

Work one of the exercises on each page with the class. Remind the children that the arrays are illustrations for the stories. Assign the remaining exercises and, when they are finished, discuss the equations the pupils have chosen for each story and array.

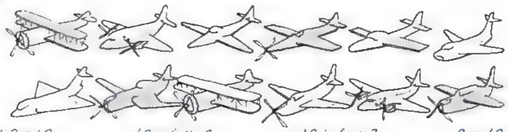
● On pages 31 and 32, the child is asked to draw an array and to write one multiplication and one division equation for each story. Discuss and complete exercise 1 on page 31 with the class; then assign the rest of the exercises on page 31 and exercises 1 through 3 on page 32 for independent work. Remind the children that the arrays they draw should only suggest the objects; they need not be realistic.

You can assign exercises 4 and 5 at the bottom of page 32 or use them as a class activity. The emphasis should be on mathematical accuracy and creativity.

Name

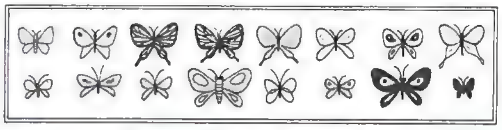
Write four multiplication equations and four division equations for each story.

1. Each of 6 boys had 2 model airplanes. They had 12 models.



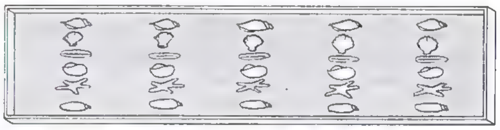
6 x 2 = 12	12 = 6 x 2	12 ÷ 6 = 2	2 = 12 ÷ 6
2 x 6 = 12	12 = 2 x 6	12 ÷ 2 = 6	6 = 12 ÷ 2

2. Rachel put 16 butterflies in a display case. She mounted 8 butterflies in each of 2 rows.



8 x 2 = 16	16 = 8 x 2	16 ÷ 2 = 8	8 = 16 ÷ 2
2 x 8 = 16	16 = 2 x 8	16 ÷ 8 = 2	2 = 16 ÷ 8

3. Herb put 30 shells in a box. He put 5 shells in each of 6 rows.



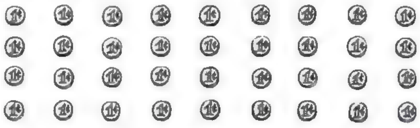
5 x 6 = 30	30 = 5 x 6	30 ÷ 6 = 5	5 = 30 ÷ 6
6 x 5 = 30	30 = 6 x 5	30 ÷ 5 = 6	6 = 30 ÷ 5

reference page

E-29

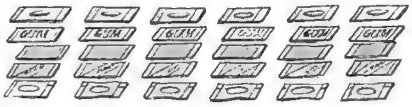
Write four multiplication equations and four division equations for each story.

1. Tom bought 4 notebooks. Each cost 9¢. Tom paid 36¢ for the notebooks.



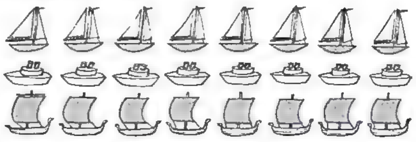
4 x 9 = 36	36 = 4 x 9	36 ÷ 9 = 4	9 = 36 ÷ 4
9 x 4 = 36	36 = 9 x 4	36 ÷ 4 = 9	4 = 36 ÷ 9

2. Sue had 6 packs of gum. There were 5 sticks of gum in each pack. Sue had 30 sticks of gum.



6 x 5 = 30	30 = 6 x 5	30 ÷ 6 = 5	5 = 30 ÷ 6
5 x 6 = 30	30 = 5 x 6	30 ÷ 5 = 6	6 = 30 ÷ 5

3. Jim arranged his collection of 24 boats in 3 rows. There were 8 boats in a row.



8 x 3 = 24	24 = 8 x 3	24 ÷ 8 = 3	3 = 24 ÷ 8
3 x 8 = 24	24 = 3 x 8	24 ÷ 3 = 8	8 = 24 ÷ 3

E-30

Name

Draw an array for each story. Write one multiplication equation and one division equation for each story.

Equations may vary. Position of arrays may vary.

1. Eighteen boys from 3 different schools belong to a baseball league. Six boys from each school joined the league.

3 × 6 = 18   *18 ÷ 6 = 3*

2. A group of 7 boys washed 56 windows. Each boy washed 8 windows.

7 × 8 = 56   *56 ÷ 7 = 8*

3. Each of 9 boys ate 4 cookies. The boys ate 36 cookies.

9 × 4 = 36   *36 ÷ 4 = 9*

4. There were 49 boxes of soap on a table. There were 7 stacks with 7 boxes in each stack.

7 × 7 = 49   *49 ÷ 7 = 7*

5. Seven girls sold 63 boxes of cookies. They sold 9 boxes apiece.

7 × 9 = 63   *63 ÷ 9 = 7*

6. Jack bought a set of 45 tin soldiers. He set them up so that there were 5 rows with 9 soldiers in each row.

5 × 9 = 45   *45 ÷ 9 = 5*

7. Jan's mother used 8 eggs to make cakes for a bake sale. She put 2 eggs in each cake. Jan's mother baked 4 cakes.

4 × 2 = 8   *8 ÷ 4 = 2*

E-31

Draw an array for each story. Write one multiplication equation and one division equation for each story.

Position of arrays may vary.

1. Mary Lou bought 4 packages of rolls. There were 8 rolls in each package. She bought 32 rolls.

8 × 4 = 32   *32 ÷ 4 = 8 **

2. Five cars were parked in each of 3 rows. There were 15 cars parked.

5 × 3 = 15   *15 ÷ 5 = 3 **

3. Seven women bought 4 spools of thread apiece. They bought 28 spools of thread.

7 × 4 = 28   *28 ÷ 4 = 7 **

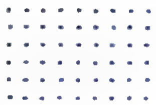
4. Write a story and draw an array for the equations.

7 × 5 = 35   5 = 35 ÷ 7

Stories will vary.

5. Write a story and draw an array for the equations.

9 × 6 = 54   54 ÷ 9 = 6



Stories will vary.

** Equations may vary.*

E-32

Supplemental Experiences

■ Write on the chalkboard the equivalent multiplication and division equations that the children wrote for exercise 2 on page 27. Ask the children to make up new stories to fit these equations. For example, for exercise 2, the children could tell stories such as these.

Carl had 56 sheets of paper. He received 8 sheets from each of 7 pupils.

Each of 8 boys had 7 pieces of candy. They had 56 pieces of candy.

Charlie had 56 marbles. He gave 8 marbles to each of his 7 friends.

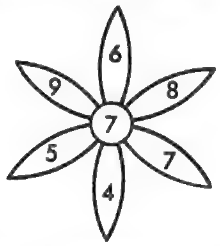
Let the children select equations to use from each exercise on page 27. Such activities as these re-emphasize the idea that different stories can suggest the same equations.

■ Write on the chalkboard the following two columns of equations that lack operation signs.

		Key
3 △ 8 = 24	7 △ 4 = 28	(×, ×)
9 △ 3 = 12	24 ▽ 3 = 8	(+, ÷)
7 = 56 △ 49	9 = 12 ▽ 3	(−, −)
28 △ 7 = 4	4 △ 4 = 16	(÷, ×)
36 ▽ 6 = 30	4 ▽ 2 = 8	(−, ×)
4 = 16 △ 4	30 △ 6 = 36	(÷, +)
8 ▽ 4 = 2	49 = 56 ▽ 7	(÷, −)

The equations in the second column should be equivalent to those in the first column. Ask what operation belongs in the first equation in the first column (multiplication). Tell someone to write the times sign in the proper place. Follow a similar procedure for each of the other equations. When both columns of equations have been completed, tell several children to connect the equivalent equations.

■ Draw a large daisy on the chalkboard as illustrated.



The children are to multiply the number on each petal by the center number, 7. Let the child who successfully "picks all the petals off the daisy" by computing each product correctly change the numeral in the center of the daisy. Repeat the activity by using several different factors in the center of the daisy.

KEY IDEA

$ab = ba$.

Scope

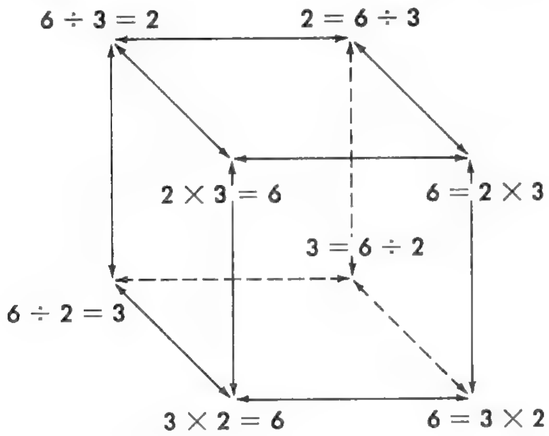
To explore relationships among multiplication and division equations.

To review basic multiplication facts.

Fundamentals

The eight multiplication and division equations that describe an array reflect the definition of division, the commutative property of multiplication, and the symmetric property of equality. Division is defined in terms of multiplication, that is, $c \div b$ is the number that when multiplied by b is c. If $ab = c$, then $a = c \div b$. The commutative property of multiplication means that the order of two factors has no effect on the product: $ab = ba$. The symmetric property of equality expresses a meaning of equality as a relationship between identical numbers so that when $a = b$, $b = a$.

A three-dimensional diagram with one of the equations at each vertex shows how the eight equations are related.



The equations connected by horizontal arrows show the symmetric property of equality.

$$6 \div 3 = 2; 2 = 6 \div 3$$
$$2 \times 3 = 6; 6 = 2 \times 3$$
$$6 \div 2 = 3; 3 = 6 \div 2$$
$$3 \times 2 = 6; 6 = 3 \times 2$$

The equations connected by vertical arrows show the commutative property of multiplication.

$$2 \times 3 = 6; 3 \times 2 = 6$$
$$6 = 2 \times 3; 6 = 3 \times 2$$
$$6 \div 3 = 2; 6 \div 2 = 3$$
$$2 = 6 \div 3; 3 = 6 \div 2$$

The equations joined by oblique arrows show the definition of division.

$$3 \times 2 = 6; 6 \div 2 = 3$$
$$2 \times 3 = 6; 6 \div 3 = 2$$
$$6 = 2 \times 3; 2 = 6 \div 3$$
$$6 = 3 \times 2; 3 = 6 \div 2$$

The ability to recognize the mathematical structure of even a simple story problem and express this structure with an equation is not mastered at any one grade level. Unit 1 and this unit should be considered exploratory in nature and the pupil should not be expected to master this ability at this time. As the child examines the mathematical structure of story problems, he begins to realize how mathematics applies to real situations. He also uses basic addition and multiplication facts in a broadened frame of reference.

Readiness for Understanding

Knowledge of product.

Developmental Experiences

tagboard cards
(3″ × 9″)
felt-tip pen
pocket chart

► Write on the chalkboard several stories similar to the following.

Each of 6 girls ate 3 cookies. How many cookies did these girls eat?

When Mrs. Luke moved, she packed 63 glasses in one box. She put 7 glasses in each row. How many rows of glasses did she pack?

Charlie has some marbles. He gave 5 of these marbles to each of his 4 friends. How many marbles did Charlie give his friends?

A group of girls baked 72 cupcakes for the picnic. Each girl baked 8 cupcakes. How many girls baked cupcakes?

Ask the class what numbers are specifically named in the first story (6 girls were eating cookies and each girl ate 3). Ask what number is not specifically named (the total number of cookies eaten by the girls is not stated).

Ask a child to come to the chalkboard and to write an equation for the story. Ask him to use a placeholder ($\square$) for the number that is not specifically named. The child may write any one of the following eight equations.

$\square = 3 \times 6$	$3 \times 6 = \square$
$\square = 6 \times 3$	$6 \times 3 = \square$
$3 = \square \div 6$	$\square \div 6 = 3$
$6 = \square \div 3$	$\square \div 3 = 6$

Call on other children to write the seven other equations equivalent to the first equation.

Continue this activity using the other stories on the chalkboard.

► Write on the chalkboard pairs of related items similar to those shown in the example.

9 shelves	12 birds	54 chairs	28 children
72 books	3 branches	6 rows	7 scissors

Ask the class to help you make up story exercises using the pairs of items. Tell the children that no product greater than 72 is to be used in any exercise. Write each story as the child dictates it. For example, the second pair of facts may result in a story similar to this:

The 12 birds lived in a tree that had only 3 branches. The same number of birds rested on each of the branches. How many birds were on each branch?

For each story, ask one child to write a multiplication equation on the chalkboard and ask another child to

write a division equation. Ask the children to use a $\square$ for the number they do not specifically name. For example, in the story about the birds, the number of birds resting on each branch is not specifically named. The children could write either of these equations.

$$\square = 12 \div 3 \qquad \square \times 3 = 12$$

After one multiplication and one division equation have been written for each story, ask for the standard name for the number that the placeholder represents. Write each result below the corresponding equation on the chalkboard.

$$\square = 12 \div 3$$
$$\square = 4$$

With each pair of equivalent equations, have the class observe that the solution is the same. Let them tell why this is the case. They should be able to explain that the relationship among the 3 numbers is the same—a product and its two factors.

► Write on the chalkboard the following four equations.

$$24 \times 3 = \square \qquad \square \div 3 = 24$$

$$\square \times 3 = 24 \qquad 24 \div 3 = \square$$

Next to these equations write the following story.

The airport manager let 24 children sit in a jet airplane. Three children sat in each row of seats. How many rows of seats did the children use?

Ask the children to examine the four equations and to identify those that are suggested by the story ($\square \times 3 = 24$ and $24 \div 3 = \square$). Ask them to explain why these equations are appropriate.

The total number of children (24) divided by the number of children seated in each row (3) equals the number of rows ($24 \div 3$, which is $\square$). The equation $24 \div 3 = \square$ expresses this idea.

The number of rows ($\square$) times the number of children seated in each row (3) equals the total number of children ($\square \times 3$, which is 24). The equation $\square \times 3 = 24$ expresses this idea.

Continue this activity with other sets of four equations and related stories.

► Write placeholder equations on 100 tagboard cards, making certain that you have an equivalent division equation for each multiplication equation that you use. Use basic multiplication facts in creating the equations. Here are some examples.

$7 \times \square = 56$	$48 = \square \times 6$	$4 \times 9 = \square$
$\square = 63 \div 9$	$42 \div \square = 7$	$\square \div 5 = 8$

Place five equations that are not equivalent in the left side of a pocket chart. In the right side of the chart place five equations, each one equivalent to one of the five equations on the left, and two more equations that are not related to the others.

	$45 \div 5 = \square$
$3 \times 6 = \square$	$6 \div 3 = \square$
$32 \div 8 = \square$	$\square \div 3 = 6$
$\square \times 7 = 21$	$32 = \square \times 8$
$\square = 56 \div 8$	$3 \times \square = 6$
$45 \div \square = 5$	$7 = 21 \div \square$
	$\square \times 8 = 56$

Call five children forward; ask them to take turns matching an equation on the left side of the chart with one equivalent to it on the right side. This matching may be shown by placing the two equations side by side in the chart. Ask the class to decide whether the equations have been matched correctly.

Continue this activity until all of the children have had an opportunity to find a pair of equivalent equations.

► Write on the chalkboard several multiplication and division equations that contain a placeholder. Discuss whether the placeholder ($\square$) in each equation represents the product or one of the factors.

$3 \times 9 = \square$ (product)	$6 \div \square = 3$ (factor)
$4 \times \square = 8$ (factor)	$\square \div 7 = 4$ (product)
$\square \times 7 = 14$ (factor)	$18 \div 2 = \square$ (factor)
$\square = 5 \times 6$ (product)	$\square = 18 \div 9$ (factor)
$12 = \square \times 3$ (factor)	$5 = \square \div 4$ (product)
$16 = 8 \times \square$ (factor)	$2 = 10 \div \square$ (factor)

Next, write on the board the equation $5 \times 4 = \square$. Ask a child to write an equivalent division equation below the original equation. He may write any one of the following equations.

$$\square \div 5 = 4 \qquad 4 = \square \div 5$$
$$\square \div 4 = 5 \qquad 5 = \square \div 4$$

Tell the class to examine both equations and to tell whether $\square$ is the product or one of the factors.

Follow a similar procedure with a division equation such as $8 \div 2 = \square$. After you write the equation on the chalkboard, call on a child to write a multiplication equation equivalent to the division equation. He should write any one of the following equations.

$$\square \times 2 = 8 \qquad 8 = \square \times 2$$
$$2 \times \square = 8 \qquad 8 = 2 \times \square$$

Have the class tell whether $\square$ is the product or one of the factors.

Continue this, using multiplication or division equations such as the following.

$2 \times 8 = \square$	$8 \div \square = 4$
$4 \times \square = 16$	$\square \div 5 = 2$
$\square \times 3 = 15$	$14 \div 7 = \square$
$\square = 9 \times 1$	$\square = 10 \div 2$
$12 = \square \times 6$	$4 = \square \div 1$
$24 = 8 \times \square$	$1 = 9 \div \square$

If a multiplication equation is given, the child should write an equivalent division equation; if a division equation is used, the child should write an equivalent multiplication equation. As the children tell whether □ is the product or one of the factors in each equation, they will reinforce their understanding of the relationship between multiplication and division.

► On the chalkboard, write a story that contains no specifically named numbers. For example:

The fourth grade garden club planted some bean plants. They put the same number of plants in each of several rows. How many bean plants did the club plant?

Tell the class that equations can be written for this story just as they were for previous stories. However, placeholders □, △, and ○ will be used for all three numbers.

Let □ stand for the number of bean plants planted.
Let △ stand for the number of rows of plants.
Let ○ stand for the number of plants put in each row.

Select eight children to come to the board. Each child should write either a multiplication or a division equation that shows the relationship among the numbers.

□ = △ × ○	△ × ○ = □
□ = ○ × △	○ × △ = □
○ = □ ÷ △	□ ÷ △ = ○
△ = □ ÷ ○	□ ÷ ○ = △

Do this with the following stories.

△ rose plants were growing in the greenhouse. Each plant had □ buds. ○ buds were on all of the rose plants.

John arranged △ chairs in several rows, with □ chairs in each row. There were ○ rows of chairs.

Jill bought △ packages of seeds. Each package held □ seeds. Jill bought ○ seeds.

Pages 33 through 38

● Pages 33 and 34 present story exercises with one number not specifically named. A placeholder □ is used in the story and in the equations to represent this number. The child is asked to write one multiplication and one division equation for each of the stories. He need not solve the equations.

Discuss the example at the top of page 33 with the class. Questions such as these may help the children to understand the relationship among the numbers in the story.

What numbers are there in the story? (20 pupils, 4 teams, □ pupils on each team)

What does the placeholder represent? (the number of pupils on each team)

Ask what eight related multiplication and division equations express the relationship among the numbers in the story. Have the children write on the chalkboard the six equations that are not given.

20 ÷ □ = 4	□ × 4 = 20
4 = 20 ÷ □	20 = □ × 4
□ = 20 ÷ 4	20 = 4 × □

Then complete exercise 1 with the class. Assign the rest of the exercises. When they finish, discuss the equations that the children wrote. Ask whether □ represents a product or a missing factor.

Work the example at the top of page 34 with the class. Assign the exercises. The vocabulary is relatively simple; most pupils can complete these exercises independently. Some children may need to draw arrays to help them visualize the equations for the stories. They should be permitted to do so. Do not ask the children to solve the equations, but merely to write them.

● The story exercises on page 35 are designed to help the children see that the number that is not specifically named in a story exercise is usually the number that is asked for in the form of a question. To help the children make the transition from incomplete sentences such as, "There were □ pupils in her class.", to questions such as, "How many pupils were in her class?", have them translate into questions the incomplete sentences given on pages 33 and 34.

Ask a child to read the example story to the class. Ask what number is not specifically named in this story (the total number of boys who played basketball). Point out that □ has been used to represent this number. Ask what other multiplication equations might be given for this story (□ = 5 × 2, 2 × 5 = □, □ = 2 × 5). Then discuss division equations that might be given (□ ÷ 5 = 2, 5 = □ ÷ 2, □ ÷ 2 = 5). Ask whether □ in each of these equations is the product or one of the factors (the product).

Work exercise 1 with the class. Assign the rest of the exercises. As the children discuss the equations that they selected for each exercise, ask whether each □ represents the product or one of the factors.

● The exercises on page 36 provide practice for the children in writing and solving equations for story exercises. The children are also asked to compose their own sentence to answer the question posed in each story.

With the class, work one exercise on page 36 so that the children understand the procedure. Then assign the remaining exercises. Encourage the children to write the equation that seems easiest to them. No attempt should be made to enforce uniformity or to lead the children to write a particular kind of equation for specific types of stories. The important thing is the relationship among the numbers. When checking the exercises, discuss only those that pupils found difficult.

After the page has been completed, let the children create other story exercises using the equations they wrote. For example, if the children wrote the equation 9 × 7 = □ for exercise 2, someone may create this story exercise for the same equation:

Marge gave 9 spools of thread to each of 7 girls.
How many spools of thread did she give away?

By creating several story exercises for the same equation, the children will learn to identify the relationship among numbers in any story.

Name

In a gym class, 20 pupils played dodgeball. These 20 pupils were divided into 4 teams. There were □ pupils on each team.

4 × □ = 20
20 ÷ 4 = □

See pupil page suggestions.

Write one division equation and one multiplication equation for each story.
Equations may vary.

1. Six groups of children were reading. There were 8 children in each group. There were □ children reading. *□ ÷ 6 = 8* / *8 × 6 = □*
2. Mary saw 9 large zips and each zip had 8 small goops. There were □ small goops. *□ ÷ 9 = 8* / *9 × 8 = □*
3. Four boys had 4 rabbits apiece. They had □ rabbits. *□ ÷ 4 = 4* / *□ = 4 × 4*
4. The children in the art class work in 7 groups. Each group has a jar of brushes. There are 9 brushes in each jar. There are □ brushes in all the jars. *□ ÷ 9 = 7* / *7 × 9 = □*
5. Coach Jones has 36 baseball gloves that he must divide equally among 4 teams. Each team will get □ gloves. *36 ÷ 4 = □* / *4 × □ = 36*
6. Tim made a toy from 18 pipe cleaners. There were 9 pipe cleaners in each package. He used □ packages. *18 ÷ □ = 9* / *9 × □ = 18*
7. There were 8 men selling popcorn at the baseball game. Each man sold 9 boxes of popcorn. They sold □ boxes of popcorn. *□ ÷ 8 = 9* / *8 × 9 = □*
8. Seven squirrels each found 9 nuts. They found □ nuts. *□ ÷ 7 = 9* / *7 × 9 = □*

reference page

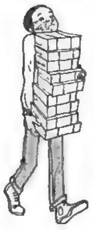
There are 6 grades at Victory School. There are 3 classes in each grade. There are □ classes in the school.

6 × 3 = □ □ ÷ 6 = 3

Write one multiplication and one division equation for each story.
Equations may vary.

1. There are 24 children in one room. Six children sit in each row. There are □ rows of desks in the room.
6 × □ = 24 *24 ÷ □ = 6*
2. The children in Miss Powell's class were grouped in 3 rows for the class picture. There were 7 children in each row. There are □ children in Miss Powell's class.
7 × 3 = □ *□ ÷ 7 = 3*

3. The 25 children who direct traffic at the school are grouped in squads of 5. There are □ squads.
5 × □ = 25 *25 ÷ 5 = □*
4. Nine boys each delivered 9 packs of paper to the art room. The boys delivered □ packs of paper.
9 × 9 = □ *□ ÷ 9 = 9*



5. A card file has 4 rows of drawers with 8 drawers in each row. There are □ drawers in the file.
4 × 8 = □ *□ ÷ 4 = 8*
6. There are 8 vims in each of 8 voms. There are □ vims.
8 × 8 = □ *□ ÷ 8 = 8*
7. Carol has 4 postcards on each of 7 pages in her scrapbook. She has □ postcards on these pages.
4 × 7 = □ *□ ÷ 4 = 7*

Name

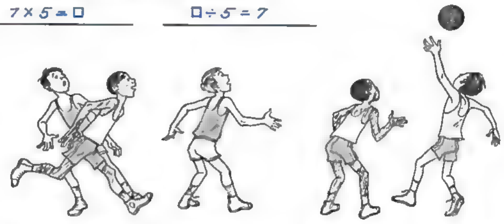
Two teams of boys played basketball. There were 5 boys on each team. How many boys played? *See pupil page suggestions.*

5 × 2 = □ 2 = □ ÷ 5

Write one division and one multiplication equation for each story.
Equations may vary.

1. Eight carloads of boys went to a basketball game. Five boys went in each car. How many boys went? *8 × 5 = □* / *□ ÷ 8 = 5*
2. The school gym had 63 seats. There were 7 seats in each row. How many rows were there? *7 × □ = 63* / *63 ÷ 7 = □*
3. Seven boys ate 2 hot dogs apiece. How many hot dogs did these boys eat? *2 × 7 = □* / *□ ÷ 2 = □*
4. The home team scored 8 points in each of the 4 quarters of the game. What was its final score? *8 × 4 = □* / *□ ÷ 8 = 4*
5. Each of the 5 boys on the other team scored 6 points in the game. What was the final score for that team? *5 × 6 = □* / *□ ÷ 5 = 6*
6. The 2 teams each had 5 cheerleaders at the game. How many cheerleaders were there?
2 × 5 = □ *□ ÷ 5 = 2*

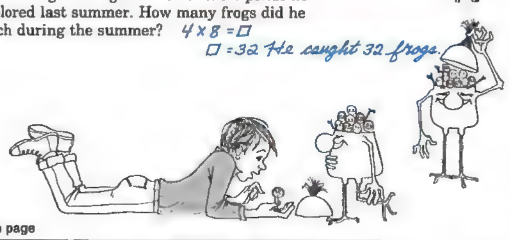
7. On the way home 7 carloads of 5 boys each stopped for milkshakes. How many boys is this?
7 × 5 = □ *□ ÷ 5 = 7*



Write an equation for each story. Solve the equation. Write your answer in a sentence.
Equations may vary.

1. Lorie collected 21 sand dollars. She collected 3 each day she went to the beach. How many days did she go to the beach? *21 ÷ □ = 3*
□ = 7 She went 7 days.
2. Mark did 9 push-ups every day for 7 days. How many push-ups did he do in one week? *9 × 7 = □*
□ = 63 He did 63 push-ups.
3. How many zoops will be needed if we want to arrange them in 8 rows with 9 zoops in each row? *8 × 9 = □*
□ = 72 72 zoops are needed.
4. Terry has 36 pick-up sticks. She has an equal number of blue, green, yellow, and red sticks. How many of each color does she have? *36 ÷ 4 = □*
□ = 9 She had 9 of each color.
5. Thirty-five dokes decided to play a game of duff. They formed 7 equal teams. How many dokes were on a team? *35 ÷ □ = 7*
□ = 5 There were 5 on a team.

6. Four delivery trucks each made 8 deliveries in one day. How many deliveries were made?
4 × 8 = □
□ = 32 32 deliveries were made.
7. Pat found 9 gloofs in each of 3 tweekles. How many gloofs did Pat find? *□ ÷ 3 = 9*
□ = 27 He found 27 gloofs.
8. Jamie caught 4 frogs in each of the 8 ponds he explored last summer. How many frogs did he catch during the summer? *4 × 8 = □*
□ = 32 He caught 32 frogs.



reference page

● Page 37 provides practice for the pupils to write a story and draw a picture for the equations in each exercise.

Assign the exercises for independent work. When the page has been completed, let pupils read their stories for the exercises and let various pupils illustrate their stories on the chalkboard.

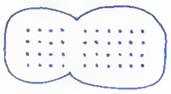
Name

Write a story and draw a picture for the equations.
Positions of pictures will vary.

1. 12 ÷ □ = 4   12 = □ × 4
Stories will vary.

2. 4 × □ = 36   36 ÷ □ = 4
Stories will vary.

3. □ + 16 = 40   40 − □ = 16
Stories will vary.



4. 63 ÷ □ = 9   63 = 9 × □
Stories will vary.

E-37

● The exercises on page 38 provide practice for the pupils in writing and solving equations for story exercises. The pupils are also asked to compose their own sentence to answer the question posed in each story.

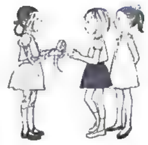
With the class, work one exercise on page 38 so that the pupils understand the procedure. Then assign the remaining exercises. Encourage the pupils to write the equation that seems easiest to them. No attempt should be made to enforce uniformity or to lead the pupils to write a particular kind of equation for specific types of stories. The important thing is the relationship among the numbers. When checking the exercises, discuss only those that pupils found difficult.

This unit should help the children develop their ability to recognize and understand product-factor relationships among numbers used in a story exercise and to write equations that will express these relationships. The child who has not developed this ability probably does not understand some of the basic multiplication and division combinations, the relationship between multiplication and division, or the relationship of stories and equations to arrays. Forthcoming units may help the pupils improve their understanding of these concepts.

Write an equation for each story exercise. Solve the equation. Then write a sentence to answer the question.
Equations may vary.

1. Neal had 36 inches of string. He cut off 20 inches to tie to his balloon. How long was the other piece of string? *36 − 20 = □*
□ = 16 The other piece is 16 inches long.

2. Vickie used 7 different colors of paper to make a birthday card for her mother. She used 5 pieces of each color. How many pieces of paper did she use? *7 × 5 = □*
□ = 35 She used 35 pieces.



3. There are 6 grades in Glenn School. Each grade has 4 classes. How many classes are there in the school? *6 × 4 = □*
□ = 24 There are 24 classes.

4. Audrey's father gave her 40 tickets for free rides at the fair. She divided them equally among 8 friends. How many tickets did she give to each friend? *40 ÷ 8 = □*
□ = 5 She gave 5 tickets to each friend.

5. Jack bought a box containing 54 pieces of candy. Eight were filled with cherries and 9 were filled with coconut. How many pieces of candy were not filled with cherries or coconut? *54 − (8 + 9) = □*
□ = 37 37 were not filled with cherries or coconut.

6. An auto dealer has 54 new cars. If he sells 6 cars a day, how long will it take him to sell all of the cars? *54 ÷ 6 = □*
□ = 9 It will take 9 days.

7. Jan put 8 cups on each of 8 trays. How many cups did she put on trays? *8 × 8 = □*
□ = 64 She put 64 cups on trays.

8. There were 49 cars in a parking lot. Seven cars were parked in each row. How many rows of cars were in the lot? *49 ÷ 7 = □*
□ = 7 There were 7 rows.

E-38

Supplemental Experiences

■ Have the children complete tables similar to these. They may be put on the chalkboard or duplicated.

	÷	=	×	=
64	8		8	
56	7		4	
63	9		3	
72	8		6	
42	7		8	
28	7		9	
45	9		7	
49	7		5	

	×	=	÷	=
8	3		6	
2	9		3	
9	9		9	
3	4		6	
2	8		4	
8	9		8	
5	6		6	
9	4		6	

Key:

8	64	24	4
8	32	18	6
7	21	81	9
9	54	12	2
6	48	16	4
4	36	72	9
5	35	30	5
7	35	36	6

■ Ask several children to draw pictures of rockets at different altitudes on the chalkboard. Write a multiplication combination on each rocket; each exercise should have a factor or a product missing.

Select a child to act as rocket booster and point to one rocket at a time. Explain to the class that each child in turn will have a chance to fire a rocket by reading the exercise and giving the product or missing factor. A correct answer is a successful space shot. Use combinations that the children need for practice.

■ Prepare a cross-numeral puzzle for the children to solve. Duplicate the puzzle so that each child will have a copy.

a 4	2		b 2		c 5	d 4
9		e 2	1			5
	f 4				g 8	
h 7	2		i 9		j 1	6
k 2		l 6	m 4			n 5
o 8	1		0		p 3	6

ACROSS

a) 6 × 7
b) 14 ÷ 7
c) 6 × 9
e) 3 × 7
f) 28 ÷ 7
g) 56 ÷ 7
h) 9 × 8
i) 54 ÷ 6
j) 8 × 2
k) 12 ÷ 6
l) 8 × 8
n) 35 ÷ 7
o) 9 × 9
p) 9 × 4

DOWN

a) 7 × 7
b) 7 × 3
c) 45 ÷ 9
d) 5 × 9
e) 16 ÷ 8
f) 6 × 7
g) 9 × 9
h) 42 ÷ 6
i) 36 ÷ 4
k) 4 × 7
m) 8 × 5
n) 7 × 8

Before the children try to complete the cross-numeral puzzle on their own, copy part of the puzzle on the chalkboard. With the class, work several of the exercises across and down so that the children understand how to complete the puzzle.

UNIT 4
THE MULTIPLICATION ALGORISM
Pages 39 Through 56

OBJECTIVE

To develop long multiplication.

The pupil observes that an array may be partitioned to represent *partial products* and that the sum of partial products is the whole product. He works with the long multiplication algorism and recognizes that it is based on the distributive property. His knowledge of basic multiplication facts and numeration is used to compute products with factors of 2 and 3 digits.

See Key Topics in Mathematics for the Intermediate Teacher: Multiplication and Division of Whole Numbers.

KEY IDEAS

$7 \times 9 = (7 \times 5) + (7 \times 4)$.

The distributive property lets us multiply digit-by-digit.

Tens times tens is ten tens or hundreds.

To find a missing factor, multiply and check.

KEY IDEA

$7 \times 9 = (7 \times 5) + (7 \times 4)$.

Scope

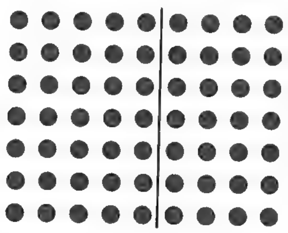
To recognize a product as a sum of partial products.

Fundamentals

The distributive property involves two operations, multiplication and addition. A partitioned array shows the distributive property of multiplication over addition. Consider a 7 by 9 array partitioned into two smaller arrays, one 7 by 5 and the other 7 by 4. Each smaller array illustrates a partial product; the sum of these partial products is the product, the number of members in the whole array.

$7 \times 9 = 7 \times (5 + 4)$

$\underline{7 \times 5}$



$\underline{7 \times 4}$

$7 \times (5 + 4)$ or $(7 \times 5) + (7 \times 4)$

An array can be partitioned in many different ways. The particular partition chosen in no way affects the product. The whole is still the sum of its parts.

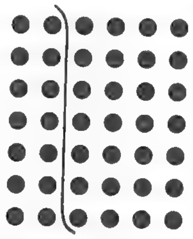
Readiness for Understanding

Understanding of product.
Understanding of sum.

Developmental Experiences

circle stickers	*for each child*
sheets of paper or tagboard (15″ × 15″)	85 counters
crayons	yarn
masking tape	

► Draw a 7 by 6 array on the chalkboard and ask a child to write the product for this array. Then draw a line to partition the array as illustrated.



Have the class tell the product shown by each of these two smaller arrays. Write on the chalkboard a sum of the products shown by the smaller arrays.

$$7 \times 6$$
$$(7 \times 4) + (7 \times 2)$$

Ask the children to comment on the relationship between the whole array and its parts. They may say things like this:

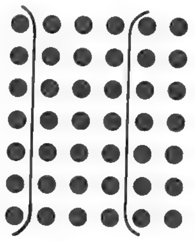
A 7 by 6 array is made up of a 7 by 4 array and a 7 by 2 array.

A 7 by 6 array contains the same number of objects as a 7 by 4 array and a 7 by 2 array.

By adding the products shown by the small arrays, you get the product shown by the large array.

The number of things in the array is the sum of the number of things in the parts.

Partition the array again, as illustrated.

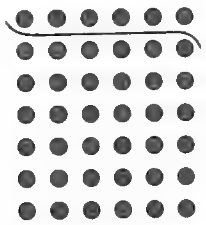


Ask the children to state the product shown by each of these arrays. Write their responses on the chalkboard below the other two expressions.

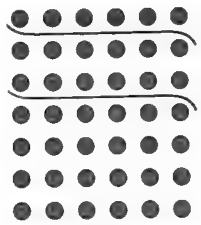
$$7 \times 6$$
$$(7 \times 4) + (7 \times 2)$$
$$(7 \times 1) + (7 \times 3) + (7 \times 2)$$

Erase the partition lines and ask a child to draw a vertical line to partition the array in a different way. Have another child write the sum of partial products for the new partition.

Erase the vertical partition line from the board. Ask some child to draw a horizontal line that partitions the array. Ask another child to record the sum of partial products shown by this partition. This example illustrates $(1 \times 6) + (6 \times 6)$.

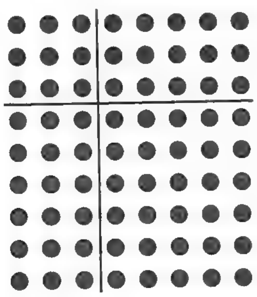


Tell some other child to draw a second horizontal line. Ask someone to record the sum of partial products shown by the partitioned array. This example illustrates $(1 \times 6) + (2 \times 6) + (4 \times 6)$.



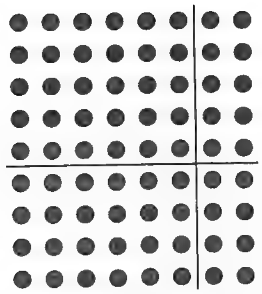
Adapt this procedure to a 9 by 7 array, an 8 by 5 array, a 7 by 8 array, and a 6 by 9 array. First, let the children partition using vertical lines and then have them partition using horizontal lines. Allow them to make any number of partitions that they choose. As each partition line is drawn, have the child record below the original product the sum of partial products.

▶ Let several children demonstrate arrays partitioned once vertically and once horizontally. Draw a 9 by 8 array on the chalkboard. Ask a child to draw a vertical line that partitions the array; ask him to draw another line that partitions the array horizontally. Then instruct the child to call on another member of the class to write the sum of the partial products illustrated by the partitioned array. Tell this second child to call on someone else to write the product illustrated by this array.



$(3 \times 3) + (3 \times 5) + (6 \times 3) + (6 \times 5)$
9×8

Erase the partition lines in the 9 by 8 array. Direct another child to partition the array with one horizontal and one vertical line in a different way. Let him call on some member of the class to write the sum of the partial products below 9×8.

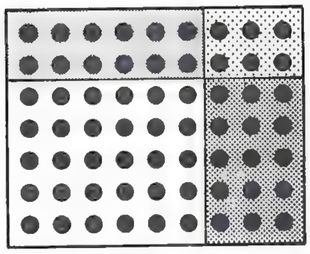


$(3 \times 3) + (3 \times 5) + (6 \times 3) + (6 \times 5)$
9×8
$(5 \times 6) + (5 \times 2) + (4 \times 6) + (4 \times 2)$

Adapt this procedure to other products such as 6×8, 7×5, and 4×9.

▶ Use circle stickers to make a 9 by 7 array on a large sheet of paper. Partition the array once horizontally and once vertically, and shade the background of each small array using different colors, for example: red, yellow, blue, and white. Fasten the array to the chalkboard.

Ask the class to help you compute the product of this array in parts. First, ask someone to give the product illustrated by the array in the red section of the card. Then, ask him to count the number of members in this section. Write on the chalkboard the equation: $2 \times 6 = 12$. Continue in this way with each of the other arrays.



Red:	$2 \times 6 = 12$
Yellow:	$2 \times 3 = 6$
White:	$5 \times 6 = 30$
Blue:	$5 \times 3 = 15$

Then let the class compute the sum of the partial products; write this number below the list of the partial products.

Red	$2 \times 6 = 12$
Yellow	$2 \times 3 = 6$
White	$5 \times 6 = 30$
Blue	$5 \times 3 = \underline{15}$
	63

Since the number of members in the array is 7×9, as well as 63, we may write this equation on the chalkboard:

$$7 \times 9 = 63$$

Select several children to count the elements in this array in the following ways:

Count the members of the array one by one.
Count by 3 (add 3s).
Count by 7 (add 7s).
Count by 9 (add 9s).

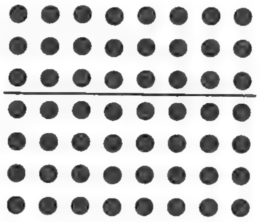
Use this procedure for other products from 6×6 through 9×9. In each instance, place an array-card

on the chalkboard; compute the partial products; compute the sum of the partial products. Ask the pupils to write an equation which says that each product is the number you get when you count the elements in the array.

Draw a 7 by 8 array on the chalkboard and ask a child to write the product below the array. Ask a child to show the class what part of the array the factor 7 represents (the number of horizontal rows). Then draw a line to partition the array to show 7 as 4 + 3.

3

4



7×8

Then, below 7×8, rewrite this product using $(4 + 3)$ instead of 7.

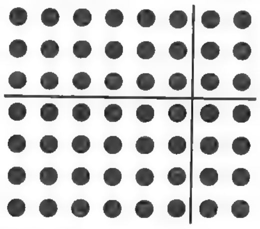
$$7 \times 8$$
$$(4 + 3) \times 8$$

Ask a pupil to point out the part of the array that the factor 8 represents (the number of vertical rows). Then partition the array to show 8 as 6 + 2.

6 2

3

4



Write the product for the array.

$$7 \times 8$$
$$(4 + 3) \times 8$$
$$(4 + 3) \times (6 + 2)$$

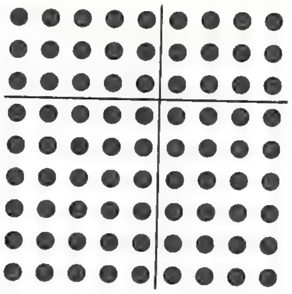
Ask for volunteers to name the partial products that are illustrated. Write each partial product as it is given.

$$7 \times 8$$
$$(4 + 3) \times 8$$
$$(4 + 3) \times (6 + 2)$$
$$(4 \times 6) + (3 \times 6) + (4 \times 2) + (3 \times 2)$$

Erase the partition lines from the array on the board. Let other children use horizontal and vertical lines to partition the array in a different way. For each partition write each factor as a sum. Then let the class write the sum of the partial products.

► Give each child 85 counters and 2 pieces of yarn about 15 inches long. (This activity could be adapted to the overhead projector, using metal washers as counters.) The children can use their counters on their desks to make a 9 by 9 array. Tell them to use pieces of yarn to partition their arrays once vertically and once horizontally.

For example, a child could partition the array like this:



Write the product for the array (9×9) on the chalkboard. Ask a child to express the product of sums illustrated by his partitioned array. His expression of the product may be any of these:

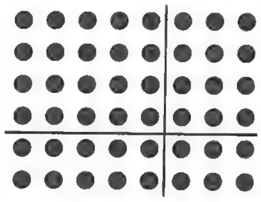
$$9 \times 9 = (3 + 6) \times (5 + 4)$$
or
$$(6 + 3) \times (4 + 5)$$
or
$$(3 + 6) \times (4 + 5)$$
or
$$(6 + 3) \times (5 + 4)$$

Ask a child to give the partial products that result from his partition (3×5, 3×4, 6×5, and 6×4). Let the children discuss their partitions. Each child should name the partial products he found.

Adapt this procedure to other products from 6×6 through 9×9.

► Have a child draw a 6 by 8 array on the chalkboard and write the product illustrated by this array.

Ask another child to use vertical and horizontal lines to partition the array. He could do it this way.



6×8

Ask the second child to write the number of things in this array as a product of sums. Remind him that the particular way he partitioned the array will give him the information he needs. Tell the child to write his expression of the product below the first expression of this number.

$$6 \times 8$$
$$(4 + 2) \times (5 + 3)$$

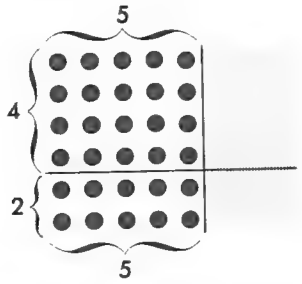
Ask another child to write a sum of partial products illustrated by this array.

$$6 \times 8$$
$$(4 + 2) \times (5 + 3)$$
$$(4 \times 5) + (2 \times 5) + (4 \times 3) + (2 \times 3)$$

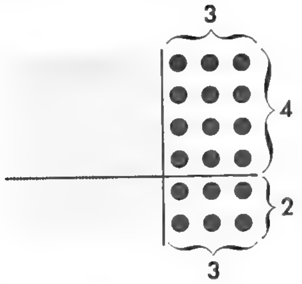
The child may have written these in a different order. Even so, he should be able to point out the parts of the array that correspond to each partial product.

Help the pupils see that the number of members in the portion of the array indicated below is

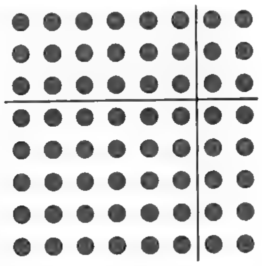
$(4 \times 5) + (2 \times 5)$. Have the class examine the array to locate the 5 that is multiplied by 4 and by 2 in the expression $(4 \times 5) + (2 \times 5)$.



Help the children see that the number of members in the portion of the array illustrated below is $(4 \times 3) + (2 \times 3)$. Ask the children to locate the 3 that is multiplied by 4 and by 2.



Draw an 8 by 8 array on the chalkboard. Have a child write the product (that is, the number of elements in the array) on the chalkboard (8×8). Ask for a vertical and a horizontal partitioning of the array.



8×8

Have the number of elements in the array (8×8) expressed as a sum of the partial products. For example, if the array is partitioned like the preceding illustration, the following sum of partial products would be written.

$$\begin{gathered} 8 \times 8 \\ (6 \times 5) + (6 \times 3) + (2 \times 5) + (2 \times 3) \end{gathered}$$

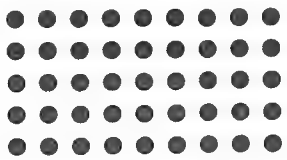
Next, have the two factors each expressed as a sum, as indicated by the partition. Ask a pupil to write the product of these sums.

$$\begin{gathered} (6 \times 5) + (6 \times 3) + (2 \times 5) + (2 \times 3) \\ (6 + 2) \times (5 + 3) \end{gathered}$$

Have the children discuss the relationship between the addends of the factors in the product $(6 + 2) \times (5 + 3)$ and the partial products in the sum $(6 \times 5) + (6 \times 3) + (2 \times 5) + (2 \times 3)$. The class may observe that 6 is multiplied by 5 and 3, and 2 also is multiplied by 5 and 3.

Adapt this procedure to examine other arrays, from 6×6 through 9×9.

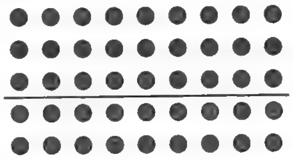
► Have a pupil draw a 5 by 9 array on the chalkboard.



Tell him to write on the board in vertical form the product and the count for this array.

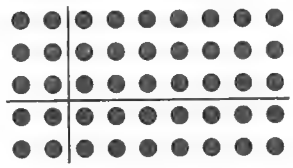
$$\begin{array}{r} 5 \\ \underline{\times\ 9} \\ 45 \end{array}$$

Have someone express 5 as a sum and write his suggestion on the chalkboard. If a child suggests that 5 is $2 + 3$, let a second child partition the array to illustrate this.



$$\begin{array}{rl} 5 &= 2 + 3 \\ \underline{\times\ 9} & \\ 45 & \end{array}$$

Ask another child to express the factor 9 as a sum. If some child suggests that 9 is $2 + 7$, direct someone to partition the array to illustrate this.



$$\begin{array}{rl} 5 &= 2 + 3 \\ \underline{\times\ 9} &= \underline{2 + 7} \\ 45 & \end{array}$$

Ask a child to write the product for each of the two parts of the array that are to the right of the vertical partitioning line given by the class (in this case, they would write 7×3 and 7×2). Record these products as indicated in the following example. Ask the pupils to compute these products. Record these numbers as indicated. Have the class observe that 7 is multiplied by 3 and by 2, demonstrating the distributive property.

Follow a similar procedure for the partitioning of the left side of the array. Have the class observe that 2 is multiplied by 3 and by 2.

$$\begin{array}{rlr} 5 &= 2 + 3 & \\ \underline{\times\ 9} &= \underline{2 + 7} & \\ 45 & & 21\,(7 \times 3) \\ & & 14\,(7 \times 2) \\ & & 6\,(2 \times 3) \\ & & \underline{4}\,(2 \times 2) \\ & & 45 \end{array}$$

Continue the activity by using several other 9 by 5 arrays partitioned in different ways. For example, 5 is $4 + 1$ and 9 is $6 + 3$.

$$\begin{array}{rl} 5 &= 4 + 1 \\ \underline{\times\ 9} &= \underline{6 + 3} \\ 45 & \end{array}$$

Another child may say that 5 is $3 + 2$ and 9 is $4 + 5$.

$$\begin{array}{rl} 5 &= 3 + 2 \\ \underline{\times\ 9} &= \underline{4 + 5} \\ 45 & \end{array}$$

Then investigate other products from 6 × 6 through 9 × 9.

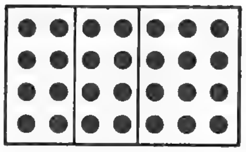
Pages 39 through 42

● Pages 39 and 40 give the children more experience in expressing a product as a sum of products. Use page 39 primarily for discussion. Ask the children to study the first array and equation in the example. Ask the pupils to explain why each partial product is the number of members in a part of the array. Then ask how the sum 18 + 18 is illustrated. Someone may explain that 18 is the same number as 2 × 9, that each part has 18 members, and that the number of members in the whole array is 18 + 18. Finally, have a pupil describe 36 as a sum of the partial products. Follow a similar procedure in discussing the other 4 by 9 array at the top of the page.

Next, refer to the exercises on this page. Tell the children that you know they can give the standard numerals for the product 7 × 4. Explain that these exercises will give them practice in computing products in parts; this will be helpful later when they use the multiplication algorism.

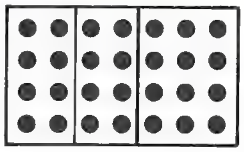
Some of the children may want to investigate other ways of partitioning the arrays on the page. Encourage this type of investigation. This will not only reinforce the children's knowledge of the basic combinations but will also strengthen their understanding of a product as a sum of products.

Draw on the chalkboard a model of the 7 by 4 array shown in exercise 2. Write the product for the array below this model. Tell the class to study the three arrays that the partition shows and to tell the product for each part (4 × 2, 4 × 2, and 4 × 3 or 2 × 4, 2 × 4, and 3 × 4). Ask a child to write on the chalkboard a sum of the products of the small arrays. Ask him to write an equation to show that the sum of the partial products is the same number as the product illustrated on the array. Note that the order of the partial products may vary.



$$7 \times 4 = (4 \times 2) + (4 \times 2) + (4 \times 3)$$

Ask the pupils to count the members in the parts (8, 8, and 12). Place another equal sign as indicated and write the sum 8 + 8 + 12. Ask a child to compute the sum of the partial products and write this number as indicated here.



$$\begin{aligned} 7 \times 4 &= (4 \times 2) + (4 \times 2) + (4 \times 3) \\ &= 8 + 8 + 12 \\ &= 28 \end{aligned}$$

If the children have not had much experience with arrays, follow a similar procedure with the remaining exercises on page 39 and with each of the exercises on page 40. If the children understand the concepts presented on these pages, work several exercises with the class. Then assign the remaining exercises. Be sure the children realize that the order of the partial products in the equation may vary. The order of the factors in each partial product may also vary. The following examples are two ways that the children might complete exercise 2 on page 40.

$$\begin{aligned} 5 \times 8 &= (2 \times 8) + (3 \times 8) \\ &= 16 + 24 \\ &= 40 \end{aligned}$$

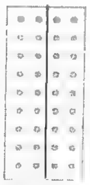
$$\begin{aligned} 5 \times 8 &= (8 \times 3) + (8 \times 2) \\ &= 24 + 16 \\ &= 40 \end{aligned}$$

Name

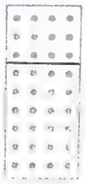
UNIT 4 THE MULTIPLICATION ALGORISM

For Class Discussion

We can partition an array into smaller arrays without changing the number of members of the array. See pupil page suggestions.



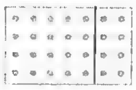
$4 \times 9 = (2 \times 9) + (2 \times 9)$
$= 18 + 18$
$= 36$



$4 \times 9 = (4 \times 3) + (4 \times 6)$
$= 12 + 24$
$= 36$

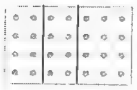
Compute as shown above. Equations may vary.

1. 7 × 4



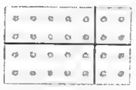
$7 \times 4 = (5 \times 4) + (2 \times 4)$
$= 20 + 8$
$= 28$

2. 7 × 4



$7 \times 4 = (2 \times 4) + (2 \times 4) + (3 \times 4)$
$= 8 + 8 + 12$
$= 28$

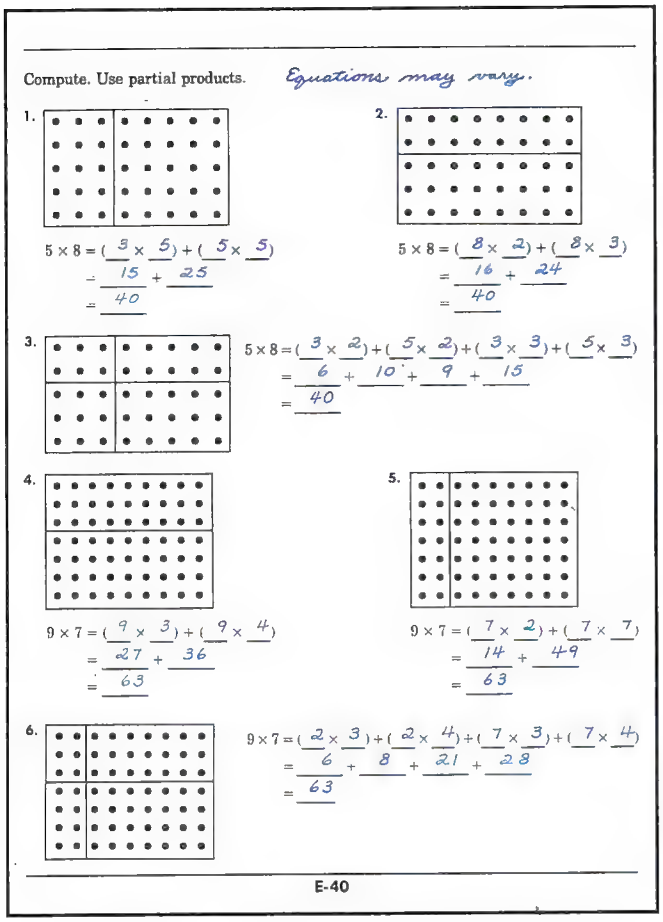
3. 7 × 4



$7 \times 4 = (5 \times 2) + (5 \times 2) + (2 \times 2) + (2 \times 2)$
$= 10 + 10 + 4 + 4$
$= 28$

reference page

E-39



Compute. Use partial products. *Equations may vary.*

1. $5 \times 8 = (3 \times 5) + (5 \times 5)$
   $= 15 + 25$
   $= 40$

2. $5 \times 8 = (8 \times 2) + (8 \times 3)$
   $= 16 + 24$
   $= 40$

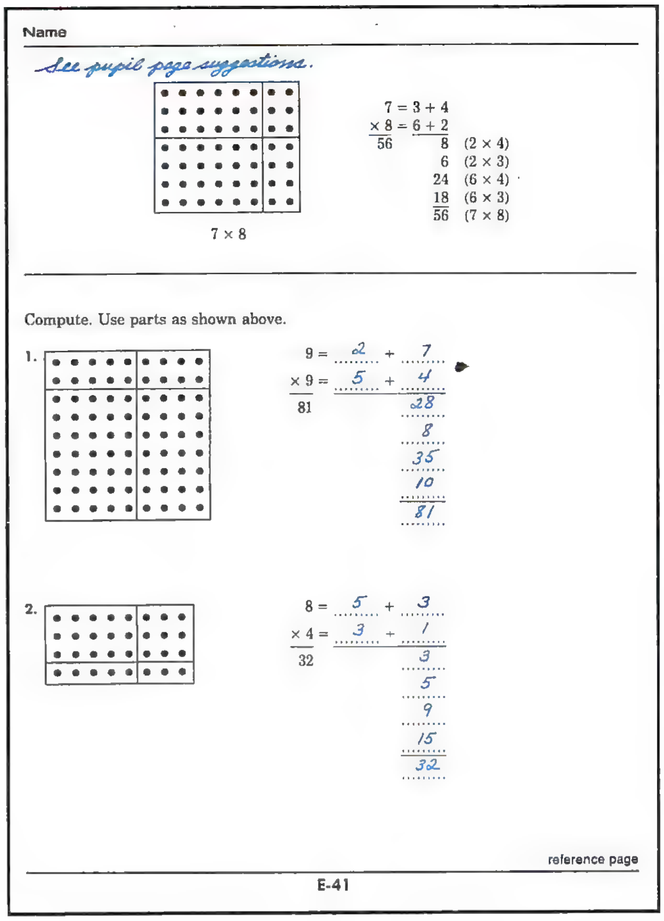
3. $5 \times 8 = (3 \times 2) + (5 \times 2) + (3 \times 3) + (5 \times 3)$
   $= 6 + 10 + 9 + 15$
   $= 40$

4. $9 \times 7 = (9 \times 3) + (9 \times 4)$
   $= 27 + 36$
   $= 63$

5. $9 \times 7 = (7 \times 2) + (7 \times 7)$
   $= 14 + 49$
   $= 63$

6. $9 \times 7 = (2 \times 3) + (2 \times 4) + (7 \times 3) + (7 \times 4)$
   $= 6 + 8 + 21 + 28$
   $= 63$

E-40



Name

See pupil page suggestions.

7 × 8

$7 = 3 + 4$
$\times 8 = 6 + 2$

56	8	(2 × 4)
	6	(2 × 3)
	24	(6 × 4)
	18	(6 × 3)
	56	(7 × 8)

Compute. Use parts as shown above.

1. $9 = 2 + 7$
   $\times 9 = 5 + 4$
   81: 28, 8, 35, 10, 81

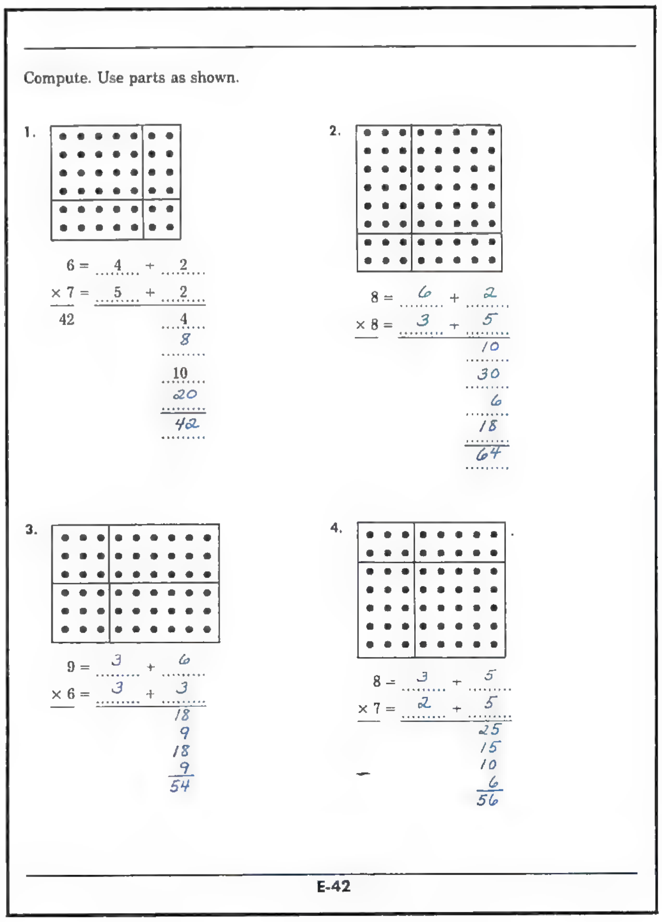
2. $8 = 5 + 3$
   $\times 4 = 3 + 1$
   32: 3, 5, 9, 15, 32

reference page

E-41

● Pages 41 and 42 give the children an opportunity to describe an array as a product of sums. Discuss the example at the top of page 41. Ask someone how the factors 7 and 8 were expressed as sums (7 was expressed as 3 + 4 and 8 was expressed as 6 + 2). Ask individual children to explain how the partial products correspond to the appropriate smaller arrays (2 × 4, or 8, is the number of members in the lower right part; 2 × 3, or 6, is the number of members in the top right part; 6 × 4, or 24, is the number of members in the lower left part; 6 × 3, or 18, is the number of members in the top left part). The class should observe that 2 is multiplied by both 3 and 4 to form two partial products, and the 6 is multiplied by both 3 and 4 to form two more partial products. The sum of these partial products is 56.

Have the products for exercises 1 and 2 computed using partial products. Remind the children that the partitioning of the arrays illustrates the way the children are to express the factors of each product. Then have them complete the exercises on page 42.



Compute. Use parts as shown.

1. $6 = 4 + 2$
   $\times 7 = 5 + 2$
   42: 4, 8, 10, 20, 42

2. $8 = 6 + 2$
   $\times 8 = 3 + 5$
   10, 30, 6, 18, 64

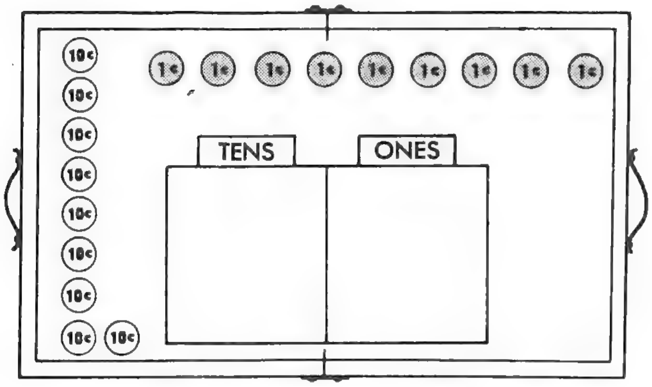
3. $9 = 3 + 6$
   $\times 6 = 3 + 3$
   18, 9, 18, 9, 54

4. $8 = 3 + 5$
   $\times 7 = 2 + 5$
   25, 15, 10, 6, 56

E-42

Supplemental Experience

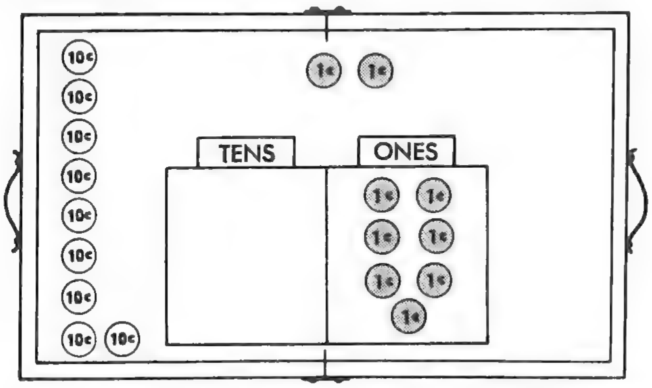
■ This activity is intended to review addition of ones and tens, which is used repeatedly in the computation of sums of partial products. Write 1¢ on 9 copper-colored disks and 10¢ on 9 silver-colored disks; place all of the simulated coins on a flannel board. Also place on the board a tagboard latticework frame that has two openings and two cards labeled TENS and ONES, as illustrated.



Explain to the children that they are going to compute sums. The rules state that the children must show the computation of each sum with coins, even though they have only 9 ones (9 pennies) and 9 tens (9 dimes) available.

Write on the chalkboard the equation $7 + 8 = 15$. Explain that the problem is to show how to compute $7 + 8$ by using only the 9 tens (9 dimes) and 9 ones (9 pennies) that are available. Tell the class that only ones (pennies) may be placed in the ONES opening and only tens (dimes) in the TENS opening of the frame to illustrate the computational steps.

They will probably begin illustrating the computation by placing 7 of the 9 pennies in the ONES opening of the frame.

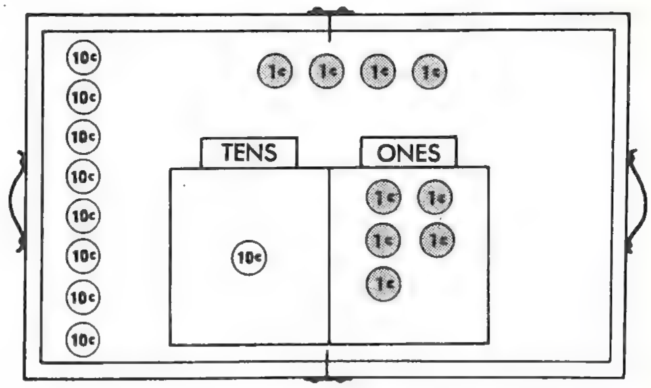


Then ask a child why you cannot put 8 ones with the 7 ones (there are not enough ones).

Put a dime in the TENS opening. A dime has the value of 10 pennies, so it is 2 cents too much. Take 2 pennies back in change.

This leaves 1 dime and 5 pennies, a value in cents of $10 + 5$, or 15.

$7 + 8$ is 15.



Demonstrate this method of computing $7 + 8$.

8 was added to 7 by adding 10. Since 10 was 2 more than needed, this 2 had to be subtracted.

$$7 + 8 = 7 + 10 - 2$$

This 2 was subtracted from the 7 ones.

$$\begin{aligned} 7 + 8 &= 7 + 10 - 2 \\ 7 + 8 &= 10 + 7 - 2 \end{aligned}$$

Since $7 - 2$ is 5, the final step in the computation results in $10 + 5$.

$$\begin{aligned} 7 + 8 &= 7 + 10 - 2 \\ 7 + 8 &= 10 + 7 - 2 \\ 7 + 8 &= 10 + 5 \end{aligned}$$

The standard name for $10 + 5$ is 15.

Do this with the sum $8 + 7$. Write $8 + 7 = 15$ on the chalkboard and ask a child to show that he will add 7 to 8 within the limitations of the game.

He will put 8 pennies in the ONES opening.

He will add 10 (1 dime) with the realization that this is 3 too much.

He will take his change from the 8 pennies he placed in the ONES opening.

He now has 1 dime and 5 pennies, a value in cents of $10 + 5$, or 15.

$8 + 7$ is 15.

Let one or two other children also show the computation of $8 + 7$. Then summarize the steps involved.

$$\begin{aligned} 8 + 7 &= 8 + 10 - 3 \\ 8 + 7 &= 10 + 8 - 3 \\ 8 + 7 &= 10 + 5 \\ 8 + 7 &= 15 \end{aligned}$$

Then do this with $6 + 8$ and $8 + 6$, $5 + 9$ and $9 + 5$, and $4 + 8$ and $8 + 4$.

KEY IDEA

The distributive property lets us multiply digit-by-digit.

Scope

To use basic multiplication combinations when computing products with factors of 10, 100, and 1000.
To develop a long multiplication algorism.

Fundamentals

The principles of numeration, basic multiplication facts, and the distributive property of multiplication with respect to addition lead to an efficient procedure for computing products. The partitioned arrays have shown that a product is a sum of partial products. The next step applies this to the decimal system.

The decimal numeration system is based on place value and additive principles. A result of this is a fundamental understanding of the numeration system: the product of a whole number and 10 can be computed by affixing a zero to the numeral.

NUMBER	NUMBER TIMES 10
1	$1 \times 10 = 10$
2	$2 \times 10 = 20$
5	$5 \times 10 = 50$
10	$10 \times 10 = 100$
15	$15 \times 10 = 150$
100	$100 \times 10 = 1000$
1492	$1492 \times 10 = 14920$

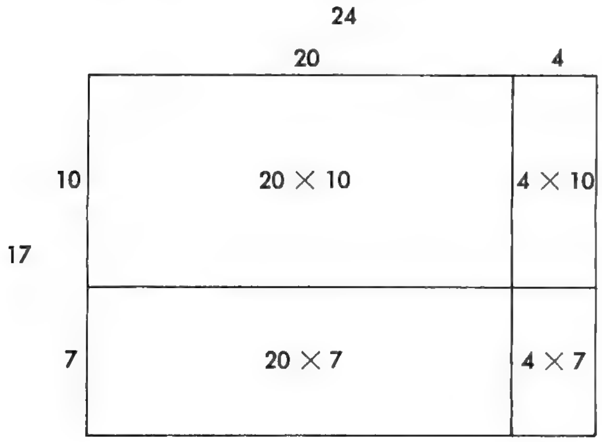
Products using other multiples of 10, such as 100 and 1000, are also computed by affixing zeros:

$$8 \times 100 = 800$$
$$7 \times 1000 = 7000$$

Other products that involve multiples of 10, 100, or 1000 can also be computed by affixing zeros:

$$6 \times 400 = 2400$$
$$60 \times 200 = 12,000$$
$$40 \times 80 = 3200$$

A procedure for computing a product such as 17×24 is shown by this illustration:



This partitioning illustrates the place value and additive principles. 24 is two tens added to four ones. That is, 24 is $20 + 4$; similarly, 17 is $10 + 7$. The partial products are thus 20×10, 4×10, 20×7, and 4×7, as shown.

The partial products are computed using only basic multiplication and addition facts and principles of numeration.

28 is 7 ones × 4 ones or (7 × 4) ones.
140 is 7 ones × 2 tens or (7 × 2) tens.
40 is 1 ten × 4 ones or (1 × 4) tens.
200 is 1 ten × 2 tens or (1 × 2) hundreds.

The computation of 17×24 may be written in vertical form as follows:

$$\begin{array}{rcl} 24 & = & 20 + 4 \\ \underline{\times 17} & = & \underline{10 + 7} \\ 28 & \longleftarrow & 4 \times 7 \\ 140 & \longleftarrow & 20 \times 7 \\ 40 & \longleftarrow & 4 \times 10 \\ \underline{200} & \longleftarrow & 20 \times 10 \\ 408 & & \end{array}$$

This algorism arranges the product as a sum of the partial products in an easily computed form.

Readiness for Understanding

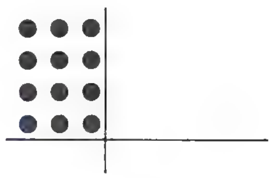
Knowledge of our numeration system.
Knowledge of basic multiplication and addition facts.
Understanding of the distributive property of multiplication with respect to addition.
Ability to compute sums.

Developmental Experiences

sheet of paper or tagboard (18″ × 24″)
felt-tip pen
masking tape
yarn

for each child
21 tagboard cards (1″ × $1\frac{1}{2}$″)

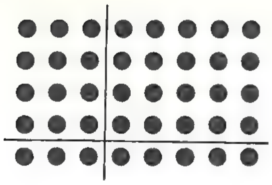
▶ Copy this diagram and product on the chalkboard.



5 × 8

Tell the class that this is a part of a 5 × 8 array. Call on someone to fill in one of the smaller arrays. He could draw a 4 by 5 array in the upper right part.

The child should be allowed to choose any smaller array that will lead to a 5 by 8 array. When he has done this, choose two other children to fill in the remaining two parts of the array:



5 × 8

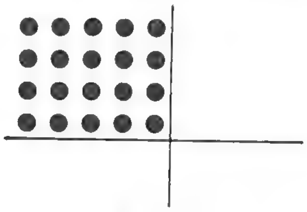
After all parts of the array have been drawn, write the product for each part of the array. Have the class compute the sum of the partial products, as shown.

$$\begin{aligned} 4 \times 5 &= 20 \\ 4 \times 3 &= 12 \\ 1 \times 5 &= 5 \\ 1 \times 3 &= \underline{3} \\ &\ 40 \end{aligned}$$

Ask the class to count the members of the array.

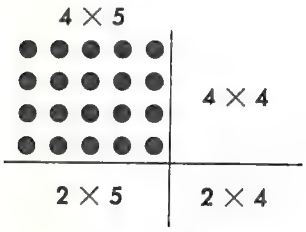
Repeat this activity with arrays that have 6, 7, 8, or 9 as factors.

Vary the activity. Draw part of a partitioned 6 by 9 array. Below the array write its product.

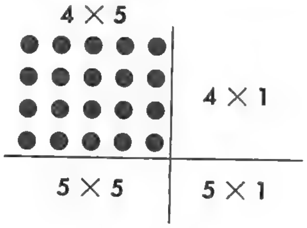


6 × 9

Tell the class that this time they will not draw the other parts of the array; they will only indicate the product that could be illustrated by each smaller array in the partition. Choose four children to write products for each part of the 6 by 9 array. The children may choose either one of the two possibilities illustrated.



6 × 9



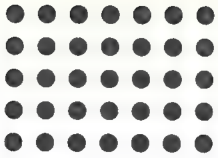
6 × 9

Then have someone write the product for each part in an equation and compute the sum of the partial products.

$$\begin{aligned} 4 \times 5 &= 20 \\ 4 \times 4 &= 16 \\ 2 \times 5 &= 10 \\ 2 \times 4 &= \underline{8} \\ &\ 54 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 4 \times 5 &= 20 \\ 4 \times 1 &= 4 \\ 5 \times 5 &= 25 \\ 5 \times 1 &= \underline{5} \\ &\ 54 \end{aligned}$$

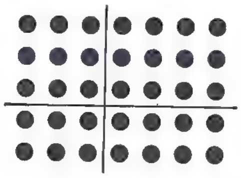
Use this procedure with other basic multiplication facts that have 6, 7, 8, or 9 as a factor.

► Ask a child to draw a 5 by 7 array on the chalkboard; then write this incomplete equation below his diagram.



$5 \times 7 = (2 + \_\_) \times (4 + \_\_)$

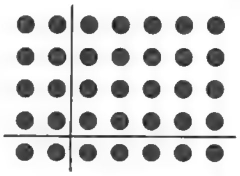
Tell another child to draw lines that partition the array according to the clues given. He might do it this way:



When he has partitioned the array, tell the child to select someone else to complete the equation on the chalkboard so that it describes the illustrated partition.

$$5 \times 7 = (2 + \underline{3}) \times (4 + \underline{3})$$

Ask whether anyone interpreted the clues differently and can show a different partition of the array. Allow any child who sees another partition to draw the array again and demonstrate his idea to the class. Some child may partition the array as illustrated.



Ask the child who demonstrated the second partition to write his interpretation of the original incomplete equation.

$$5 \times 7 = (2 + \underline{3}) \times (4 + \underline{3})$$
$$5 \times 7 = (2 + \underline{5}) \times (4 + \underline{1})$$

Select two children to come to the chalkboard and to compute 5×7 in parts. Direct one child to use $(2 + 3) \times (4 + 3)$ and to show the partial products. Tell the second child to use $(2 + 5) \times (4 + 1)$ and to show the partial products.

$$\begin{array}{rl} 5 = & 2 + 3 \\ \underline{\times 7} = & \underline{4 + 3} \\ 9 & (3 \times 3) \\ 6 & (3 \times 2) \\ 12 & (4 \times 3) \\ \underline{8} & (4 \times 2) \\ 35 & \end{array} \qquad \begin{array}{rl} 5 = & 4 + 1 \\ \underline{\times 7} = & \underline{2 + 5} \\ 5 & (5 \times 1) \\ 20 & (5 \times 4) \\ 2 & (2 \times 1) \\ \underline{8} & (2 \times 4) \\ 35 & \end{array}$$

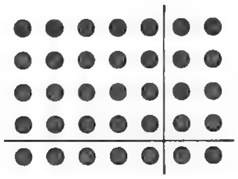
Have the class compare both of the computed sums with the count for the 5 by 7 array.

Leave the 5 by 7 array on the board, but erase the partition lines. Then write the following incomplete equation on the chalkboard.

$$5 \times 7 = (4 \times 5) + (1 \times 5) + (\_\_ \times \_\_) + (\_\_ \times \_\_)$$

Ask a child to draw lines that partition the array in a way that illustrates the partial products given in

the equation. The child might partition the array as illustrated.



Have the child complete the equation on the chalkboard so that it describes 5×7 using the partial products of the partition that he just drew.

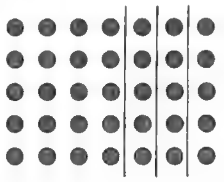
$5 \times 7 = (4 \times 5) + (1 \times 5) + (\underline{4} \times \underline{2}) + (\underline{1} \times \underline{2})$

Next ask the child to express the product 5×7 as a product of sums as illustrated by the partitioned array.

$5 \times 7 = (4 \times 5) + (1 \times 5) + (\underline{4} \times \underline{2}) + (\underline{1} \times \underline{2})$

$5 \times 7 = (4 + 1) \times (5 + 2)$

Ask the children to do this with a different partition. Someone might partition the array as shown in the following illustration.



Call on someone to write an equation that represents 5×7 as a sum of the partial products as indicated by this partition.

$5 \times 7 = (4 \times 5) + (1 \times 5) + (\underline{1} \times \underline{5}) + (\underline{1} \times \underline{5})$

Then choose another child to express the product 5×7 as a product of sums according to this partition.

$5 \times 7 = (4 \times 5) + (1 \times 5) + (\underline{1} \times \underline{5}) + (\underline{1} \times \underline{5})$

$5 \times 7 = 5 \times (4 + 1 + 1 + 1)$

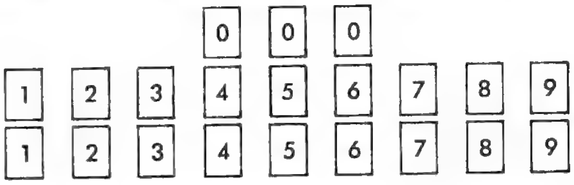
Ask for two volunteers to compute 5×7 in parts. One child should use the product of sums, $(4 + 1) \times (5 + 2)$. The second child should use the other product of sums, $5 \times (4 + 1 + 1 + 1)$.

$$\begin{array}{rl} 5 = & 4 + 1 \\ \underline{\times 7} = & \underline{5 + 2} \\ 2 & (1 \times 2) \\ 8 & (4 \times 2) \\ 5 & (1 \times 5) \\ \underline{20} & (4 \times 5) \\ 35 & \end{array}$$

$$\begin{array}{rl} 7 = & 4 + 1 + 1 + 1 \\ \underline{\times 5} = & \underline{\qquad\qquad 5} \\ 5 & (1 \times 5) \\ 5 & (1 \times 5) \\ 5 & (1 \times 5) \\ \underline{20} & (4 \times 5) \\ 35 & \end{array}$$

Use this procedure for several other products such as 7×8, 8×9, 9×6, 6×8, and 7×6.

► Give each child 21 tagboard cards (1 inch by $1\frac{1}{2}$ inches) and have him write these digits on the cards.



Write one of these products on the chalkboard:

1×1	10×1	100×10
1×10	10×10	10×100

The children are to use their numeral-cards to show on their desks the standard numeral for the product on the board. Have this standard numeral read aloud. One at a time, do this with each of the other five products. The standard numerals for 10×10, 100×10, and 10×100 may be read in various ways.

$10 \times 10 = 100$ (1 hundred, 10 tens)

$$\left.\begin{array}{l} 100 \times 10 \\ 10 \times 100 \end{array}\right\} = 1000 \left\{\begin{array}{l} \text{1 thousand} \\ \text{1 hundred tens} \\ \text{10 hundreds} \\ \text{10 ten tens} \end{array}\right.$$

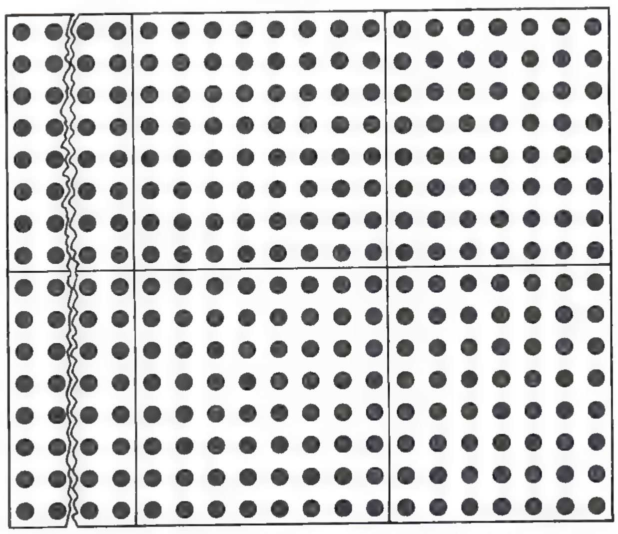
Next, write on the chalkboard, one at a time, products such as the following:

6×7	80×80
7×8	60×30
90×3	700×40
9×80	50×600

Have the children show and read the count for each product.

► Use the following array for this activity.

8×8 8×8 7×8



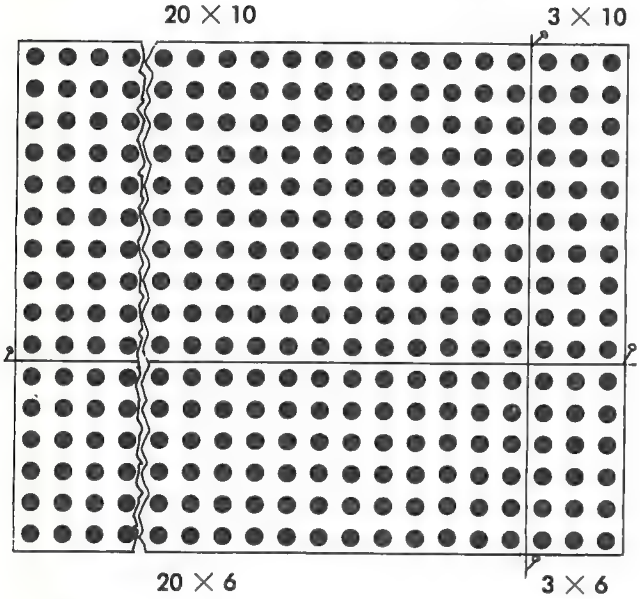
8×8 8×8 7×8

Draw a 16 by 23 array on a large sheet of paper; tape the paper to the chalkboard. Have three or four children take turns taping pieces of yarn to the

paper to show different partitions for this array. After each child demonstrates his partition, ask him to write the illustrated partial products on the board above or below the part of the array that each represents. For example, someone could partition the array like this; if so, he should write each partial product as shown.

Then have the child compute the partial products and the sum of the partial products.

If no child chooses to show the array partitioned into tens and ones, suggest this idea to the class. Explain that when the factors of a product are greater than 10 it is often more efficient and convenient to think of the factors as tens and ones. Have someone partition the 16 by 23 array to show each factor as a sum of tens and ones ($23 = 20 + 3$, and $16 = 10 + 6$). Let another child write the partial products for each part.



Then ask for a volunteer to compute the partial products as well as the sum of the partial products, showing the steps used.

$$\begin{array}{rl}
23 = 20 + 3 & \text{(20 + 3 is the expanded form for 23)} \\
\underline{\times 16} = \underline{10 + 6} & \text{(10 + 6 is the expanded form for 16)} \\
18 & \text{(3 ones × 6 ones is 18 ones)} \\
120 & \text{(2 tens × 6 ones is 12 tens)} \\
30 & \text{(3 ones × 1 ten is 3 tens)} \\
\underline{200} & \text{(2 tens × 1 ten is 2 ten tens or 2 hundreds)} \\
368 &
\end{array}$$

As the children examine the multiplication algorism on the chalkboard, ask them these questions:

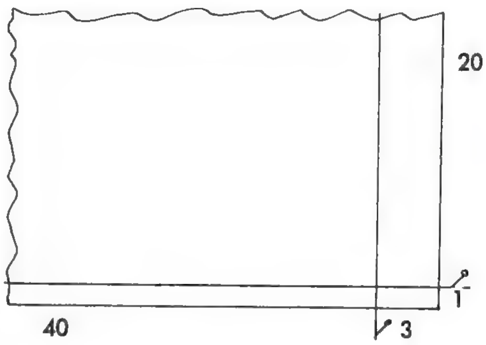
What digits do you multiply to get the ones in the result? (3 and 6)

What digits do you multiply to get the tens? (2 and 6, 3 and 1)

What digits do you multiply to get the ten tens or hundreds? (2 and 1)

As each question is answered, ask the child to indicate the corresponding partial product in the array. Adapt this to other products such as 43 × 21, 62 × 54, 58 × 19, and 97 × 38.

With these products it becomes cumbersome to show all the members of the array. Use a rectangle to show partial products for 43 × 21. Do not try to explain this diagram in terms of area; rather, the diagram should be thought of as an array that is not filled with its individual members.



Select someone to write the product vertically for this diagram.

Ask another child to write the product vertically, using expanded notation for each factor. Then he should compute each partial product. As he does, let someone else show the part of the illustration that each partial product represents.

$$\begin{array}{r}
43 = 40 + 3 \\
\underline{\times 21} = \underline{20 + 1} \\
3 \\
40 \\
60 \\
\underline{800} \\
903
\end{array}$$

► Write the product 79 × 62 on the chalkboard. Choose four children to compute the four partial products. Ask a fifth child to compute the sum of partial products.

$$\begin{array}{r}
79 \\
\underline{\times 62} \\
18 \\
140 \\
540 \\
\underline{4200} \\
4898
\end{array}$$

Select four other children to describe how each partial product is computed:

18 (18 ones) is the product 9 ones times 2 ones; 18 ones is the product of the ones in both factors.

140 (14 tens) is the product 7 tens times 2 ones; 14 tens is one product of tens and ones.

540 (54 tens) is the product 6 tens times 9 ones; 54 tens is another product of ones and tens.

4200 (42 ten tens) is the product 7 tens times 6 tens; 42 ten tens, or 42 hundreds, is the product of the tens in both factors.

In this way, have the children compute the partial products and products 26 × 41, 43 × 37, and 63 × 98. In each instance, have some of the children describe the way the partial products were computed.

Pages 43 through 48

● Use page 43 for class discussion. Copy the example 24 × 32 on the chalkboard. As you and the children discuss the example, let individuals come to the chalkboard to point to the part they are discussing. For example when a child says there are 6 ten tens, or 6 hundreds, in the array and that he multiplied 3 tens and 2 tens to get 6 ten tens, have him point to these numerals.

Have the children complete exercise 6 on their papers while a volunteer does it on the chalkboard. Ask the child who worked at the board to explain the steps involved in the computation. Encourage the other children to correct the explanation if they feel it is inadequate. Follow a similar procedure for exercise 7.

● Before proceeding with the study of the multiplication algorism, the children will review computing with multiples of 10, 100, and 1000. Discuss the examples at the top of page 44. The children should understand that the computation of 3 × 20 and 30 × 20 involves only the basic multiplication fact, 3 × 2 = 6. The basic multiplication facts combined with the principles of numeration—in this instance the fact that ones × tens = tens, and tens × tens = ten tens or hundreds—are sufficient to compute products for multiples of 10, 100, and 1000. Work exercises 1 and 2 with the class, and assign the remaining exercises as independent work. After the children have completed the page, ask individuals to describe how they computed specific exercises. For example, to compute 60 × 900, a child might explain, "6 × 9 = 54; tens × hundreds = thousands; the product is 54 thousand or 54,000."

● Page 45 uses the Math Machine to give additional practice in computing products for multiples of 10, 100, and 1000. Discuss the illustrations at the top of the page. Tell the children to study the first Math Machine; ask them to examine the list below the machine and to select the result that occurs when tens are put into both openings in the machine (ten tens or hundreds). Similarly, discuss the other two machines. Then select from the list one of the possible results for each machine. Ask what would have to be put into the Math Machine so that ten thousands would result (tens and thousands or hundreds and hundreds). Then, tell the children to complete each of the exercises.

● Use page 46 for class discussion. Copy the algorism shown at the top of the page and let volunteers complete the example on the chalkboard.

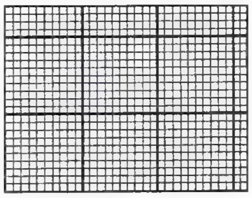
For the other exercises, ask a child to complete the expanded notation for the given factors. Then ask volunteers to explain and compute each partial product and the sum of the partial products. Try to give each child an opportunity to explain a computation.

Next, ask for volunteers to work some of these exercises on a sheet of tagboard. Post these examples in the classroom and tell the class to refer to them when necessary.

Name

For Class Discussion

How could you compute 24 × 32?

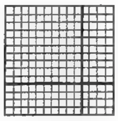


24 =	20 + 4	
× 32 =	30 + 2	
	8	(2 ones × 4 ones)
	40	(2 ones × 2 tens)
	120	(3 tens × 4 ones)
	600	(3 tens × 2 tens)
	768	(32 × 24)

1. How many ten tens in the array? 6
2. Which digits do you multiply to get ten tens? 2 and 3
3. Which digits do you multiply to get tens? 3 and 4
4. What other digits do you multiply to get tens? 2 and 2
5. Which digits do you multiply to get ones? 2 and 4

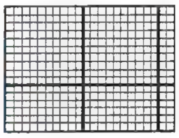
Compute.

6.



14 = 10 + 4
× 14 = 10 + 4
16
100
40
40
196

7.



22 = 20 + 2
× 16 = 10 + 6
12
120
20
200
352

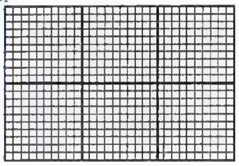
reference page

E-43

See pupil page suggestions.

3 ones × 2 tens = 6 tens

3 × 20 = 60



3 tens × 2 tens = 6 ten tens

30 × 20 = 600

Compute.

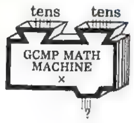
1. 2 × 40 = 80; 20 × 40 = 800; 200 × 40 = 8000
2. 600 × 7 = 4200; 60 × 70 = 4200; 6 × 700 = 4200
3. 7 × 8000 = 56,000; 70 × 800 = 56,000; 700 × 80 = 56,000
4. 400 × 4 = 1600; 400 × 40 = 16,000; 400 × 400 = 160,000
5. 8 × 200 = 1600; 80 × 20 = 1600; 800 × 2 = 1600
6. 500 × 400 = 200,000; 50 × 400 = 20,000; 5 × 400 = 2000
7. 15 × 10 = 150; 150 × 10 = 1500; 1500 × 10 = 15,000
8. 170 × 100 = 17,000; 170 × 10 = 1700; 170 × 1 = 170
9. 9000 × 6 = 54,000; 900 × 60 = 54,000; 90 × 600 = 54,000
10. 50 × 70 = 3500
11. 800 × 50 = 40,000
12. 60 × 900 = 54,000
13. 90 × 400 = 36,000
14. 800 × 7 = 5600
15. 700 × 90 = 63,000

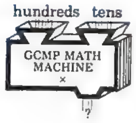
reference page

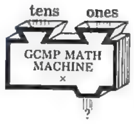
E-44

Name

What product? *See pupil page suggestions.*







tens ones thousands ten tens ten thousands ten ten tens

hundreds ten ten ten tens ten hundreds

Compute.

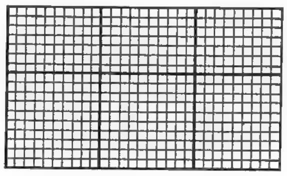
1. 6 tens × 3 tens = *18* ten tens
2. 3 ones × 5 hundreds = *15 hundreds*
3. 2 tens × 5 hundreds = *10 ten hundreds*
4. 5 tens × 7 tens = *35 ten tens*
5. 4 hundreds × 5 hundreds = *20 hundred hundreds*
6. 5 ones × 5 ones = *25 ones*
7. 7 tens × 6 tens = *42 ten tens*
8. 5 tens × 6 ones = *30 tens*
9. 8 ones × 8 tens = *65 tens*
10. 700 × 90 = *63,000*
11. 2 × 900 = *1800*
12. 7 × 500 = *3500*
13. 800 × 50 = *40,000*
14. 20 × 60 = *1200*
15. 80 × 700 = *56,000*
16. 50 × 90 = *4500*
17. 70 × 10 = *700*
18. 200 × 7 = *1400*
19. 40 × 900 = *36,000*
20. 600 × 5 = *3000*
21. 3 × 4 = *12*
22. 600 × 7 = *4200*
23. 90 × 50 = *4500*
24. 6 × 60 = *360*

E-45

For Class Discussion

How could you compute 17 × 29?

```
   29 = 20 + 9
 × 17 = 10 + 7
   63   (7 ones × 9 ones)
  140   (7 ones × 2 tens)
   90   (1 tens × 9 ones)
  200   (1 tens × 2 tens)
  493   (17 × 29)
```



Complete. Compute the product.

```
1.   35 = 30 + 5       2.   63 = 60 + 3       3.   46 = 40 + 6
   × 47 = 40 + 7          × 51 = 50 + 1          × 68 = 60 + 8
     35                        3                     48
    210                       60                    320
    200                      150                    360
   1200                     3000                   2400
   1645                     3213                   3128

4.   28 = 20 + 8       5.   76 = 70 + 6       6.   37 = 30 + 7
   × 65 = 60 + 5          × 92 = 90 + 2          × 86 = 80 + 6
     40                       12                     42
    100                      140                    180
    480                      540                    560
   1200                     6300                   2400
   1820                     6992                   3182
```

E-46

● Page 47 provides practice for the children in computing products of 2 two-digit factors. Discuss the example at the top of the page. The children should notice that the order in which they compute partial products will not affect the result. Work several of the exercises with the class. For each exercise, let individuals show the expanded form for the factors and give the explanation of each partial product.

Name

```
   23 = 20 + 3                        23 = 20 + 3
 × 17 = 10 + 7                      × 17 = 10 + 7
  200  (tens × tens)                  21  (ones × ones)
   30  (tens × ones)                 140  (ones × tens)
  140  (ones × tens)                  30  (tens × ones)
   21  (ones × ones)                 200  (tens × tens)
  391                                391
```

Compute.

```
1.   27 = ...... + ......    2.   58 = ...... + ......    3.   65 = ...... + ......
   × 43 = ...... + ......       × 31 = ...... + ......       × 26 = ...... + ......

4.   69 = ...... + ......    5.   97 = ...... + ......    6.   45 = ...... + ......
   × 96 = ...... + ......       × 81 = ...... + ......       × 37 = ...... + ......

7.   88 = ...... + ......    8.   71 = ...... + ......    9.   37 = ...... + ......
   × 69 = ...... + ......       × 35 = ...... + ......       × 79 = ...... + ......

10.  85 = ...... + ......    11.  42 = ...... + ......    12.  84 = ...... + ......
   × 47 = ...... + ......       × 69 = ...... + ......       × 58 = ...... + ......
```

E-47

```
20  7        50  8        60  5
40  3        30  1        20  6
   21            8           30
   60           50          360
  280          240          100
  800         1500         1200
 1161         1798         1690

60  9        90  7        40  5
90  6        80  1        30  7
   54            7           35
  360           90          280
  810          560          150
 5400         7200         1200
 6624         7857         1665

80  8        70  1        30  7
60  9        30  5        70  9
   72            5           63
  720          350          270
  480           30          490
 4800         2100         2100
 6072         2485         2923

80  5        40  2        80  4
40  7        60  9        50  8
   35           18           32
  560          360          640
  200          120          200
 3200         2400         4000
 3995         2898         4872
```

● The children have an opportunity to test themselves using the exercises on page 48. Work the example at the top of the page with the class. Tell the children to write the partial products in the position shown in the example. Before the children do the exercises, ask for volunteers to give the expanded forms that they will use. As the children work each exercise, they need not write the expanded forms on their papers. The example at the top of the page will be helpful to explain the way a partial product is computed. The children should check their multiplication of ones times ones, ones times tens, tens times ones, and tens times tens.

When the children understand the procedure, instruct them to complete the other exercises. Assign no more than two rows of exercises at one time. Use the results of this page to decide which children need additional work with the long multiplication algorisms.

```
  87 = 80 + 7
× 37 = 30 + 7
  49   (ones × ones)
 560   (ones × tens)
 210   (tens × ones)
2400   (tens × tens)
3219
```

Compute.

1.	2.	3.	4.
33 × 79	81 × 25	27 × 45	49 × 51
27	5	35	9
270	400	100	40
210	20	280	450
2100	1600	800	2000
2607	2025	1215	2499

5.	6.	7.	8.
84 × 48	25 × 95	37 × 27	23 × 61
32	25	49	3
640	100	210	20
160	450	140	180
3200	1800	600	1200
4032	2375	999	1403

9.	10.	11.	12.
64 × 27	25 × 18	54 × 36	36 × 85
28	40	24	30
420	160	300	150
80	50	120	480
1200	200	1500	2400
1728	450	1944	3060

Copy and compute.

13.	14.	15.	16.
69 × 43	69 × 86	47 × 39	76 × 47
2967	5934	1833	3572

reference page

E-48

Supplemental Experiences

■ Review multiplication by 10. Tell the class that computing the product of a whole number and 10 is just a matter of writing a 0 after the numeral. For example, 26 × 10 = 260. Similarly, the product of a whole number and 100 can be computed by writing two 0's after the numeral (52 × 100 = 5200), and the product of a whole number and 1000 can be computed by writing three 0's after the numeral (78 × 1000 = 78,000). Place in the pocket chart a card that contains the phrase 70 hundreds.

70 hundreds

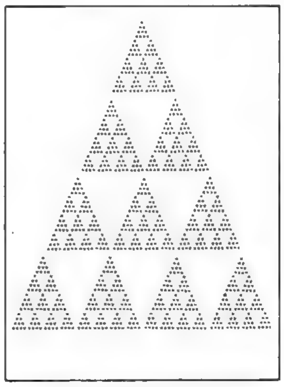
The children should see that another way to say 70 hundreds is "70 times 100." Ask them to help you show the standard numeral for numbers like 70 hundreds. Show a child an assortment of cards that contain the numerals 0, 00, or 000. Tell him to place a card over the word *hundreds* that will show the standard numeral for 70 hundreds.

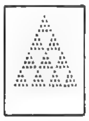
70	00

Ask a second child to read aloud the standard numeral for 70 hundreds (seven thousand).

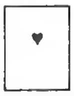
Continue in this way with 700 tens, 5 thousands, 800 hundreds, 90 tens, 30 thousands, and 400 thousands. In each instance, ask a child to select a card to show the standard numeral for the given number, and ask a second child to read the standard numeral aloud.

■ In the pocket chart, place 4 thousand-cards, 7 hundred-cards, 3 ten-cards, and 6 one-cards. Samples of these cards are shown here.


thousand-card


hundred-card


ten-card


one-card

Ask the children how many of each card there are. Ask for a volunteer to write the standard numeral for the number of this set on the chalkboard. Write an equal sign to the right of 4736 and ask another person to write the expanded form for the numeral 4736 to the right of the equal sign.

4736 = 4000 + 700 + 30 + 6

Write another equal sign below the first one and ask the children how many hundreds are in 4736. Someone will probably respond, "7." If so, point to the 4000 in 4000 + 700 + 30 + 6 and ask how many hundreds are in 4000. When a child replies "40," have him add the 40 hundreds and the 7 hundreds and write 4736 in terms of hundreds, tens, and ones.

$$\begin{aligned} 4736 &= 4000 + 700 + 30 + 6 \\ &= 4700 + 30 + 6 \end{aligned}$$

Follow a similar line of questioning to help the children show the meaning of 4736 in each of these ways.

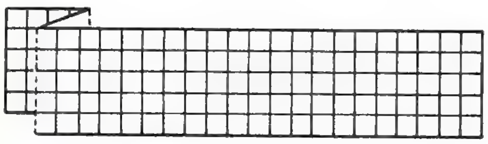
thousands, hundreds, and ones
4000 + 700 + 36
thousands, tens, and ones
4000 + 730 + 6
thousands and ones
4000 + 736
hundreds and ones
4700 + 36
tens and ones
4730 + 6
ones
4736

Next, have the class read aloud each of the statements written on the chalkboard.

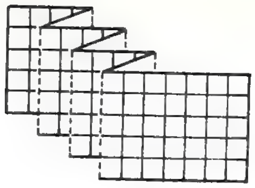
4000 + 700 + 30 + 6: "4 thousands plus 7 hundreds plus 3 tens plus 6 ones."
4700 + 30 + 6: "47 hundreds plus 3 tens plus 6 ones."
4000 + 700 + 36: "4 thousands plus 7 hundreds plus 36 ones."
4000 + 730 + 6: "4 thousands plus 73 tens plus 6 ones."
4000 + 736: "4 thousands plus 736 ones."
4700 + 36: "47 hundreds plus 36 ones."
4730 + 6: "473 tens plus 6 ones."
4736: "4736 ones."

Continue to use set cards that will help the children visualize other numbers such as 3029 and 5304.

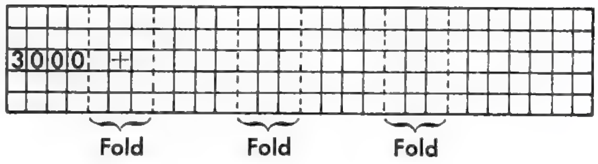
■ Give each child a strip of $\frac{1}{4}$-inch squared graph paper. Each strip should be 28 squares long and 5 squares wide. Direct the children to count four squares from the left end of the strip and to fold the strip forward along the line at the end of the fourth column of squares. Then tell them to count three more squares and to fold the strip back along the line at the end of these three columns of squares.



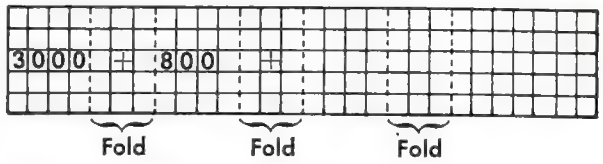
The children should use this same pattern of folding until they have made six accordion folds and the paper looks like this:



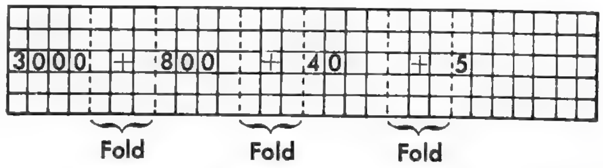
Next, write the numeral 3845 on the chalkboard and tell the children to write the expanded form for the numeral on their strips of paper. Explain that they should use a red crayon to write the numeral 3 in the first column of squares at the left end of the strip. Then they are to use a black crayon to write a zero in each of the three remaining columns in this first set of squares. They should also use their black crayon to write a plus sign in the middle of the neighboring set of three columns of squares.



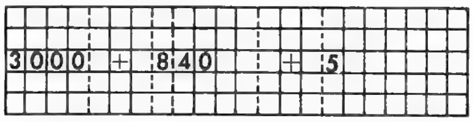
Now tell the children to use their red crayon and write the 8 of 800 in the first column of the next set of four columns of squares. Have them use their black crayon and write a zero in each of the next two columns in this set of squares and a plus sign in the middle of the second set of three columns of squares.



Follow a similar procedure to guide the children to complete the notation on their strips as shown.



The class will see that when the strip is stretched out flat the meaning of 3845 is shown as a sum of thousands, hundreds, tens, and ones. But the strip can be folded in other ways to show the meaning of 3845. For example, fold the strip in such a way that the meaning of 3845 is seen as a sum of thousands, tens, and ones:



Ask several of the children to show the class the meaning of 3845 in other ways. It is possible to show the meaning of 3845 in the following terms:

thousands, hundreds, and ones
3000 + 800 + 45
thousands and ones
3000 + 845
hundreds, tens, and ones
3800 + 40 + 5
hundreds and ones
3800 + 45
tens and ones
3840 + 5
ones
3845

Give the children additional strips of graph paper and have them show several meanings of other four-digit numerals.

KEY IDEA

Ten times tens is ten tens or hundreds.

Scope

To develop the idea that it is not necessary to record partial products of 0.

To provide practice in multi-digit multiplication.

To provide practice with story exercises.

Fundamentals

The number of partial products in the expanded algorism is always the product of the number of digits in each factor. For example, the computation of 234 × 567 involves 3 × 3, or 9, partial products.

$$\begin{array}{r} 234 = 200 + 30 + 4 \\ \underline{\times\ 567} = \underline{500 + 60 + 7} \end{array}$$

The partial products to be computed are as follows:

7 ones times the value of each digit in 234:
7 ones × 4 ones
7 ones × 3 tens
7 ones × 2 hundreds

6 tens times the value of each digit in 234:
6 tens × 4 ones
6 tens × 3 tens
6 tens × 2 hundreds

5 hundreds times the value of each digit in 234:
5 hundreds × 4 ones
5 hundreds × 3 tens
5 hundreds × 2 hundreds

The computed sum of these partial products is the computed product of 234 × 567.

Now, consider the product 500 × 200. The computation of 500 × 200 involves only one basic multiplication fact. The computation of 5 × 2 gives 10 and the product is 10 hundred hundreds.

$$\begin{array}{r} 500 \\ \underline{\times\ 200} \\ 100{,}000 \end{array}$$

To compute the result, the child does not need to know that another name for 10 hundred hundreds is one hundred thousand.

Since the number of partial products involved in a multi-digit computation is the product of the number of digits in each factor, it is possible to look at 500 × 200 as the sum of nine partial products.

$$\begin{array}{r} 500 = 500 + 00 + 0 \\ \underline{\times\ 200} = \underline{200 + 00 + 0} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \underline{100{,}000} \\ 100{,}000 \end{array}$$

While this is entirely correct, it is clearly possible to obtain the same result if the first 8 partial products are omitted. This procedure is also helpful in computing products such as 304 × 102.

$$\begin{array}{r} 304 = 300 + 4 \\ \underline{\times\ 102} = \underline{100 + 2} \end{array}$$

Note that the 0 tens is omitted in the expansion of each factor. For computational purposes this example is the same as multiplying 2 two-digit numbers.

$$\begin{array}{rl} 304 & \\ \underline{\times\ 102} & \\ 8 & \text{(2 ones × 4 ones)} \\ 600 & \text{(2 ones × 3 hundreds)} \\ 400 & \text{(1 hundred × 4 ones)} \\ \underline{30000} & \text{(1 hundred × 3 hundreds)} \\ 31{,}008 & \end{array}$$

It is not necessary to include the five partial products that have 0 as a factor.

Readiness for Understanding

Knowledge of numeration.

Knowledge of basic multiplication and addition facts.

Understanding of the properties of 0.

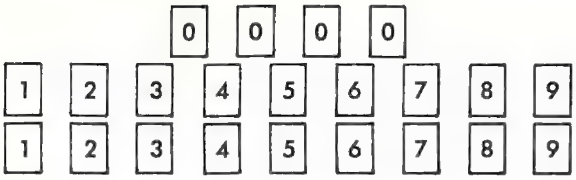
Understanding of the distributive property of multiplication.

Ability to compute sums.

Developmental Experiences

tagboard cards ($2\frac{1}{2}'' \times 2''$)	*for each child*
tagboard times sign	22 tagboard cards
2 boxes	($1'' \times 1\frac{1}{2}''$)

► The children should have a complete set of $1'' \times 1\frac{1}{2}''$ tagboard cards labeled as shown.



Choose one of the following products to write on the chalkboard.

4×8
4×80
4×800
40×80
40×800
400×800

Direct the children to use their tagboard numeral-cards on their desks to show the computed product. Have this read aloud. One at a time, do this with the other five products. After each example is written, the children should show the computed product and read it aloud. The computed products greater than 32 may be read in several ways; as the children read them, write the products on the chalkboard.

320 { 3 hundred 20; 32 tens

3200 { 3 thousand 2 hundred; 32 hundreds; 32 ten tens; 320 tens

32,000 { 32 thousands; 32 hundred tens; 32 ten ten tens; 320 hundreds; 320 ten tens; 3200 tens

320,000 { 320 thousands; 320 ten ten tens; 32 hundred hundreds

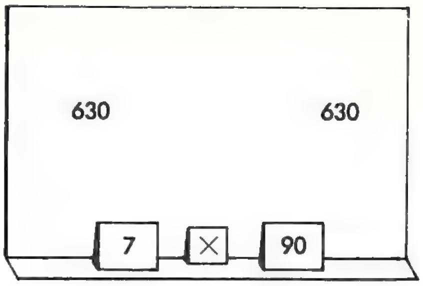
Do this with several other sets of products that involve basic multiplication combinations and multiples of 10.

► Use $2\frac{1}{2}'' \times 2''$ pieces of tagboard and make two sets of the number cards illustrated.

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900

Place each set of cards in a separate box. Place a times sign in the center of the chalktray.

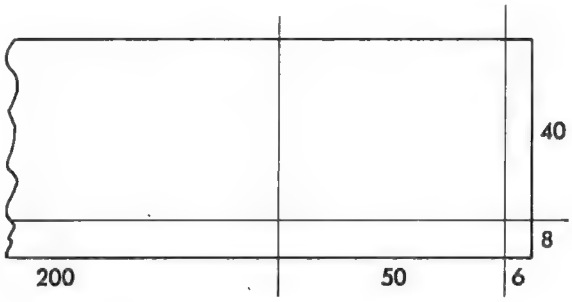
Direct the class to form two teams. Ask a member of each team to come forward; tell one child to choose a card from one box and the second child to choose a card from the other box. Instruct these children to place their cards on each side of the times sign to form a product; then each child may write the computed product on the board.



1 point is earned for computing the product correctly; 1 additional point is earned for finishing first. Do not return the cards to the box.

Draw two more cards and continue in this way until all members of both teams have had a turn. Then let each team total its points and declare a winner.

► Write on the chalkboard the product 256×48 in vertical form and ask a child to show the expanded notation for each factor. Then draw the following illustration to show the parts of each factor in the given product.



As the class computes each partial product and the sum of these products, write the results of the computations below the expanded form for the factors. Ask some child to describe the steps used to compute each partial product.

$$\begin{array}{rcr} 256 & = & 200 + 50 + 6 \\ \underline{\times\ \ 48} & = & \underline{40 + 8} \end{array}$$

48	(8 ones × 6 ones is 48 ones)
400	(8 ones × 5 tens is 40 tens)
1600	(8 ones × 2 hundreds is 16 hundreds)
240	(4 tens × 6 ones is 24 tens)
2000	(4 tens × 5 tens is 20 ten tens, 20 hundreds, or 2 thousands)
8000	(4 tens × 2 hundreds is 8 ten hundreds or 8 thousands)
12,288	

The children should refer to the partitioned diagram on the chalkboard for reference during the discussion. Ask the children the following questions, and have a

child point to the part of the illustration that represents his answer.

What digits do you multiply to get ones? (8 and 6)

What digits do you multiply to get tens? (8 and 5, 4 and 6)

What digits do you multiply to get ten tens or hundreds? (8 and 2, 4 and 5)

What digits do you multiply to get ten hundreds or thousands? (4 and 2)

Use this procedure in computing products such as 213 × 22, 643 × 11, 578 × 9, 431 × 2, and 85 × 4.

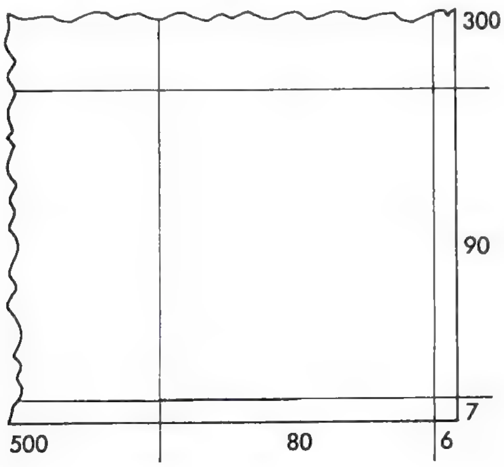
► Write 437 × 86 on the chalkboard and select six pupils to take turns computing the partial products for this product; each child should compute one partial product. Call on another child to compute the sum of the partial products.

$$\begin{array}{r} 437 \\ \underline{\times \quad 86} \\ 42 \\ 180 \\ 2400 \\ 560 \\ 2400 \\ \underline{32000} \\ 37{,}582 \end{array}$$

Choose six other children; direct them to take turns describing how each partial product was derived.

Then have the children take turns computing products such as 121 × 26, 734 × 14, 111 × 98, 438 × 9, and 56 × 7. In each instance, give several children a chance to describe the way the partial products were derived.

► Write the product 586 × 397 on the chalkboard and ask a child to show the expanded notation for each factor. Then draw the following illustration to show the parts of the factors in the given product.



As the class computes each partial product and then the sum of these products, write the result of the computations below the expanded form for the factors. Ask a child to describe the steps he used to compute each partial product.

$$\begin{array}{r} 586 = 500 + 80 + 6 \\ \underline{\times\ 397} = \underline{300 + 90 + 7} \end{array}$$

42	7 ones × 6 ones is 42 ones)
560	(7 ones × 8 tens is 56 tens)
3500	(7 ones × 5 hundreds is 35 hundreds)
540	(9 tens × 6 ones is 54 tens)
7200	(9 tens × 8 tens is 72 ten tens or 72 hundreds)
45000	(9 tens × 5 hundreds is 45 ten hundreds or 45 thousands)
1800	(3 hundreds × 6 ones is 18 hundreds)
24000	(3 hundreds × 8 tens is 24 ten hundreds or 24 thousands)
150000	(3 hundreds × 5 hundreds is 15 hundred hundreds or 150 thousands)
232,642	

As the class examines the multiplication algorism that is on the chalkboard, ask the following questions: They should point out the part of the diagram that represents their partial product.

What digits do you multiply to get ones? (7 and 6)

What digits do you multiply to get tens? (7 and 8, 9 and 6)

What digits do you multiply to get ten tens or hundreds? (7 and 5, 3 and 6, 9 and 8)

What digits do you multiply to get ten hundreds or hundred tens? (9 and 5, 3 and 8)

What digits do you multiply to get hundred hundreds? (3 and 5)

Do this with other products such as 213 × 321, 433 × 241, 635 × 191, and 476 × 921.

► Write 576 × 493 on the chalkboard and choose nine children to take turns computing the partial products for this product; each child should compute just one partial product. Ask a tenth child to compute the sum of the partial products.

$$\begin{array}{r} 576 \\ \underline{\times \quad 493} \\ 18 \\ 210 \\ 1500 \\ 540 \\ 6300 \\ 45000 \\ 2400 \\ 28000 \\ \underline{200000} \\ 283{,}968 \end{array}$$

Ask nine other children to take turns describing how each partial product was derived. Their comments may

be similar to the following.

18 is the product 3 ones times 6 ones; 18 is the product of the ones in both factors.

210 (21 tens) is the product 3 ones times 7 tens; 540 (54 tens) is the product 9 tens times 6 ones; 210 and 540 are products of tens and ones.

1500 (15 hundreds) is the product 3 ones times 5 hundreds; 6300 (63 ten tens or 63 hundreds) is the product 9 tens times 7 tens; 2400 (24 hundreds) is the product 4 hundreds times 6 ones; 1500, 6300, and 2400 are products of ones and hundreds or tens and tens.

45,000 (45 hundred tens or 45 thousands) is the product 9 tens times 5 hundreds; 28,000 (28 ten hundreds or 28 thousands) is the product 4 hundreds times 7 tens; 45,000 and 28,000 are products of hundreds and tens.

200,000 (20 hundred hundreds or 200 thousands) is the product 4 hundreds times 5 hundreds; 200,000 is the product of hundreds and hundreds.

Proceed in this way to have the children compute such products as 427 × 178, 312 × 295, and 897 × 798. In each instance, give other children a chance to describe the partial products.

► Separate the class into two teams and assign a panel of the chalkboard to each team. Write 720 on one panel of the chalkboard and 4800 on the other. One player from each team should go to the board and express the number written for his side as a product. Explain that they are to use a basic multiplication fact in their product. With 720 a child might write 8 × 90, 9 × 80, or the commuted form of either product. With 4800 a child might write 8 × 600, 6 × 800, 80 × 60, or the commuted form of any one of these products.

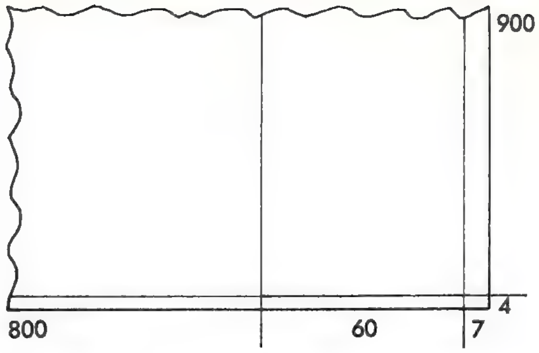
The children earn points for their team in the following ways: 1 point is earned for writing the product correctly; 1 point is earned for being first to finish correctly.

Continue in this way—using basic multiplication facts and multiples of 10—until all team members have participated. Then let each team total its points and declare a winner.

► On the chalkboard write the number 453; then have a child write 453 in expanded notation (400 + 50 + 3). Point out to the class that the value of each digit can be observed in this notation: hundreds + tens + ones.

Ask someone to write two expanded notations for 206. This child may write 200 + 00 + 6, or he may write 200 + 6. Remind the children that the notation 200 + 00 + 6 gives the value of each digit: hundreds + tens + ones.

Have the class compute a product that has 0 as a middle digit in one of its factors. Write on the chalkboard the product 867 × 904 in vertical form. Ask a child to write each factor in expanded notation. Draw the following illustration to show the partial products for the given products.



As the class computes each partial product and then the sum of these products, write the results of the computations below the expanded form for the factors. Ask for a volunteer to describe the steps used in computing each partial product.

867 =	800 + 60 + 7	
× 904 =	900 + 4	
	28	(4 ones × 7 ones is 28 ones)
	240	(4 ones × 6 tens is 24 tens)
	3200	(4 ones × 8 hundreds is 32 hundreds)
	6300	(9 hundreds × 7 ones is 63 hundreds)
	54000	(9 hundreds × 6 tens is 54 ten hundreds or 54 thousands)
	720000	(9 hundreds × 8 hundreds is 72 hundred hundreds or 720 thousands)
	783,768	

Ask the children to indicate what part of the diagram on the board illustrates their answers to the following questions:

What digits do you multiply to get ones? (4 and 7)

What digits do you multiply to get tens? (4 and 6) Why not record the product of 0 and 7? (The result, 0 tens, does not affect the sum of partial products and is not recorded. It does not appear in the diagram.)

What digits do you multiply to get ten tens or hundreds? (4 and 8, 9 and 7) Why not record the product of 0 and 6? (The result is 0 ten tens)

What digits do you multiply to get ten hundreds or thousands? (9 and 6) Why not record the product of 0 and 8? (The result is 0 ten hundreds)

What digits do you multiply to get hundred hundreds? (9 and 8)

Use this procedure in computing products such as 526 × 400, 489 × 30, 607 × 98, 240 × 63, 370 × 604, 208 × 506, and 700 × 903.

► Read some story exercises to the class. Ask the children to tell whether the relationship among the numbers in the story is additive or multiplicative.

Each of the 28 members of my class made 12 favors for the Red Cross. How many favors did

these pupils make?

My club sold 498 boxes of cookies in 6 days. The same number of boxes were sold each day. How many boxes were sold on one day?

The letter I mailed was going 238 miles. The letter my mother mailed was going 519 miles. How much farther than my letter did my mother's letter go?

There are 446 children in my school. 289 of the children ride the bus. How many children in my school do not ride the bus?

I counted the books on the shelves in the front of the room. There were 29 spelling books, 32 readers, 30 science books, and 31 arithmetic books. How many books are on these shelves altogether?

Ask the children to create several story exercises from their own experiences. Tell them that their exercises may use either multiplication or addition. Read the stories as they are given.

Let the creator of each story exercise choose someone to come to the chalkboard and, using a placeholder, to write an equation that expresses the relationship between the numbers in the story. Then let the writer of the equation select a child to solve his equation. Ask the class to check the accuracy of the completed work.

Pages 49 through 54

● Page 49 provides practice in computing products that involve two through six partial products. Work the example at the top of the page. As you discuss the partial products that result when a two-digit factor and a three-digit factor are multiplied, help the children see that there is only one way to get ones (ones × ones), two ways to get tens (ones × tens and tens × ones), two ways to get ten tens or hundreds (ones × hundreds and tens × tens), and one way to get ten hundreds or thousands (tens × hundreds).

Call on several children to give the expanded form for the numerals in each exercise. See if the children can tell how many partial products will be needed in each exercise.

Number of Partial Products

1. 4	2. 6	3. 4	4. 3
5. 4	6. 3	7. 6	8. 6
9. 2	10. 3	11. 6	12. 3
13. 6	14. 4	15. 4	16. 4

Explain that it is important to keep the columns in line when listing the partial products; there is less chance that addition mistakes will be made. Then assign the exercises, but do not assign more than five exercises at any one time.

● Page 50 provides practice in computing products that involve three through nine partial products. Work the example at the top of the page. The children should observe that when multiplying two three-digit factors there is only one way to get ones (ones × ones), two ways to get tens (ones × tens and tens × ones), three ways to get ten tens or hundreds (ones × hundreds, tens × tens, and hundreds × ones), two ways to get ten hundreds or thousands (tens × hundreds and hundreds × tens), and one way to get hundred hundreds or ten thousands (hundreds × hundreds).

Work exercise 1 with the class. Then have the children complete exercise 2 independently. After they have had sufficient time to compute exercise 2, ask someone to work the exercise at the chalkboard and to describe his computations. This may help other children check their work. Next, have the children give the expanded form for the factors in each of the other exercises. Ask them to complete the remaining exercises. Do not assign more than three exercises at any one time. The children who have difficulty keeping their columns in line may find it helpful to turn their paper sideways and use the lines on the paper as guidelines.

			3	3	7
		×	5	8	2
				1	4
				6	0
			6	0	0
			5	6	0
		2	4	0	0
	2	4	0	0	0
		3	5	0	0
	1	5	0	0	0
1	5	0	0	0	0
1	9	6,	1	3	4

Name

See pupil page suggestions.

```
   346
 × 52
    12  (ones × ones)
    80  (ones × tens)
   600  (ones × hundreds)
   300  (tens × ones)
  2000  (tens × tens)
 15000  (tens × hundreds)
 17992
```

Compute.

```
1.   44      2.  257      3.   25      4.  639
   × 23        × 63         × 25         ×  7
     12          21           25           63
    120         150          100          210
     80         600          100         4200
    800         420          400         4473
   1012        3000          625
              12000
             16,191

5.   37      6.  281      7.  498      8.  139
   × 61        ×  8         × 17         × 52
      7           8           56           18
     30         640          630           60
    420        1600         2800          200
   1800        2248           80          450
   2257                      900         1500
                            4000         5000
                            8466         7228
```

Copy and compute.

```
9.   45     10.  117     11.  621     12.  247
   ×  8        ×  5         × 85         ×  6
    360         585       52,785         1482

13.  673    14.   89     15.   29     16.   78
   × 26        × 69         × 93         × 24
  17,498       6141         2697         1872
```

reference page


```
   714
 × 984
    16  (ones × ones)
    40  (ones × tens)
  2800  (ones × hundreds)
   320  (tens × ones)
   800  (tens × tens)
 56000  (tens × hundreds)
  3600  (hundreds × ones)
  9000  (hundreds × tens)
630000  (hundreds × hundreds)
702,576
```

Compute.

```
1.   297     2.   852
   × 173        × 958
      21           16
     270          400
     600         6400
     490          100
    6300         2500
   14000        40000
     700         1800
    9000        45000
   20000       720000
  51,381      816,216

3.   586     4.  468     5.   527     6.  954
   × 746       × 52        × 897        ×  8
      36          16           49          32
     480         120          140         400
    3000         800         3500        7200
     240         400          630        7632
    3200        3000         1800
   20000       20000        45000
    4200       24,336        5600
   56000                    16000
  350000                   400000
  437,156                  472,719
```

Copy and compute.

```
7.   986     8.  468     9.   345    10.   865
   × 293       × 86        × 494        × 719
  288,898     40,248      170,430      621,935
```

reference page

● Use page 51 to give the children practice and review in computing products when one or both factors contain a zero digit. Work the example at the top of the page with the class. Review the fact that partial products of zero need not be shown since they will not affect the final result (zero times any number is zero). Ask how many partial products would be shown if zero products were written (0). Ask the children to describe the partial products that are zero in the example on page 51: ones × ones, ones × tens, ones × hundreds in one instance; tens × tens in one instance; and hundreds × tens in one instance. Next have the children tell how many partial products will be written to compute each product on page 51.

Number of Partial Products

1. 2	2. 4	3. 2	4. 6
5. 4	6. 4	7. 4	8. 6
9. 6	10. 2	11. 6	12. 4
13. 4	14. 4	15. 4	16. 6

Assign no more than two rows of the exercises at a time for independent work. After the children have completed a given assignment, let them tell which exercises have partial products that are 0.

Name

```
   209
 × 370
   630  (tens × ones)
 14000  (tens × hundreds)
  2700  (hundreds × ones)
 60000  (hundreds × hundreds)
77,330
```

Compute.

```
1.  109      2.   24     3.   31     4.  234
   ×  7        × 97        × 40        × 23
     63          28          40          12
    700         140        1200          90
    763         360        1240         600
               1800                      80
               2328                     600
                                       4000
                                       5382

5.  407      6.   73     7.   93     8.  264
   × 67        × 28        × 69        × 16
     49          24          27          24
   2800         560         810         360
    420          60         180        1200
  24000        1400        5400          40
  27,269       2044        6417         600
                                       2000
                                       4224
```

Copy and compute.

```
9.   968    10.  300    11.  356    12.  406
   × 408       × 65        × 78        × 503
  394,944     19,500      27,768      204,218

13.  204    14.  730    15.  809    16.  298
   × 97        × 28        × 890       × 47
   19,788     20,440      720,010      14,006
```

reference page

● Page 52 provides the children with additional practice in computing products. Assign the exercises, but do not ask the children to do more than eight exercises at any one time. Before the children begin an assignment, ask them how many partial products will be involved in computing each product.

Number of Partial Products

1. 4	2. 2	3. 4	4. 4
5. 6	6. 3	7. 2	8. 9
9. 6	10. 6	11. 1	12. 4
13. 4	14. 3	15. 9	16. 6
17. 9	18. 4	19. 9	20. 6

Compute. *See pupil page suggestions.*

1. 41×25 = 1025	2. 29×8 = 232	3. 85×16 = 1360	4. 709×13 = 9217
5. 111×97 = 10,767	6. 123×4 = 492	7. 650×30 = 19,500	8. 614×239 = 146,746

Copy and compute.

9. 406×573 = 232,638	10. 251×87 = 21,837	11. 90×60 = 5400	12. 809×98 = 79,282
13. 13×69 = 897	14. 654×300 = 196,200	15. 555×897 = 497,835	16. 270×574 = 154,980
17. 481×841 = 404521	18. 307×908 = 278,756	19. 987×987 = 974,169	20. 367×62 = 22,754

E-52

● Pages 53 and 54 provide practice for the children in writing equations to describe story exercises. No attempt is made to solve the equations at this time. Read all the stories on page 53 with the class. Ask the children to tell what numbers are given and what the question is in each story. Ask them to use $\square$ to represent a number which is not specifically named. Then have them write equations for the first three exercises. When everyone has finished, call on someone to tell what equation he selected for each exercise. Remember, the equation $24 \times 65 = \square$ (or any of its commuted and symmetric forms) and the equation $\square \div 65 = 24$ (or any of its commuted and symmetric forms) would be appropriate for exercise 2.

$24 \times 65 = \square$	$\square \div 65 = 24$
$65 \times 24 = \square$	$\square \div 24 = 65$
$\square = 24 \times 65$	$65 = \square \div 24$
$\square = 65 \times 24$	$24 = \square \div 65$

Assign the remaining exercises.

Tell the children to read the story exercises on page 54. Then ask them to look at the first story and think about these questions.

Can you tell the relationship among the numbers in the story?

What is the question for the story?

Instruct the children to complete the exercises on this page. Then ask individuals to give the equations they wrote for specific exercises.

Although the children are not required to solve the equations, you may want to have them write answers for two or three of the exercises on pages 53 and 54. The other exercises can be solved at a later time if the children need practice in reading and solving story exercises.

Name

Write an equation for each question. *Equations may vary.*

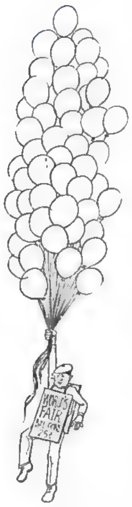
1. Michael and Marcia went to a World's Fair. They saw 19 exhibits. The admission fee for each exhibit was 25¢. How much money did each child need for admission fees? $\square \div 19 = 25¢$

2. They saw 24 model rockets. Each rocket carried 65 people. How many people did the rockets carry? $24 \times 65 = \square$

3. There were 45 seats in each row of the theater. Thirty-six rows were filled with people. How many people were in the theater? $\square \div 45 = 36$

4. In the Oriental exhibit there were 45 pictures of children. Each picture showed 36 children. How many children were in the pictures? $\square = 45 \times 36$

5. A man with 108 big gas-filled balloons floated a few inches above the ground. He said he always floated when he had more than 59 balloons. How many balloons did he have to sell to reach the ground? $59 + \square = 108$



6. The fair grounds had 6 entrances. One day 352 people went through each entrance. How many people went into the fair that day? $\square \div 352 = 6$

E-53

Write an equation for each story exercise.
Equations may vary.

1. Robert is a delivery boy. He rides his bike 12 miles each day. How many miles does he ride in 17 days? $12 \times 17 = \square$

2. Ruth read 25 pages of her book yesterday. Today she read 15 pages before school and 73 pages after school. How many pages did Ruth read in two days? $25 + 15 + 73 = \square$

3. Teddy's little brother sleeps 11 hours every night. How many hours does he sleep in a year? (365 days = 1 year) $\square \div 365 = 11$

4. To pass a physical fitness test each of Miss Marshall's 32 pupils earned 16 points. Mr. Brigg's class earned 301 points. How many points did the two classes earn? $\square = (32 \times 16) + 301$

5. Mary bought 16 sets of foreign stamps. Each set contained 144 stamps. How many stamps did Mary buy? $\square \div 16 = 144$

6. Woods Camp has 35 cabins. 28 people can stay in each cabin. How many people can stay at Woods Camp? $35 \times 28 = \square$

E-54

Supplemental Experiences

■ Establish two teams of seven members each; the remaining children may serve as judges. Have each team choose a captain. Give each captain one set of 3 by 5 inch cards labeled as illustrated.

ones × ones	tens × ones
ones × tens	tens × tens
ones × hundreds	tens × hundreds

Ask each captain to shuffle the cards face down and to distribute them to his teammates. Then write on the chalkboard an exercise such as 78 × 359 for each team.

Team I	Team II
359	359
× 78	× 78

Instruct the team captains to call on each of their teammates to record the partial product indicated on their cards. When all of the partial products have been recorded, the team captain should compute the sum. For example, the partial products might be recorded as follows:

Team I

359		
× 78		
630	(or 400)	(tens × ones)
2,400		(ones × hundreds)
3,500		(tens × tens)
400	(or 630)	(ones × tens)
72		(ones × ones)
21,000		(tens × hundreds)
28,002		

Team II

359	
× 78	
2,400	(ones × hundreds)
3,500	(tens × tens)
630	(tens × ones)
400	(ones × tens)
72	(ones × ones)
21,000	(tens × hundreds)
28,002	

Note that the order of partial products may vary.
The team that finishes first with the correct computation is declared the winner.

■ Have each child write a story exercise involving multiplication. Divide the class into two teams. Make sure each child has a chance to read his story for a member of the opposing team to answer and that each child also has a turn to answer. Keep score to determine the winning team.

■ To provide the children with practice in multiplying one-, two-, and three-digit factors by single-digit factors, draw on the chalkboard a table that involves chain exercises. Have the children copy the table on paper and complete it by writing the numeral for each required product in the chain.

	×	Product	×	Product	×	Product
8	9	72	4	288	5	1440
6	8		3		9	
7	4		6		3	
3	5		9		8	
2	7		5		6	
4	3		2		7	
5	6		7		4	

KEY IDEA

To find a missing factor, multiply and check.

Scope

To review multiplication by solving missing-factor equations.

Fundamentals

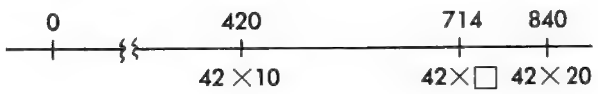
This problem-solving technique involves testing some possible solutions and closing in on the correct solution by using a repetitive process. Missing-factor equations may be solved by this testing procedure. Consider $42 \times \square = 714$. Be testing, the child can find the solution in a relatively short time. He may test 10 and 20.

$$42 \times 10 = 420$$
$$42 \times 20 = 840$$

He discovers that $\square$ is between 10 and 20.

$$42 \times 10 = 420$$
$$42 \times \square = 714$$
$$42 \times 20 = 840$$

Then he may observe that 714 is nearer to 840 than to 420.



Thus the possibilities have been narrowed to the whole numbers between 15 and 20. He may test these numbers by multiplying each one by 42. If one of the products is 714, the child has the solution.

Problem solving that uses recently reviewed knowledge and skills helps to develop a broader concept of number and number relationships. In a later unit, the division algorism will be developed as another way to solve the missing-factor equation.

Readiness for Understanding

Knowledge of numeration.
Knowledge of basic multiplication facts.

Developmental Experiences

for each child
set of mathematical symbols:
a tagboard placeholder ($\square$)
a times sign
an equal sign
26 tagboard cards ($2'' \times 2\frac{1}{2}''$)

► Give each child a tagboard placeholder ($\square$), a times sign, an equal sign, and a set of twenty-six 2 by $2\frac{1}{2}$ inch tagboard cards. Instruct the children to label their set of cards in the following manner:

0 0 0 0 0 0 0 0
1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9

Read each of these sentences to the pupils.

10 times some number equals 10 tens.
($10 \times \square = 100$)
10 times some number equals 100 tens.
($10 \times \square = 1000$)
10 tens times some number equals 100 tens.
($100 \times \square = 1000$)
Some number times 10 equals 10 tens.
($\square \times 10 = 100$)
Some number times 10 equals 100 tens.
($\square \times 10 = 1000$)
Some number times 10 tens equals 100 tens.
($\square \times 100 = 1000$)
10 tens equals 10 times some number.
($100 = 10 \times \square$)
100 tens equals 10 times some number.
($1000 = 10 \times \square$)
100 tens equals 10 tens times some number.
($1000 = 100 \times \square$)
10 tens equals some number times 10.
($100 = \square \times 10$)
100 tens equals some number times 10.
($1000 = \square \times 10$)
100 tens equals some number times 10 tens.
($1000 = \square \times 100$)

After each sentence is read, have the children use the cards on their desks to build a placeholder equation

that expresses the statement they just heard. Ask the pupils to analyze the facts given in the statement and then to complete the sentence by replacing the placeholder with cards that show the correct number. After the equation has been completed, ask individuals to tell how they computed the replacement number. For example, someone might say that he completed the sentence $100 \times \square = 1000$ in the following way:

> By first observing that tens are given in the factor as well as in the product (100 is 10 tens and 1000 is 100 tens). All that has to be supplied is the number that multiplied by 10 is 100. That number is 10; 10×10 tens is 100 tens.

Vary the activity by reading these sentences to the class.

6 tens times some number equals 54 tens.
($60 \times \square = 540$)
Some number times 7 tens equals 42 ten tens.
($\square \times 70 = 4200$)
32 hundreds equals 4 tens times some number.
($3200 = 40 \times \square$)
8 tens times some number equals 48 hundreds.
($80 \times \square = 4800$)
Some number times 6 hundreds equals 42 hundreds.
($\square \times 600 = 4200$)
64 ten tens equals 8 tens times some number.
($6400 = 80 \times \square$)

Again, have the children write an appropriate placeholder equation for each exercise. Tell them to analyze the facts given in the statement and then to complete the sentence. Ask various children to explain how they found the replacement number in each instance. For example, in completing the sentence $3200 = 40 \times \square$, someone may describe his thinking in the following manner:

> 4 times some number is 32; that number is 8. Tens times some number is hundreds; that number is tens. By combining these two facts we get the replacement number: 8 times tens or 8 tens. 4 tens times 8 tens equals 32 ten tens or 32 hundreds; $3200 = 40 \times 80$.

▶ Write $53 \times \square = 901$ on the chalkboard. Have the children help compute to find the missing factor. Ask the class if 10 could be the missing factor. Someone may point out that it could not be, because 53×10 is 530. Ask the class if 20 could be the missing factor; someone may point out that the 5 tens of 53 times 2 tens is 10 hundreds—10 hundreds is greater than the 9 hundreds of 901. Encourage the children to discuss any conclusions they have drawn concerning the missing factor. Someone might point out that it is a number between 10 and 20.

Ask a child to suggest a number that he thinks might be the missing factor. Suppose he suggests 15. Tell him to come to the chalkboard and compute 53×15.

$$\begin{array}{r} 53 \\ \times\ 15 \\ \hline 15 \\ 250 \\ 30 \\ 500 \\ \hline 795 \end{array}$$

Then ask him what conclusion he has drawn concerning the missing factor. He should point out that it is a number between 15 and 20.

Ask another child what number might be the missing factor. Suppose he suggests 17. Have him come to the chalkboard and compute 53×17.

$$\begin{array}{r} 53 \\ \times\ 17 \\ \hline 21 \\ 350 \\ 30 \\ 500 \\ \hline 901 \end{array}$$

Ask this child to explain why he thought the missing factor was 17.

Direct the class to find the missing factor in equations such as these:

$15 \times \square = 405$	$72 \times \square = 1656$
$28 \times \square = 1512$	$164 \times \square = 4756$
	$406 \times \square = 4872$

Pages 55 and 56

● Use page 55 to give the children an opportunity to test themselves and to judge their ability to find a missing factor. Discuss the examples at the top of the page with the class. Then, at the chalkboard, work exercises 1 and 2 with the class. When you are sure that the children understand the procedure, assign the other exercises. You may need to allow more than one class period for this page. Ask individuals to tell how they attempted to find missing factors for specific exercises. The pupils should not be expected to become efficient at finding missing factors at this time; pages 55 and 56 should be considered developmental and exploratory.

Name

For Class Discussion

Mr. Mitchell, the fire chief, has 735 tickets for the fireman's ball. There are 35 firemen. Mr. Mitchell will give each man the same number of tickets to sell. How many tickets will he give each fireman?

$735 = \square \times 35$

To find the missing factor, try some numbers.

First try 10.	$10 \times 35 =$ 350	What is the product?
Then try 20.	$20 \times 35 =$ 700	What is the product?
Then try 30.	$30 \times 35 =$ 1050	What is the product?

Which factors give a product less than 735? 10 and 20
Which factors give a product greater than 735? factors over 20
What is the missing factor? 21

Try to find the missing factor in this exercise: $640 = \square \times 20$.

Try 10.	$10 \times 20 =$ 200	What is the product?
Try 20.	$20 \times 20 =$ 400	What is the product?
Try 30.	$30 \times 20 =$ 600	What is the product?
Try 40.	$40 \times 20 =$ 800	What is the product?

Should we try 50? no What number should we try next? a number between 30 and 40
What is the missing factor? 32

Find the missing factor.

1.	$43 \times \square = 774$	$\square =$ 18	2.	$1284 = 214 \times \square$	$\square =$ 6
3.	$32 \times \square = 1600$	$\square =$ 50	4.	$425 = \square \times 85$	$\square =$ 5
5.	$\square \times 18 = 576$	$\square =$ 32	6.	$26 \times \square = 1274$	$\square =$ 49
7.	$9270 = 309 \times \square$	$\square =$ 30	8.	$\square \times 19 = 399$	$\square =$ 21

reference page

E-55

● Page 56 also provides practice in finding the missing factor. Work the example with the class; then instruct the children to complete the page. Some pupils will require more time than others to complete the exercises. It might be best to assign only three exercises to the children who work more slowly.

What is the missing factor?

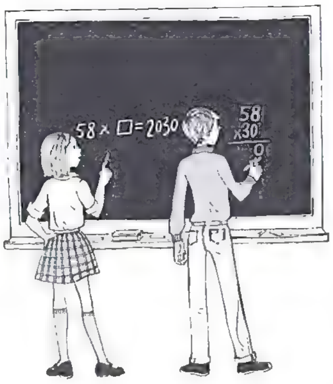
58 × □ = 2030

```
  58        58
× 30      × 40
 240       320
1500      2000
1740      2320
```

□ is between 30 and 40.

```
  58        58
× 34      × 35
  32        40
 200       250
 240       240
1500      1500
1972      2030
```

□ is 35.



Find the missing factor.

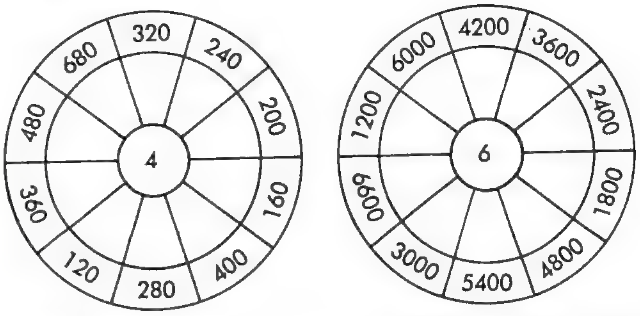
1. 82 × 23 = 1886
2. 39 × 44 = 1716
3. 79 × 59 = 4661

4. 64 × 38 = 2432
5. 43 × 37 = 1591
6. 56 × 106 = 5936
7. 58 × 13 = 754
8. 30 × 28 = 840
9. 45 × 20 = 900
10. 40 × 40 = 1600
11. 31 × 31 = 961
12. 22 × 22 = 484

E-56

Supplemental Experiences

■ To help the children maintain their skill with multiples of 10, 100, and 1000, draw number wheels on the chalkboard.



Ask individuals to name the missing factors for the class, or have the whole class copy and complete the wheels on paper.

■ **Units 1, 2, and 3 might be termed concept units; they** have exposed the child to ideas rather than to skills for mastery. You can best judge the children's understanding of concepts by observing which pupils readily apply the concepts when confronted with situations requiring such application. Another indication of whether the children understand a concept is the way they answer questions, or the way they illustrate the idea on the chalkboard.

In Unit 4 concepts are also important, but the emphasis is on a skill; the skill is the ability to use the multiplication algorism which is essential to division and later computational skills. At this time you may wish to have each pupil take a short written quiz to help you locate those who need additional practice. Following is a suggested quiz that can be used for this purpose.

SUGGESTED QUIZ

Compute.

1. 58×3 = 174
2. 394×56 = 22,064
3. 763×6 = 4578
4. 508×27 = 13,716
5. 409×807 = 330,063
6. 673×528 = 355,344

Find the missing factor.

7. $1012 = \square \times 23$

$\square = 44$

The steps used in computation may vary.

UNIT 5
THE MISSING-FACTOR PROBLEM
Pages 57 Through 64

OBJECTIVE

To explore the solution of the missing-factor equation $ax = b$, in the Set of Whole Numbers.

The pupil reviews multiplication equations and related division equations. He represents the mathematical structure of a story problem. He learns that sometimes a missing-factor equation, $ax = b$, does not have a whole-number solution.

See Key Topics in Mathematics for the Intermediate Teacher: Multiplication and Division of Whole Numbers.

KEY IDEAS

$3 \times (6 \div 3) = 6.$
If $3a = 6$, then $a = 6 \div 3$.

CONCEPT

missing factor

KEY IDEA

$3 \times (6 \div 3) = 6.$

Scope

To explore solutions to some missing-factor equations.

To present some missing-factor equations that have no whole-number solutions.

To review basic multiplication facts.

Fundamentals

The product of any two whole numbers is a whole number. This is the *closure property* of multiplication. Since 4 and 3 are whole numbers, the product 4×3 is a whole number.

Can the equation $ax = b$ always be solved for x, if x must be a whole number? For example, can 11 pencils be distributed equally into 5 boxes? The answer is obviously no. But the story suggests the equation $5x = 11$. By testing the factors 2 and 3 in the equation, we see that no whole number will satisfy the equation.

$$5 \times 2 = 10$$
$$5x = 11$$
$$5 \times 3 = 15$$

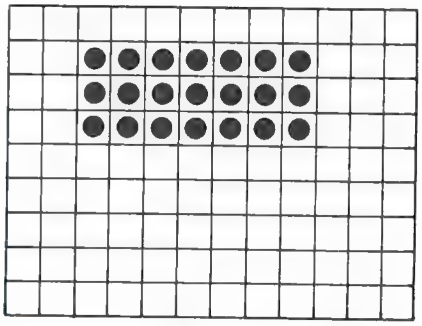
Readiness for Understanding

Understanding of product.
Knowledge of basic multiplication facts.

Developmental Experiences

tagboard cards (3″ × 9″)	*for each child*
felt-tip pen	sheet of 1-inch squared paper (9″ × 12″)
paper punch	counters
notebook rings	
tagboard strips (1″ × 3″)	
pocket chart	

► Give each child a 9 by 12 inch sheet of 1-inch squared paper and 85 counters. Ask the children to show an array for 3×7.

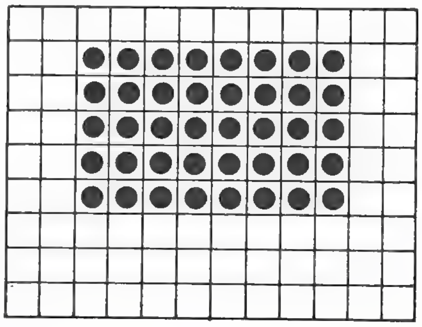


Ask someone to give the product illustrated by this array (3×7 or 7×3). Ask someone to count the members of this array (21). Have someone else come to the chalkboard and write the equation ($3 \times 7 = 21$). Continue in this way to have the class illustrate several basic multiplication facts.

Then tell the children to arrange an array that has 40 members so that the array has 5 rows. Ask someone to come to the chalkboard and to write a multiplication placeholder equation that represents this array.

$\square \times 5 = 40$ or $5 \times \square = 40$, $40 = \square \times 5$, $40 = 5 \times \square$

Tell the children that this sentence can be completed if they build an array.

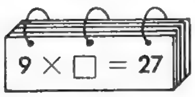


Use this approach with other basic multiplication facts.

Finally, use placeholder equation cards with the class. On 3 by 9 inch tagboard cards, write equations such as the following:

$4 \times 9 = \square$	$6 \times \square = 42$	$72 = \square \times 9$
$\square = 7 \times 5$	$\square \times 8 = 56$	$64 = 8 \times \square$

Punch three holes along the top edge of each card and assemble all of the cards in one pack. Insert three notebook rings into the holes; the cards should swing freely back and forth on these rings.



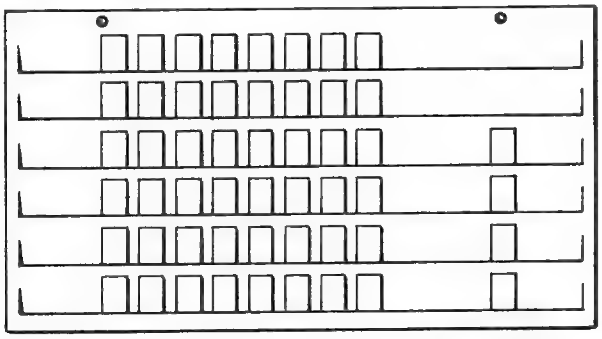
Show the first child in the first row one of the equations, for example, $\square \times 7 = 28$. Have him read the equation as a sentence. He may read the statement in any one of the following ways:

"Placeholder times 7 equals 28."
"Box times 7 equals 28."
"A certain number times 7 equals 28."
"Some number times 7 equals 28."
"The missing factor times 7 equals 28."

Direct him to read this equation a second time, replacing the placeholder (4 times 7 equals 28).

Then show the next card to a second child. Instruct this child to read the placeholder equation and then to reread the equation, replacing the placeholder. Continue in this way until every child in the room has completed an equation. If time permits, continue the activity for two or three more rounds, giving each child an opportunity to complete a different equation.

► On the chalkboard, write the placeholder equation, $52 = \square \times 8$. Call on a child to come to the pocket chart and give him 52 tagboard strips (1 inch by 3 inches). Ask the child if he can arrange the 52 strips into rows that have exactly 8 strips in each row. He will demonstrate that this cannot be done.



Let other children use tagboard strips to show whether or not the following equations have a solution in the set of whole numbers.

$7 \times \square = 84$	$72 = 3 \times \square$
$\square \times 9 = 108$	$83 = \square \times 6$
$8 \times \square = 96$	$52 = 4 \times \square$

► Suggest to the children that a letter can be used as a placeholder. Write on the chalkboard several multiplication equations in which a letter has been used for the missing factor.

$4r = 8$	$y \times 4 = 16$
$7b = 14$	$a \times 2 = 10$
$18 = 9c$	$12 = t \times 3$
$21 = 3m$	$15 = s \times 5$

Encourage the children to examine the use of the times sign in these equations. The children should observe that, when a letter is used for the second factor in a multiplication equation, the times sign is omitted. They should also observe that the times sign is not omitted when the letter is used for the first factor.

Let several children take turns reading these equations to the class. Then ask for volunteers to complete these equations, and others like them, that you write on the board. The children may read the first 4 of the 8 suggested equations in the following way:

4 times $r = 8$.
($4r$ means 4 times r)
7 times $b = 14$.
($7b$ means 7 times b)
$18 = 9$ times c.
($9c$ means 9 times c)
$21 = 3$ times m.
($3m$ means 3 times m)

Write several more placeholder equations. This time include a few equations that do not have a whole-number solution.

$44 = 4r$	$8h = 48$
$48 = 6s$	$7k = 55$
$35 = 5t$	$3q = 21$
$72 = 9m$	$9y = 279$
$120 = 60a$	$80b = 560$
$150 = 30c$	$40c = 290$

Ask the children to tell you whether or not each equation has a whole-number solution. When the children decide that an equation has a whole-number solution, ask someone to complete the equation. For example, with the equation $35 = 5t$, a pupil may write $35 = 5 \times 7$. Help the children check their solution for each equation.

► Write several story exercises such as these.

Mary has 64 shells. She wants to mount them on a card and put exactly 9 shells in each row. Can this be done?

Eight children sold 54 tickets for the school play. Could each person have sold the same number of tickets?

All 180 books are to be placed in 20 boxes. Could the same number of books be placed in each box?

Ask the class to give a missing-factor equation for each story exercise. The equations should be written with a letter as a placeholder. Call on several children to write the equations that are suggested by the stories.

Pages 57 through 60

● Use page 57 for class discussion. Give each child 24 counters. Ask the children to read exercise 1 and write an equation that represents this situation. Write on the chalkboard the equation that the class decides to use.

Tell the children to put their 24 counters in 2 equal piles to see how many checkers will be in each pile. Ask a child to write an equation that shows the solution. Follow a similar procedure with exercises 2 through 6.

Finally, direct the children to the story at the bottom of page 57. They should use their counters to put 24 checkers in 5 equal piles. The children will observe that Nancy could not do this. This demonstrates also that there is no whole-number solution to the equation $5 \times \square = 24$.

Name

UNIT 5 THE MISSING-FACTOR PROBLEM

For Class Discussion

Nancy wants to put 24 checkers into equal piles.

How many checkers will there be in a pile if Nancy puts—

1. 24 checkers in 2 equal piles? 12
2. 24 checkers in 3 equal piles? 8
3. 24 checkers in 4 equal piles? 6
4. 24 checkers in 6 equal piles? 4
5. 24 checkers in 8 equal piles? 3
6. 24 checkers in 12 equal piles? 2

Nancy would like to put 24 checkers in 5 equal piles. When she tries, what will happen? The piles will not be equal. Answers may vary.

THE PROBLEM

What whole number times 5 equals 24?

$5 \times \square = 24$ No whole-number solution

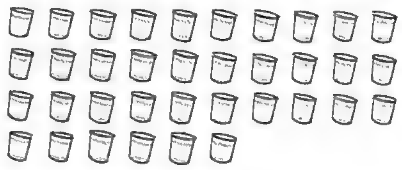
reference page

E-57

● Use page 58 for class discussion. Give each child 36 counters. Then follow a procedure similar to that suggested for page 57. The story at the bottom of page 58 suggests a missing-factor equation with no whole-number solution.

For Class Discussion

Sara wants to put 36 paper cups into equal stacks.



How many cups will there be in each stack if Sara puts—

1. 36 cups in 2 equal stacks? 18
2. 36 cups in 3 equal stacks? 12
3. 36 cups in 4 equal stacks? 9
4. 36 cups in 6 equal stacks? 6
5. 36 cups in 9 equal stacks? 4
6. 36 cups in 12 equal stacks? 3
7. 36 cups in 18 equal stacks? 2
8. 36 cups in 1 stack? 36

Sara would like to put 36 paper cups in 7 equal stacks. When she tries, what will happen? The stacks will not be equal. Answers may vary.

THE PROBLEM

What whole number times 7 equals 36?

$7 \times \square = 36$ No whole-number solution

E-58

● Complete page 59 as a class activity. If possible, give each child or each pair of children 64 counters. Ask the class to read the information at the top of the page; then have the children try to arrange 64 counters in 2 rows. Ask how many counters are in each row. Tell a child to write the appropriate equations on the chalkboard.

$$64 = 2d$$
$$64 \div 2 = d$$
$$64 \div 2 = 32$$

Ask someone to read the solution. He should say that the missing factor is 32.

Next, read with the class the directions for each exercise in the middle of the page. Choose someone to write on the chalkboard the missing-factor equations for one of the exercises. Then tell the children to group their counters into an array using the given number of rows. Let a child write the solution on the chalkboard.

After the children have completed these exercises, examine the last question on the page. The class will see that Jim cannot arrange the 64 dots in 3 equal rows. This is another problem that has no whole-number solution.

Name

For Class Discussion

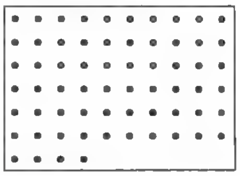
There are 64 dots. Arrange them in a rectangular array.

If Jerry makes 2 rows, how many dots will be in each?

$64 = 2 \times d$

$64 = 2d$

$64 \div 2 = d$ $\quad$ $64 \div 2 =$ 32



How many will be in a row if he makes 4 rows?

$64 = 4d$

$64 \div 4 = d$ $\quad$ $64 \div 4 =$ 16

How many will be in a row if he makes 8 rows?

$64 = 8d$

$64 \div 8 = d$ $\quad$ $64 \div 8 =$ 8

How many will be in a row if he makes 16 rows?

$64 = 16d$

$64 \div 16 = d$ $\quad$ $64 \div 16 =$ 4

How many will be in a row if he makes 32 rows?

$64 = 32d$

$64 \div 32 = d$ $\quad$ $64 \div 32 =$ 2

Jim would like to arrange the 64 dots in 3 equal rows. Can this be done?

$64 = 3d$ $\qquad$ $64 \div 3 = d$

THE PROBLEM

What whole number times 3 equals 64?

$3 \times \square = 64$ no whole-number solution

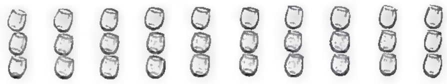
reference page

E-59

● Complete page 60 as a class activity. Give each child 30 counters and follow a procedure similar to that suggested for page 59. After this page has been completed, the children should know that there are missing-factor equations that do not have whole-number solutions.

For Class Discussion

Mrs. Hull at the gift shop wants to put 30 glasses into boxes with the same number of glasses in each box.



How many will be in a box if she puts them in 2 boxes?

$2 \times b = 30$

$b = 30 \div 2$ $\quad$ $b =$ 15

How many will be in a box if she puts them in 3 boxes?

$3 \times b = 30$

$b = 30 \div 3$ $\quad$ $b =$ 10

How many will be in a box if she puts them in 5 boxes?

$5b = 30$

$b = 30 \div 5$ $\quad$ $b =$ 6

How many will be in a box if she puts them in 6 boxes?

$6b = 30$

$b = 30 \div 6$ $\quad$ $b =$ 5

How many will be in a box if she puts them in 10 boxes?

$10b = 30$

$b = 30 \div 10$ $\quad$ $b =$ 3

How many will be in a box if she puts them in 15 boxes?

$15b = 30$

$b = 30 \div 15$ $\quad$ $b =$ 2

Mrs. Hull would like to put the 30 glasses into 9 boxes with the same number in each box. Can this be done?

$9b = 30$ $\qquad$ $b = 30 \div 9$

THE PROBLEM

What whole number times 9 equals 30?

$9 \times \square = 30$ no whole-number solution

E-60

Supplemental Experiences

■ Write simple missing-factor exercises on the chalkboard. Instruct the children to copy and solve them.

$$\begin{array}{r}9\\ \times\square\\ \hline 72\end{array}\qquad\begin{array}{r}\square\\ \times 8\\ \hline 56\end{array}\qquad\begin{array}{r}7\\ \times\square\\ \hline 63\end{array}\qquad\begin{array}{r}\square\\ \times 5\\ \hline 45\end{array}$$

$$\begin{array}{r}\square\\ \times 8\\ \hline 24\end{array}\qquad\begin{array}{r}4\\ \times\square\\ \hline 36\end{array}\qquad\begin{array}{r}8\\ \times\square\\ \hline 56\end{array}\qquad\begin{array}{r}3\\ \times\square\\ \hline 27\end{array}\qquad\begin{array}{r}\square\\ \times 6\\ \hline 54\end{array}$$

■ Prepare cards such as those pictured.

20q = 100	8q = 56	110 = 10q
9q = 63	11q = 110	9q = 81
10q = 1000	15q = 900	50q = 2000

Give one card to each child. Call out a number such as 5. If a child's equation has 5 as a solution, let him read the equation to the class. Now try other missing-factor equations.

KEY IDEA

If $3a = 6$, then $a = 6 \div 3$.

Scope

To solve missing-factor equations.
To review basic multiplication facts.

Fundamentals

Since division and multiplication are inverse operations, the missing factor may be expressed in terms of either multiplication or division. For example, t represents the same missing factor in each of the following equations.

$$7t = 14$$
$$14 \div 7 = t$$
$$14 \div t = 7$$

The parts of a multiplication equation are product, factor, and factor. In the above equations, 14 is the product and 7 and t are the factors. The order of the factors does not affect the product.

Readiness for Understanding

Knowledge of basic multiplication facts.
Knowledge of numeration.

Developmental Experiences

for flannel board	pins
array cards	sheet of tagboard
sheet of tagboard	(24″ × 36″)
(18″ × 24″)	yarn

► On the flannel board, pin a card that shows a 6 by 10 array. Cover all but the first row of 6 in this array. Tell the children to examine the array on the flannel board. Explain that there are more rows that the children will uncover to show an array you describe.

Ask someone to write on the chalkboard a missing-factor equation that represents an array with 56 members arranged in 6 rows. Suggest that this child use a letter as a placeholder. Then ask another child to write a division equation that expresses this same idea for the same array; for example:

$$56 = 6t$$
$$t = 56 \div 6$$

Choose someone else to try to show an array of 56 members arranged in 6 rows. This child may decide that the closest he can come to 56 is either a 9 by 6 array or a 10 by 6 array. He cannot uncover the array on the flannel board in such a way that there are 56 members arranged in 6 rows.

Continue to have the children write one division equation and one multiplication equation for each situation you describe; use arrays to demonstrate whether or not there is a whole-number solution. Questions such as the following can be posed.

Is it possible to build an array with 63 members in 9 rows? (Use a 9 by 10 array card).

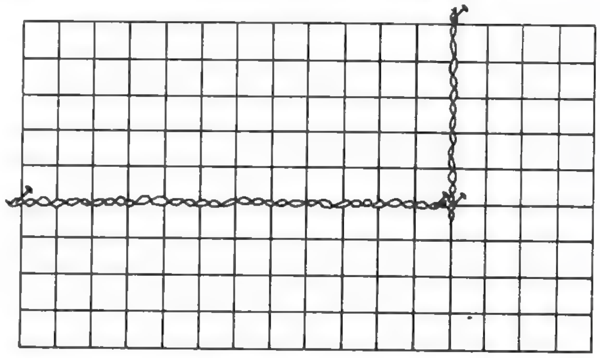
Is it possible to build an array with 47 members in 7 rows? (Use a 7 by 10 array card).

Is it possible to build an array with 45 members in 5 rows? (Use a 5 by 10 array card).

Is it possible to build an array with 44 members in 11 rows? (Use a 4 by 12 array card).

► Write $108 \div 12 = t$ on the chalkboard and have a child write a multiplication equation that expresses the same relationship between 108, 12, and t. The child may choose to write $108 = 12t$.

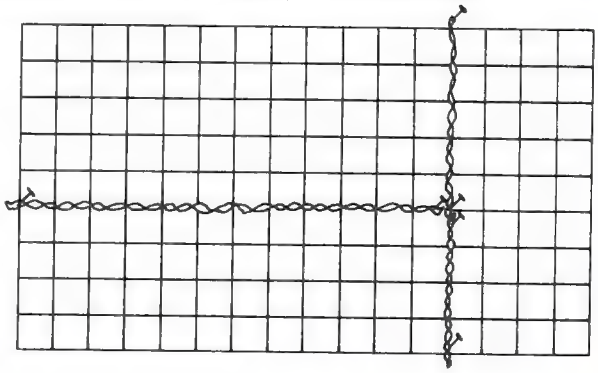
Pin on a bulletin board a 24 by 36 inch sheet of tagboard partitioned into $1\frac{1}{2}$ -inch squares. Ask a child to outline with yarn an array that illustrates the equations $108 = 12t$ and $108 \div 12 = t$. Perhaps the child will begin by outlining a 12 by 5 array.



Ask someone to write an equation that indicates the product for this array.

$$12 \times 5 = 60$$

The first child may want to show a larger array with 12 rows. Tell him to leave the outline of the small array in place. Let him outline any other array that can be added to this first array to make an array of 108 members arranged in 12 rows. A 12 by 4 array adjacent to the 12 by 5 array will work best.



Ask someone to write, below $12 \times 5 = 60$, an equation that indicates the product for the second array on the bulletin board. Then add the products of the two arrays.

$$\begin{array}{r} 12 \times 5 = 60 \\ 12 \times 4 = \underline{48} \\ 108 \end{array}$$

Encourage the children to describe their conclusions at this point. They may offer comments such as: "The number of a 12 by 9 array is 108," or "The missing factor in the equations $108 \div 12 = t$ and $108 = 12t$ is 9."

Continue to have several children determine the solutions to missing-factor equations. In most cases, it should be possible to find a whole-number solution.

$120 \div t = 20$	$64 \div 4 = a$
$76 \div 9 = r$	$75 \div b = 15$
$10 = 210 \div s$	$11 = 220 \div c$
$m = 91 \div 7$	$y = 136 \div 8$

► Write on the chalkboard the following missing-factor equations:

$660 \div t = 80$	$18 = 370 \div m$
$730 \div s = 9$	$750 \div 50 = r$
$29 = 319 \div a$	$b = 442 \div 7$

Ask several volunteers to write a related multiplication equation below each of the division equations.

Ask the class to suggest ways to determine whether or not there is a whole-number solution. For example, with the equation $29 = 319 \div a$, someone may suggest that 29 tens is 290.

Someone else may point out that 29×11 could be tried: this number is $290 + 29$. Write this idea on the chalkboard.

$$29 \times 10 = 290$$
$$29 \times 11 = 290 + 29$$

Ask someone to compute the sum $290 + 29$. Since $29 \times 11 = 319$, the missing factor is the whole number 11.

With the equation $730 \div s = 9$, someone may suggest that 9×8 tens is 72 tens ($9 \times 80 = 720$).

Someone else may suggest trying 9×81; this is the number $720 + 9$ or 729. Write this information on the chalkboard.

$$9 \times 80 = 720$$
$$9 \times 81 = 720 + 9 \text{ (or 729)}$$

If someone suggests 9×82, have a member of the class write this product on the chalkboard.

$$9 \times 80 = 720$$
$$9 \times 81 = 720 + 9 \text{ (or 729)}$$
$$9 \times 82 = 729 + 9 \text{ (or 738)}$$

Now the children may observe that $730 \div s = 9$ has no whole-number solution. Continue to work with the other missing-factor equations in the original list.

► Write on the chalkboard several story exercises such as the following:

Mr. Jolly, the druggist, has to put 118 boxes of cotton on his shelves. Will he be able to arrange these in equal stacks of 9 boxes each?

Mrs. Goodman has 128 tulip bulbs to plant. Is it possible for her to plant all these bulbs in rows that contain exactly 8 bulbs each?

Ask the children to suggest a division equation and a multiplication equation for each story exercise. Remind the children to use a letter as a placeholder in each equation and to determine whether or not there is a whole-number solution.

Pages 61 through 64

● Use page 61 to give the children an opportunity to test their ability to use multiplication and division equations in determining a missing whole-number factor. Work several exercises with the children. In several cases the children will indicate that there is no whole-number solution. Then ask the children to complete the page. Have counters available for the students who find arrays helpful. In correcting the exercises, you will be able to identify those children who may need additional help.

●

Name

If the missing factor is a whole number, find it. If the missing factor is not a whole number, write "No Solution."

1. $5f = 15$ 3	2. $64 = 8m$ 8	3. $g = 32 \div 4$ 8
4. $54 = 6m$ 9	5. $14 \div 2 = c$ 7	6. $9a = 63$ 7
7. $36 \div g = 4$ 9	8. $42 \div 5 = t$ No solution	9. $70 \div 7 = b$ 10
10. $81 = 10s$ No solution	11. $7 = 56 \div h$ 8	12. $7d = 63$ 9
13. $9 = 72 \div v$ 8	14. $7p = 48$ No solution	15. $450 \div 5 = j$ 90
16. $9k = 540$ 60	17. $100 \div q = 50$ 2	18. $800 \div c = 9$ No solution

19. Mrs. Olsen made 36 blueberry pancakes. She wanted to serve the same amount to 12 people. Could each person get the same number of pancakes? What whole number times 12 equals 36? Yes, 3

20. Bonnie had 460 beads. She wanted to make bead belts. Each belt had to have 70 beads. Can she use all the beads? What whole number times 70 equals 460? No, no whole-number solution

21. Ken has a total of 560 marbles. He put the same number of marbles into each of 8 bags. How many marbles did he put in each bag? 70

22. Pete and Mike have a total of 23 books. Could each boy have the same number of books? No, no whole-number solution

reference page

E-61

● Use page 62 to give the children practice in finding the missing factor in the set of whole numbers. Give each child 40 counters. Discuss the question posed by the equation $5b = 40$ or $b = 40 \div 5$ at the top of the page. Suggest to the children that they think of an array for this equation. If an array of 40 can be made with 5 rows, how many will be in each row? The solution $b = 8$ indicates that the 40 counters can be arranged in a 5 by 8 array.

Next, work exercise 1 with the class. Discuss whether it is possible to arrange 40 objects in an array that has 3 rows. Suggest to the children that they use their counters to investigate this question. They should conclude that there is no whole-number solution to the equation $3c = 40$. Tell them to write the words *no solution* on their papers to show this. Solve exercise 2 with the class. When the children find the solution (10), you may want them to use this form.

$$4d = 40$$
$$d = 10$$
$$4 \times 10 = 40$$

Then direct the children to complete exercises 3 through 18. The children who do not need counters should not be required to use them.

Finally, read the remaining exercises with the class. Let the children suggest several multiplication or division equations for each story. For example, $3p = 90$ or any of the related equivalent equations would be appropriate for exercise 19.

$3p = 90$ $90 = 3p$
$90 \div 3 = p$ $p = 90 \div 3$

$p \times 3 = 90$ $90 = p \times 3$
$90 \div p = 3$ $3 = 90 \div p$

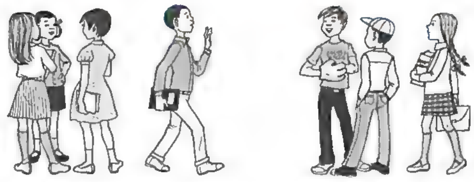
Discuss the ways the children obtained their answers in each story. In their answers for exercise 20, the children should include the idea that there is no solution in the set of whole numbers.

● Page 63 tests the children's ability to find a missing factor. Work one exercise from each section with the class. Then assign the remaining exercises. Discuss the whole-number solution (if there is one) for each exercise.

$5b = 40$ $b = 40 \div 5$ 8 $= 40 \div 5$

If the missing factor is a whole number, find it. Otherwise, write "No Solution."

1. $3c = 40$ no solution
2. $4d = 40$ d = 10
3. $5e = 40$ e = 8
4. $6f = 40$ no solution
5. $7g = 40$ no solution
6. $8h = 40$ h = 5
7. $9j = 40$ no solution
8. $10k = 40$ k = 4
9. $20m = 40$ m = 2
10. $2n = 20$ n = 10
11. $3p = 20$ no solution
12. $4q = 20$ q = 5
13. $5r = 20$ r = 4
14. $6s = 20$ no solution
15. $7t = 20$ no solution
16. $8u = 20$ no solution
17. $9v = 20$ no solution
18. $10w = 20$ w = 2



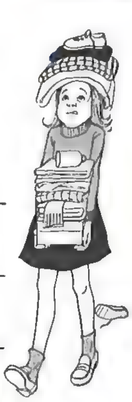
19. Warren had 90 pieces of candy to share with his class. He brought the candy to school in 3 boxes. Each box contained the same number of pieces. How many pieces of candy were in each box? 30
20. The 20 children in Miss Smith's class have a total of 47 pencils. Do all of the children have the same number of pencils? What whole number times 20 equals 47? no, no whole-number solution
21. After a snowstorm 12 boys shoveled 48 driveways. Each boy shoveled the same number of driveways. How many driveways did each boy shovel? 4

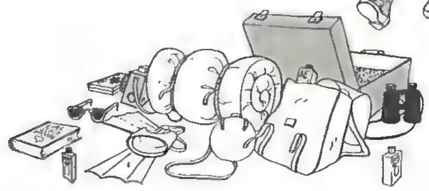
E-62

Name

If the missing factor is a whole number, find it. Otherwise, write "No Solution."

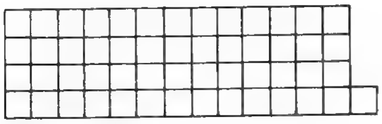
1. $6 = 3m$ m = 2
2. $18 = 9b$ b = 2
3. $40 = 8r$ r = 5
4. $14 = 3s$ no solution
5. $80 = 20c$ c = 4
6. $40 = 5k$ k = 8
7. $35 = 7f$ f = 5
8. $80 = 40g$ g = 2
9. $30 = 5h$ h = 6
10. $4n = 28$ n = 7
11. $80k = 150$ no solution
12. $40 = 10m$ m = 4
13. $48 = 6q$ q = 8
14. $80 = 1s$ s = 80
15. $8q = 32$ q = 4
16. $9q = 63$ q = 7
17. $20b = 190$ no solution
18. $75 = 25n$ n = 3
19. Bob has 30 eggs to put into cartons. Each carton holds 12 eggs. Can he fill 3 cartons? no
20. Kay needs 72 dollars to go on a camping trip. She saves 8 dollars each week. For how many weeks must Kay save before she can go on this trip? 9
21. A milkman delivers 25 quarts of milk each time he stops at Mr. Town's store. How many times must the milkman stop to deliver 75 quarts of milk? 3



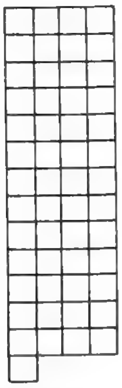


E-63

● Use the story exercises on page 64 as the basis for a class discussion. Give each child several sheets of $\frac{1}{2}$-inch graph paper. Use the graph paper to show arrays for each story question. For example, in exercise 4, show an array of 53 squares arranged in 4 rows.



or



The children should conclude that they cannot find a whole-number solution for this problem. The children should observe that for some story situations there is no whole-number solution.

Assign the exercises at the bottom of page 64. Some of the children may want to use graph paper for making arrays. When the children have completed these exercises, they should identify the product in the equations that have a solution. You might also want the children to give a multiplication equation for each division equation and a division equation for each multiplication equation.

If the missing factor is a whole number, find it. Otherwise, write "No Solution."

The Turpinville shopping center is always busy.

1. Six men purchased 48 cans of paint. Each man bought the same number of cans. How many cans of paint did each man buy? 8

2. A bookstore has 148 copies of one book. Twenty-four books can fit on one shelf. How many shelves will be needed to hold these books? no solution

3. A grocery store has 144 jars of peanut butter. There are 8 rows of jars, the same number in each row. How many jars in a row? 18

4. Fifty-three donuts were sold to 4 women. Could each woman have purchased the same number of donuts? no

5. A flower shop had 379 tulips. Seventeen people bought all the tulips. Could each person have bought the same number of tulips? no

6. $20n = 5000$ $n = 250$	7. $2400 \div 400 = f$ $f = 6$	8. $3 = 15 \div h$ $h = 5$
9. $490 \div 6 = d$ no solution	10. $9p = 8100$ $p = 900$	11. $720 = 80k$ $k = 9$
12. $6s = 359$ no solution	13. $29j = 320$ no solution	14. $280 = 7a$ $a = 40$

E-64

Supplemental Experiences

■ The children who need help with basic multiplication combinations might use the following activity.

A. Multiply.	B. Compute the product of each pair.
6 by 8 ____	3 and 9 ____
7 by 5 ____	7 and 2 ____
9 by 2 ____	8 and 8 ____
8 by 7 ____	6 and 4 ____
8 by 4 ____	5 and 3 ____
5 by 6 ____	9 and 6 ____
2 by 3 ____	

■ Write multiplication and division placeholder equations on tagboard cards. There should be no whole-number solution for some of these equations.

$6m = 45$	$32 = 4w$
$56 \div 7 = c$	$40 \div 3 = t$

Pin the cards on the bulletin board. Instruct the children to compose their own story situations for any one of the equations on the bulletin board. Post on the bulletin board several of the stories that are written. The children might solve the equations and give their solutions to the child who wrote the story. He can check the computation and the sentence that answers his story.

UNIT 6
QUOTIENT AND REMAINDER
Pages 65 Through 80

OBJECTIVE
To develop the quotient-remainder equation.

The child explores physical situations that are represented by equations of the form $b = aq + r$. He learns that a solution for $28 = 3q + r$ consists of two whole numbers, one for q and one for r. In such equations there is always a greatest partial quotient.

See Key Topics in Mathematics for the Intermediate Teacher: Multiplication and Division of Whole Numbers.

KEY IDEAS
$b = aq + r$ always has a solution.

$b = aq + r$ can have more than one whole-number solution for q and r.

There is always a greatest partial quotient.

KEY IDEA
$b = aq + r$ always has a solution.

Scope
To introduce the quotient-remainder equation, $b = aq + r$.

To explore solutions to the quotient-remainder equation.

Fundamentals
The missing-factor equation $ax = b$ does not always have a whole-number solution for x. When $ax = b$ does not have a whole-number solution, a different question is asked. This question is represented by the equation, $b = aq + r$. It is the quotient-remainder equation; q represents a partial quotient, and r stands for the remainder.

If we compare 23 and 4 using the missing-factor equation $ax = b$, we obtain the equation $23 = 4x$ which has no whole-number solution. However, if we use the quotient-remainder equation $b = aq + r$, we obtain the equation $23 = 4q + r$. This equation has six solutions for q and r.

$q = 0, r = 23$:	$23 = 4 \times 0 + 23$
$q = 1, r = 19$:	$23 = 4 \times 1 + 19$
$q = 2, r = 15$:	$23 = 4 \times 2 + 15$
$q = 3, r = 11$:	$23 = 4 \times 3 + 11$
$q = 4, r = 7$:	$23 = 4 \times 4 + 7$
$q = 5, r = 3$:	$23 = 4 \times 5 + 3$

Each solution gives a partial quotient, q, and a remainder, r. Note that the greatest partial quotient is 5. The greatest partial quotient is customarily called the quotient.

A quotient-remainder equation represents the following story problem.

> Jack has 43 crayons to put into boxes. Each box can hold 10 crayons. How many of these boxes can Jack fill? How many crayons will be left over?

The greatest partial-quotient solution of the quotient-remainder equation, $43 = 10q + r$, gives the number of boxes that can be filled and specifies the number of crayons left over.

$$q = 4, r = 3 \qquad 43 = 10 \times 4 + 3$$

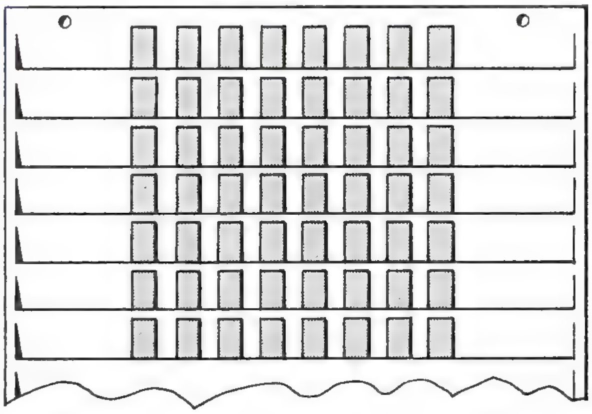
Four boxes can be filled. There will be 3 crayons left over.

Readiness for Understanding
Knowledge of basic multiplication facts.

Ability to subtract.

Developmental Experiences
tagboard strips ($\frac{1}{2}'' \times 3''$)	*for each child*
pocket chart	100 counters
	tagboard set outline

▶ On the chalkboard, write the equation $56 = 8x$. Give a child 56 tagboard strips ($\frac{1}{2}$ by 3 inches) and ask him whether or not it is possible to arrange them in an array that has 8 rows one way. He should use the pocket chart for his array. The child may point out that $56 = 8 \times 7$, so the array will have 7 rows the other way.



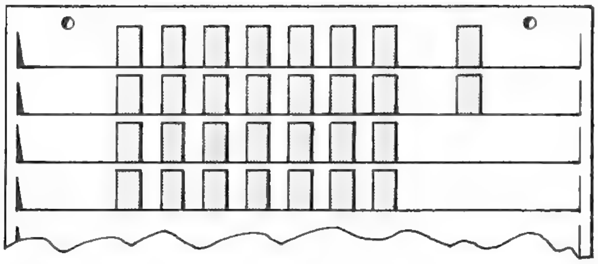
Point out that the missing factor is 7.

Now write the equation $30 = 4x$. Have a second

child try to arrange 30 tagboard strips in an array with 4 strips in each row. The child will probably make some observations such as these:

> When someone tries to arrange 30 objects in an array with 4 rows, the closest he can come is a 4 by 7 array. A 4 by 8 array would take more than 30 counters. Therefore, using all 30 objects, it is not possible to build an array having 4 rows. A 4 by 7 array does not use all the objects, while a 4 by 8 array requires more than 30 objects.

Ask this child to place the unused strips in the chart, but separated from the 4 by 7 array.



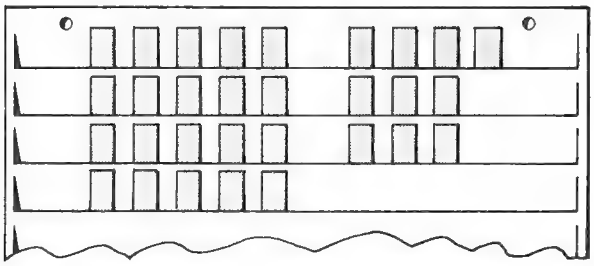
Point out that there are $(4 \times 7) + 2$ objects in the chart. Write the equation that shows this.

$$30 = (4 \times 7) + 2$$

Refer to the equation $30 = 4x$ and ask whether it has a solution. Tell the class that, while there is no whole-number solution for $30 = 4x$, 30 may be expressed as 4 times 7 plus 2. Write the equation $30 = 4q + r$ on the chalkboard between the two equations. Explain that while the equation $30 = 4x$ has no whole-number solution, the equation $30 = 4q + r$ has the solution shown in the bottom equation. Tell the children that $30 = 4q + r$ is called a quotient-remainder equation.

$$30 = 4x$$
$$30 = 4q + r$$
$$30 = (4 \times 7) + 2$$

Now suggest the possibility of another solution to the equation $30 = 4q + r$. Ask a child to try to arrange another array with 4 rows one way, using some of the 30 strips. He could decide to show a 4 by 5 array and place the remaining strips in the chart at the side.



Ask for a volunteer to write an equation that shows the solution for $30 = 4q + r$ that is suggested by this arrangement.

$$30 = 4x$$
$$30 = 4q + r$$
$$30 = (4 \times 7) + 2$$
$$30 = (4 \times 5) + 10$$

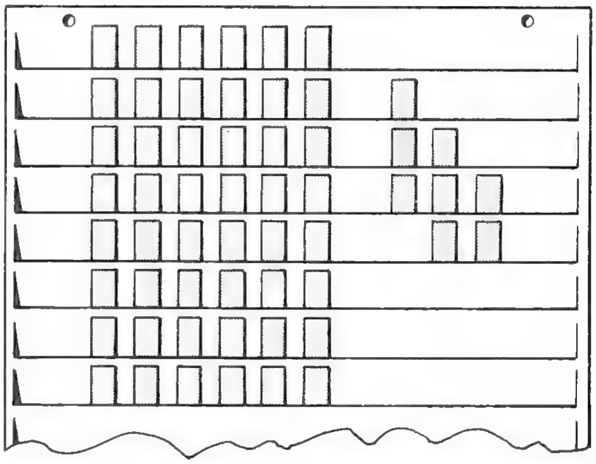
Let the pupils use 4-row arrays to suggest any other solutions to $30 = 4q + r$. Have them write an equation on the chalkboard to show each solution.

Ask the class to study all the solutions for $30 = 4q + r$ and then to identify the greatest quotient, q, and least remainder, r.

$$30 = 4q + r$$
$$30 = (4 \times 7) + 2$$
$$30 = (4 \times 5) + 10$$
$$30 = (4 \times 3) + 18$$
$$30 = (4 \times 6) + 6$$
$$30 = (4 \times 2) + 22$$
$$30 = (4 \times 4) + 14$$
$$30 = (4 \times 1) + 26$$
$$30 = (4 \times 0) + 30$$

Ask the children what happens to the remainder as the quotient increases (examination reveals that as q increases r decreases). Ask what they observe about the remainder when the greatest quotient is used in the solution. They should note that the equation that shows the greatest q also shows the least r.

Now write the equation $56 = 8q + r$. Ask a child to arrange 56 strips in an 8 by 6 array with some strips left over.



Ask for a volunteer to write the solution to $56 = 8q + r$ that is suggested by this arrangement.

$$56 = 8q + r$$
$$56 = (8 \times 6) + 8$$

Do this with the other possible arrangements of 56 objects until all solutions for $56 = 8q + r$ are shown:

$56 = 8q + r$

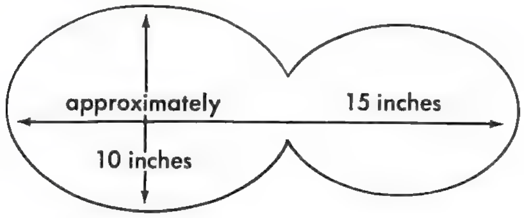
$56 = (8 \times 6) + 8$	$56 = (8 \times 4) + 24$
$56 = (8 \times 3) + 32$	$56 = (8 \times 2) + 40$
$56 = (8 \times 1) + 48$	$56 = (8 \times 7) + 0$
$56 = (8 \times 5) + 16$	$56 = (8 \times 0) + 56$

Ask the class to identify the greatest q and the least r.

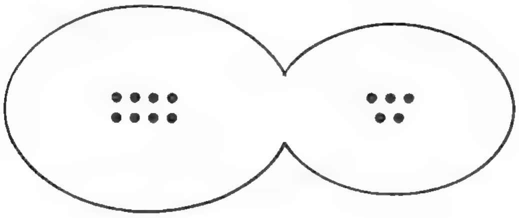
Use arrays of tagboard strips to suggest solutions for these equations:

$$35 = 10x \text{ and } 35 = 10q + r$$
$$90 = 6x \text{ and } 90 = 6q + r$$
$$29 = 12x \text{ and } 29 = 12q + r$$
$$99 = 11x \text{ and } 99 = 11q + r$$

▶ Provide each child with 100 counters and a large tagboard outline cut as illustrated.



Show the class how 13 counters may be used to represent a solution for $13 = 4q + r$.

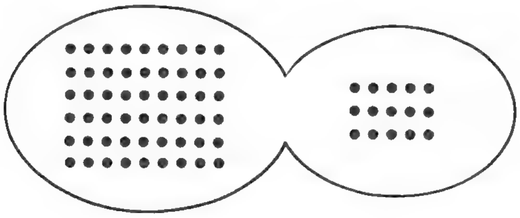


$$13 = 4q + r$$
$$13 = (4 \times 2) + 5$$

Tell the children to use their counters at their desks to find other solutions to the equation $13 = 4q + r$. As solutions are suggested, write them on the board.

$$13 = 4q + r$$
$$13 = (4 \times 2) + 5$$
$$13 = (4 \times 3) + 1$$
$$13 = (4 \times 1) + 9$$
$$13 = (4 \times 0) + 13$$

Then write the equation $69 = 9q + r$. Have each pupil use 69 counters to find a solution to this equation. Each is to make an array with 9 rows in the left side of his set outline and place the remaining counters in the right side of the outline. For example, a pupil could use this arrangement to show that 69 can be arranged in a 9 by 6 array with 15 remaining.

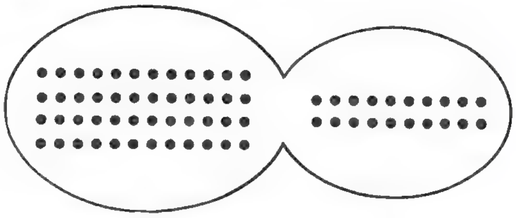


Ask various children for the equation represented by their arrangement and write these equations on the chalkboard. If some solutions are not mentioned, tell the class to find these solutions. When a complete list of solutions has been shown, direct the class to identify the greatest quotient.

Use counters and sets to find solutions for equations such as these:

$$97 = 13q + r \qquad 75 = 15q + r$$
$$83 = 18q + r \qquad 56 = 14q + r$$
$$68 = 17q + r \qquad 84 = 8q + r$$

▶ Write the equation $68 = 12q + 20$ on the chalkboard. Ask the children to arrange 68 counters in their set outlines so that the equation describes the arrangement. They should begin by placing 20 counters on the right side of the outline. On the left side they should form an array with 12 rows, using the remaining 48 counters.

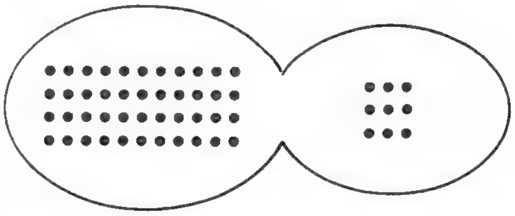


Have one child come to the chalkboard and write an equation that shows the replacement for q.

Do this with several more quotient-remainder equations in which r is given. Such equations as these may be used.

$$74 = 6q + 14 \qquad 65 = 3q + 5$$
$$96 = 20q + 16 \qquad 59 = 10q + 9$$
$$88 = 7q + 4 \qquad 77 = 13q + 25$$

Then write the equation $53 = (11 \times 4) + r$ on the chalkboard. Ask the children to illustrate this equation with 53 counters. They should begin by making a 4 by 11 array on the left side. Then place the remaining counters on the right side.



Now ask a child to write an equation that shows the replacement for r.

$$53 = (11 \times 4) + 9$$

Do this with several quotient-remainder equations in which q is given. Such equations as these may be used.

$$92 = (13 \times 6) + r \qquad 63 = (9 \times 7) + r$$
$$87 = (15 \times 4) + r \qquad 105 = (20 \times 5) + r$$
$$71 = (10 \times 5) + r \qquad 94 = (18 \times 4) + r$$

▶ On the chalkboard, write a story exercise such as this.

On Thursday, 16 children stayed after school to practice folk dancing. The teacher called two couples to the floor for the first dance. How many children danced the first dance, and how many children sat out?

After the class has read the story, have a pupil write the equation that the situation suggests.

$$16 = (2 \times 2) + 12$$

Have pupils write other equations that express the relation among the numbers of the story when 4, 6, and

8 couples dance.

$$16 = (2 \times 4) + 8$$
$$16 = (2 \times 6) + 4$$
$$16 = (2 \times 8) + 0$$

Write another story exercise, such as this.

After school one day, 32 boys reported for basketball practice. There are 5 players on a basketball team and only 2 teams can play at one time. After 2 teams have been formed, how many of the 32 children are on teams, and how many remain to be chosen?

Ask a child to write an equation that expresses the relation among the numbers in the story.

$$32 = (5 \times 2) + 22$$

Then ask for volunteers to write equations that describe how many of the 32 children are on teams, how many remain to be chosen after 4 teams have been chosen, and how many are left after the greatest number of teams has been chosen.

$$32 = (5 \times 4) + 12$$
$$32 = (5 \times 6) + 2$$

Pages 65 through 69

● Work page 65 with the class. Provide each child with 24 counters with which to depict the situation described on the page. Write the equation $24 = 5 \times \square + \triangle$ on the chalkboard. Below it, list the equations that result each time Nancy puts another checker in each pile.

$$24 = 5 \times \square + \triangle$$
$$24 = 5 \times 1 + 19$$
$$24 = 5 \times 2 + 14$$
$$24 = 5 \times 3 + 9$$
$$24 = 5 \times 4 + 4$$

Ask the children whether there are other solutions to the equation $24 = 5 \times \square + \triangle$. If a pupil suggests the solution $24 = 5 \times 0 + 24$, add this solution to the list on the chalkboard.

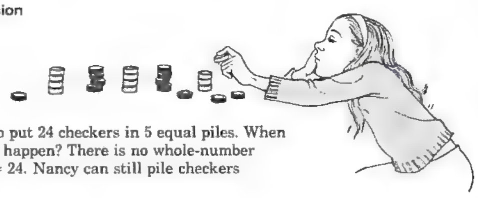
● Page 66 provides more practice in working with equations of the form $b = aq + r$. Give each child or each pair of children 42 counters. Let the children use these counters to imitate each step of Susan's solution. Solve each equation and answer each question on the page with the class. Ask for volunteers to list the solutions they found to the equation $42 = 8 \times \square + \triangle$.

$$42 = 8 \times \square + \triangle$$
$$42 = 8 \times 1 + 34$$
$$42 = 8 \times 2 + 26$$
$$42 = 8 \times 3 + 18$$
$$42 = 8 \times 4 + 10$$
$$42 = 8 \times 5 + 2$$

Name

UNIT 6 QUOTIENT AND REMAINDER

For Class Discussion



Nancy would like to put 24 checkers in 5 equal piles. When she tries, what will happen? There is no whole-number solution for $5 \times \square = 24$. Nancy can still pile checkers in 5 equal piles.

First Nancy puts 1 checker in each pile.
$24 = (5 \times 1) + 19$

Then she puts another checker in each pile.
$24 = (5 \times 2) + 14$

She puts another checker in each pile.
$24 = (5 \times \square) + \triangle$
How many checkers are in each pile? 3
How many of the 24 checkers are left? 9
$24 = (5 \times \underline{3}) + \underline{9}$

She puts another checker in each pile, and puts the rest on the table.
$24 = (5 \times \square) + \triangle$
How many checkers are in each pile? 4
How many of the 24 checkers are left? 4
$24 = (5 \times \underline{4}) + \underline{4}$

Nancy put 4 checkers in each pile and has 4 left.
$24 = (5 \times \square) + \triangle$
$24 = (5 \times \underline{4}) + \underline{4}$

E-65

For Class Discussion

Susan decided to arrange 42 star-shaped noodles in a rectangular array. Can she make 8 rows of stars? Is there a whole-number solution for $42 = 8 \times \square$?

Even though there is no whole-number solution for $42 = 8 \times \square$, Susan can put some of the stars in 8 rows.

First she put 1 star in each row.
$42 = (8 \times 1) + \triangle$
How many stars are left? 34

She put another star in each row.
$42 = (8 \times 2) + \triangle$
How many stars are left? 26

She put another star in each row.
$42 = (8 \times \square) + \triangle$
How many stars are in each row? 3
How many stars are left? 18

She put another star in each row.
$42 = (8 \times \square) + \triangle$
How many stars are in each row? 4
How many stars are left? 10

She put another star in each row.
$42 = (8 \times \square) + \triangle$
How many stars are in each row? 5
How many stars are left? 2

Susan can put 5 stars in each row and have 2 stars left.
$42 = (8 \times \underline{5}) + \underline{2}$

E-66

● The equation $b = aq + r$ can be used to solve the story exercises on page 67; but if there is a remainder of 0, the equation $ax = b$ could also be used. When everyone has finished, discuss each story and its solutions.

Exercise 4 has more than one possible answer. Some possibilities are:

1 sandwich, 22 slices left
2 sandwiches, 20 slices left
. .
. .
. .
12 sandwiches, 0 slices left

In exercise 5, the equation will depend on the answer to the question in exercise 4. Some possibilities of equations are:

$$24 = (2 \times 1) + 22$$
$$24 = (2 \times 2) + 20$$
$$\vdots$$
$$24 = (2 \times 12) + 0$$

● Additional work in solving equations of the form $b = aq + r$ is provided on page 68. Notation such as $22 = 5q + r$ appears on the page, and the children will work with this and other equations that are not related to story exercises. The emphasis on this page is on finding the greatest partial quotient. Be sure they understand that when they are asked to find the quotient, they are to find the greatest partial quotient—not just any partial quotient.

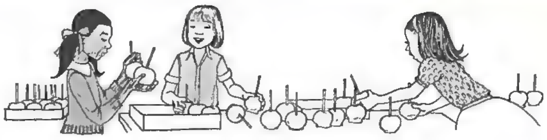
Work exercise 1 with the class. Encourage the children to try as many partial quotients as necessary to obtain the greatest partial quotient. Have them show this work on their papers. Use this method in working and discussing exercises 2 through 6 with the class. The number of equations required to solve each exercise will vary. For example, the quotient and remainder in exercise 2 may be found in any of these ways:

$93 = 9q + r$
$93 = 9 \times 10 + 3$

$93 = 9q + r$
$93 = 9 \times 7 + 30$
$93 = 9 \times 10 + 3$

$93 = 9q + r$
$93 = 9 \times 5 + 48$
$93 = 9 \times 7 + 30$
$93 = 9 \times 9 + 12$
$93 = 9 \times 10 + 3$

Name



Rita, Carol, and Sandy each made 52 taffy apples to take to the children's hospital. They packed boxes with 12 apples in each box. By 9 o'clock Rita had packed 24 apples, Carol had packed 36, and Sandy had packed 48.

1. How many boxes had Rita filled and how many apples were left? $52 = (12 \times$ 2 $) +$ 28
   Rita filled 2 boxes. 28 apples were left.

2. How many boxes had Carol filled and how many apples were left? $52 = (12 \times$ 3 $) +$ 16
   Carol filled 3 boxes. 16 apples were left.

3. How many boxes had Sandy filled and how many apples were left? $52 = (12 \times$ 4 $) +$ 4
   Sandy filled 4 boxes. 4 apples were left.

Mrs. Travis bought a loaf of bread to use to make sandwiches for a family picnic. The loaf had been cut into 24 slices. Mrs. Travis used 2 slices of bread for each sandwich.

4. How many sandwiches could she make from the loaf of bread and how many slices would be left? *

5. Write an equation to show your solution to exercise 4. *

E-67

** See pupil page suggestions.*

See pupil page suggestions.

How many 5's are in 22? What is the remainder? We can write this equation about the number of 5's (q) and the remainder (r).

$22 = 5q + r$

What does the equation mean? It means, "How many 5's are in 22 and what is the remainder?"

Let's solve the equation.

$22 = 5q + r$

$22 = (5 \times 2) + r$
$r =$ 12

$22 = (5 \times 3) + r$
$r =$ 7

$22 = (5 \times 4) + r$
$r =$ 2

$22 = (5 \times 2) + 12$
$22 = (5 \times 3) + 7$
$22 = (5 \times 4) + 2$

2, 3, and 4 are partial quotients for the equation $22 = 5q + r$.

4 is the greatest partial quotient for the equation $22 = 5q + r$.

When we say *quotient* we mean greatest partial quotient.

Find the quotient and remainder in each exercise.

1. $64 = 7q + r$ — $q =$ 9 $r =$ 1
2. $93 = 9q + r$ — $q =$ 10 $r =$ 3
3. $56 = 6q + r$ — $q =$ 9 $r =$ 2
4. $75 = 8q + r$ — $q =$ 9 $r =$ 3
5. $58 = 8q + r$ — $q =$ 7 $r =$ 2
6. $53 = 7q + r$ — $q =$ 7 $r =$ 4

reference page

E-68

● Page 69 provides further practice with quotient-remainder equations. Given a quotient q, the pupils are asked to find the resulting remainder, r; or given a possible r, they are asked to find q. Instruct the children to solve each equation in each column. When the children complete these exercises, they should see that each equation in a column shows a possible solution for the equation at the top of the column, but only one equation in each column gives the quotient (greatest partial quotient) and the least remainder.

The children may enjoy making up stories for some of the equations; give them an example. The equation $26 = 5 \times 3 + 11$ in exercise 7 suggests a story like this.

> John had 26 marbles. He planned to give 5 marbles to each of the boys he had invited to a party at his home. Only 3 boys were able to come to the party. After John gave each of these boys 5 marbles, he had 11 marbles left.

Name ______________________

See pupil page suggestions

Solve the equations. Circle the equation that shows the quotient.

1. $19 = 8q + r$

$19 = (8 \times 0) + 19$
$19 = (8 \times 1) + 11$
$19 = (8 \times 2) + 3$

2. $21 = 9q + r$

$21 = (9 \times 0) + 21$
$21 = (9 \times 1) + 12$
$21 = (9 \times 2) + 3$

3. $24 = 8q + r$

$24 = (8 \times 0) + 24$
$24 = (8 \times 1) + 16$
$24 = (8 \times 2) + 8$
$24 = (8 \times 3) + 0$

4. $25 = 6q + r$

$25 = (6 \times 0) + 25$
$25 = (6 \times 1) + 19$
$25 = (6 \times 2) + 13$
$25 = (6 \times 3) + 7$
$25 = (6 \times 4) + 1$

5. $38 = 6q + r$

$38 = (6 \times 0) + 38$
$38 = (6 \times 1) + 32$
$38 = (6 \times 2) + 26$
$38 = (6 \times 3) + 20$
$38 = (6 \times 4) + 14$
$38 = (6 \times 5) + 8$
$38 = (6 \times 6) + 2$

6. $24 = 4q + r$

$24 = (4 \times 1) + 20$
$24 = (4 \times 2) + 16$
$24 = (4 \times 3) + 12$
$24 = (4 \times 4) + 8$
$24 = (4 \times 5) + 4$
$24 = (4 \times 6) + 0$

7. $26 = 5q + r$

$26 = (5 \times 0) + 26$
$26 = (5 \times 1) + 21$
$26 = (5 \times 2) + 16$
$26 = (5 \times 3) + 11$
$26 = (5 \times 4) + 6$
$26 = (5 \times 5) + 1$

8. $35 = 6q + r$

$35 = (6 \times 1) + 29$
$35 = (6 \times 0) + 35$
$35 = (6 \times 5) + 5$
$35 = (6 \times 2) + 23$
$35 = (6 \times 4) + 11$
$35 = (6 \times 3) + 17$

9. $21 = 4q + r$

$21 = (4 \times 0) + 21$
$21 = (4 \times 1) + 17$
$21 = (4 \times 2) + 13$
$21 = (4 \times 3) + 9$
$21 = (4 \times 4) + 5$
$21 = (4 \times 5) + 1$

E-69

Supplemental Experiences

■ Make a set of cards that show quotient-remainder equations with either the q or the r not specified.

$53 = 5 \times 10 + r$	$65 = 8q + 9$
$39 = 7q + 11$	$70 = 9 \times 6 + r$

Divide the class into two teams, and have them come to the front of the room. Hold up a card and let the first child on one team supply the unspecified number. If he gives the correct answer, he can sit down. If he gives an incorrect answer, he must go to the end of his team's line. Hold up another card and let the first pupil on the other team supply the answer. As soon as all of the pupils on one team have answered correctly and returned to their desks, they are declared the winning team and the game is over.

■ Have the pupils complete tables of solutions for quotient-remainder equations. For each q named in the table, they should identify the corresponding r that makes the given sentence a true sentence.

$5q + r = 19$

q	1	3	0	2
r	14	4	19	9

$7q + r = 34$

q	4	3	2	1	0
r	6	13	20	27	34

$9q + r = 79$

q	0	1	2	3	4	5	6	7	8
r	79	70	61	52	43	34	25	16	7

The children should complete their tables in any order they choose. Order will eventually appear to the children, but let them discover this for themselves.

You may want to have small teams of children work together to complete the tables. This will give the children an opportunity to check each other's work.

KEY IDEA

$b = aq + r$ can have more than one whole-number solution for q and r.

Scope

To practice building a sequence of quotient-remainder equations using the r of each equation as the b of the next equation.

Fundamentals

The greatest quotient and least remainder of the equation $b = aq + r$ may be found by working through all solutions. However, it is possible to begin working with any one solution. Consider the equation $50 = 8q + r$ and this solution: $q = 3$, $r = 26$.

$$50 = (8 \times 3) + 26$$

This solution shows that there are 3 eights in 50 and 26 ones remaining. But how many eights are there in 26? To attack this question, use the remainder, 26, as the b in the next equation, $26 = 8q + r$. Again choose any solution of this equation—for example, $q = 2, r = 10$.

$$26 = (8 \times 2) + 10$$

Now use this remainder, 10, as the b in the next equation, $10 = 8q + r$. The only possible solution of this equation (other than $q = 0$) is $q = 1$, $r = 2$.

$$10 = (8 \times 1) + 2$$

The building of new equations must end at this point since the remainder, 2, is less than 8.

In this example, a sequence of equations has been developed in which the b of each new equation is the same as the r of the previous equation:

$$(1) \quad 50 = (8 \times 3) + \underline{26}$$

$$(2) \quad 26 = (8 \times 2) + \underline{10}$$

$$(3) \quad 10 = (8 \times 1) + 2$$

If we now substitute $(8 \times 1) + 2$ for 10 in equation (2) we obtain:

$$26 = (8 \times 2) + (8 \times 1) + 2$$

Finally, if we substitute $(8 \times 2) + (8 \times 1) + 2$ for 26 in equation (1) we have:

$$50 = (8 \times 3) + (8 \times 2) + (8 \times 1) + 2$$

The distributive property gives a simpler form of this equation:

$$50 = 8 \times (3 + 2 + 1) + 2$$

The result is this solution of $50 = 8q + r$:

$$50 = (8 \times 6) + 2$$

The solution, $q = 6$, $r = 2$, is the greatest quotient and least remainder.

Readiness for Understanding

Knowledge of basic multiplication facts.
Understanding of the distributive property.

Developmental Experiences

tagboard cards (12″ × 15″)
felt-tip pen

► Write the following story on the chalkboard.

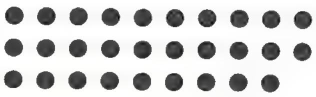
Bob has 29 gallons of paint. He uses 9 gallons each time he paints a house. How many houses can he paint? When he has painted as many houses as possible, will he have any gallons of paint remaining?

Use the story as the basis for developing a sequence of quotient-remainder equations.

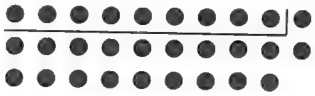
Call a pupil to the chalkboard to write a quotient-remainder equation that expresses the number structure of the story:

$$29 = 9q + r$$

Have another child draw an array on the chalkboard to show the number of gallons of paint involved in the story.



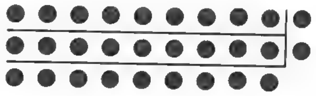
Have this same pupil partition the array to show the number of gallons of paint Bob used on the first house and how many of the 29 gallons remain.



Have someone else come to the chalkboard and write an equation that describes this model of 29:

$$29 = (9 \times 1) + 20$$

Let another pupil partition the array to show the number of gallons of paint Bob has used after he paints the second house, and how many of the 29 gallons remain.



Help still another member of the class write, under the preceding equation written on the chalkboard, an equation related to the model of 29 now visible on the chalkboard.

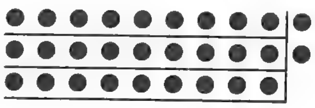
$$29 = (9 \times 1) + 20$$
$$29 = (9 \times 1) + (9 \times 1) + 11$$

At this point have the children direct their attention to the first equation and consider only the 20 gallons of paint remaining after Bob paints the first house. Ask someone to tell you an equation that shows how many of these 20 gallons Bob uses on the second house and how many of them remain. The equations can be arranged in two lists like this (or the lists may be placed beside each other), with an outline to show the source of the second equation in the second list:

$$29 = (9 \times 1) + \boxed{20}$$
$$29 = (9 \times 1) + \boxed{(9 \times 1) + 11}$$

$$29 = (9 \times 1) + 20$$
$$20 = (9 \times 1) + 11$$

Direct another pupil to partition the array to show on the chalkboard how many of the 29 gallons of paint Bob has used after he paints the third house.



Another member of the class should write, under the last equation of the first series of equations written on the chalkboard, an equation related to the model of 29 now visible on the chalkboard.

$$29 = (9 \times 1) + \boxed{20}$$
$$29 = (9 \times 1) + \boxed{(9 \times 1) + 11}$$
$$29 = (9 \times 1) + (9 \times 1) + (9 \times 1) + 2$$

$$29 = (9 \times 1) + 20$$
$$20 = (9 \times 1) + 11$$

At this point have the class direct its attention to the 11 gallons of paint that remain after Bob had painted 2 houses. Use an equation to show how many of these 11 gallons Bob used on the third house and how many of them remained. Outline this information in the first list, and add the equation to the second list.

$$29 = (9 \times 1) + \boxed{20}$$
$$29 = (9 \times 1) + \boxed{(9 \times 1) + \boxed{11}}$$
$$29 = (9 \times 1) + (9 \times 1) + \boxed{(9 \times 1) + 2}$$

$$29 = (9 \times 1) + 20$$
$$20 = (9 \times 1) + 11$$
$$11 = (9 \times 1) + 2$$

Now the class can be asked the following questions:

How many times did Bob use 9 gallons of paint?

How many houses did Bob paint?

How many of the 29 gallons remained?

Give the children ample time to discuss the relationship between the sum of products in $29 = (9 \times 1) + (9 \times 1) + (9 \times 1) + 2$ and the products in the second list:

$$29 = \boxed{(9 \times 1) + (9 \times 1) + (9 \times 1) + 2}$$

$$29 = \boxed{(9 \times 1)} + 20$$
$$20 = \boxed{(9 \times 1)} + 11$$
$$11 = \boxed{(9 \times 1) + 2}$$

Tell the class that it is possible (using the distributive property of multiplication with respect to addition) to write the sum of products $(9 \times 1) + (9 \times 1) + (9 + 1)$ as one product:

$$29 = (9 \times 1) + (9 \times 1) + (9 \times 1) + 2$$
$$29 = 9 \times (1 + 1 + 1) + 2$$

Have a pupil rewrite the second equation, using the standard numeral for the sum $1 + 1 + 1$:

$$29 = (9 \times 1) + (9 \times 1) + (9 \times 1) + 2$$
$$29 = 9 \times (1 + 1 + 1) + 2$$
$$29 = (9 \times 3) + 2$$

Be sure the children realize that the greatest quotient and least remainder for the equation $29 = 9q + r$ have been found. The greatest q is 3 and the least r is 2.

Continue in this way with one or two similar stories. For example:

> Mother puts 8 peaches into every jar of preserves. If mother has 100 peaches, how many jars will she fill, and how many peaches will be left over?

If you use this story you may find that the children will discover the possibility of considering more than 1 jar at a time. It would be appropriate to alternate between considering 2 jars, 3 jars, or 4 jars at a time.

▶ On the chalkboard, write the equation $164 = 13q + r$. Have the pupils take turns in helping to find the greatest q for this equation. The children should build a sequence of equations, using the remainder of each one as the b of the next. Let them continue to form new equations until they obtain an r that is less than 13.

Make it clear that partial quotients greater than 1 may be used.

Perhaps the first pupil will try 10 for q. He might show his work like this:

$$\begin{array}{r} 13 \\ \times\ 10 \\ \hline 130 \end{array} \qquad \begin{array}{r} 164 \\ -\ 130 \\ \hline 34 \end{array}$$

$$164 = 13 \times 10 + 34$$

A second pupil might make the following attempt, making use of the remainder 34:

$$\begin{aligned} 164 &= 13 \times 10 + 34 \\ 34 &= 13 \times \ \ 1 + 21 \end{aligned} \qquad \begin{array}{r} 34 \\ -\ 13 \\ \hline 21 \end{array}$$

Then a third pupil could make this attempt. (He may comment that no more attempts can be made, for his r is less than 13.)

$$\begin{aligned} 164 &= 13 \times 10 + 34 \\ 34 &= 13 \times \ \ 1 + 21 \\ 21 &= 13 \times \ \ 1 + \ \ 8 \end{aligned}$$

Have the class decide on the replacements for q and r in the original equation; choose someone to write the completed equation.

$$164 = 13q + r$$
$$164 = 13 \times 12 + 8$$

Have several children go to the board and check the solution. They should compute the product 13×12 and add the result to 8.

$$\begin{array}{r} 13 \\ \times\ 12 \\ \hline 6 \\ 20 \\ 30 \\ 100 \\ \hline 156 \\ +\ \ 8 \\ \hline 164 \end{array}$$

Use this procedure to find the quotient and remainder for the following equations:

$$\begin{aligned} 189 &= 7q + r & \qquad 205 &= 15q + r \\ 236 &= 14q + r & \qquad 152 &= 8q + r \\ 141 &= 9q + r & \qquad 137 &= 17q + r \end{aligned}$$

▶ On tagboard cards (12 by 15 inches), write a quotient-remainder equation with a sequence of equations that will show its solution for the greatest q and least r. Make one card for each member of the class.

$123 = 6q + r$ $123 = (6 \times 10) + 63$ $63 = (6 \times 10) + 3$	$146 = 17q + r$ $146 = (17 \times 2) + 112$ $112 = (17 \times 5) + 27$ $27 = (17 \times 1) + 10$

Separate the class into two teams. Assign one panel of the chalkboard to each team. One member of each team should select a card and place it on his team's chalktray. Each of these two pupils should be asked to write an equation that shows the greatest quotient q and the least remainder r possible in the quotient-remainder equation on his card. For example, if a team member had the card shown here, he would write the equation $269 = (13 \times 20) + 9$.

$$269 = 13q + r$$
$$269 = (13 \times 10) + 139$$
$$139 = (13 \times 10) + 9$$

The class should decide which pupils completed their assignment correctly, and which pupil was first to do it.

Pupils earn points for their team in the following ways: one point for correctly completing the assignment; one point for being first to complete the assignment correctly.

Continue the activity until every pupil has had an opportunity to participate. Then have each team's points totaled and a winner declared.

Pages 70 through 74

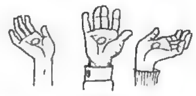
● Page 70 presents a story that leads the children to further investigate the equation $b = aq + r$. As this page is discussed, let the children explain how each step of the solution is related to the previous step. When the children combine these equations to determine the greatest q and the least r, point out that the distributive property is used.

● Page 71 further extends the children's practice in solving equations of the form $b = aq + r$. Let one pupil read aloud the story on the page. Have the class solve and discuss the first exercise. Then assign the remaining exercises to be completed independently. Use this page as a basis for class discussion after the children have completed the exercises. Help the children understand that $(4 \times 2) + (4 \times 3) + (4 \times 2)$ is $4 \times (2 + 3 + 2)$.

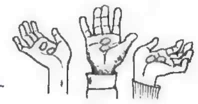
For Class Discussion

Wyatt has 11 jelly beans. He wants to give his jelly beans to his 3 friends. Help Wyatt make a record each time he gives 1 jelly bean to each friend.

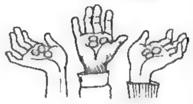
1. Wyatt gives one jelly bean to each of his 3 friends.
   $11 = (3 \times 1) + r$
   How many jelly beans are left in the bag? 8



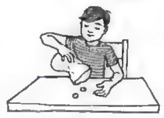
2. Then he gives one more jelly bean to each friend.
   $8 = (3 \times 1) + r$
   Now how many jelly beans are left in the bag? 5



3. Then he gives one more jelly bean to each friend.
   $5 = (3 \times 1) + r$
   How many jelly beans are left in the bag now? 2



4. How many jelly beans did he give to each friend? 3
   $11 = (3 \times 1) + (3 \times 1) + (3 \times 1) +$ 2
   $11 = (3 \times 3) +$ 2
   How many jelly beans are left in the bag? 2



$11 = (3 \times 3) + 2$

5. Complete the equations to show how many times Wyatt gave jelly beans to his friends.
   $11 = (3 \times 1) +$ 8
   $11 = (3 \times 1) + (3 \times 1) +$ 5
   $11 = (3 \times 1) + (3 \times 1) + (3 \times 1) +$ 2
   If $11 = (3 \times$ 1$) + (3 \times$ 1$) + (3 \times$ 1$) +$ 2
   then $11 = 3 \times ($1 + 1 + 1$) +$ 2
   or $11 = (3 \times$ 3$) +$ 2

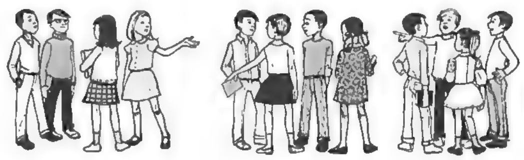
6. The equation
   $11 = 3q + r$
   was used to express the number relation in the story of Wyatt and his jelly beans.
   In $11 = 3q + r$, what is the partial quotient when—
   the remainder is 5? q=2
   the remainder is 2? q=3

reference page

E-70

Name

For Class Discussion



Mrs. Jordan asked the 30 children in her class to form teams with 4 children on each team. Each team would have a booth at the school carnival.

Complete these sentences to determine how many teams were formed.

1. First 2 teams of 4 children were formed. How many children remained? $30 = (4 \times 2) +$ 22
2. From the remaining children, 3 teams of 4 children were formed. How many children remained? $22 = (4 \times 3) +$ 10
3. Then 2 teams of 4 children were formed. How many children remained? $10 = (4 \times 2) +$ 2
4. How many teams of 4 children were formed from Mrs. Jordan's class of 30 children? 7
5. She asked the children who were not on these teams to help her plan the decorations. How many children were not on a team? 2

These equations show the number of teams that were formed.

$$30 = (4 \times 2) + (4 \times 3) + (4 \times 2) + 2$$
$$30 = 4 \times (2 + 3 + 2) + 2$$
$$30 = (4 \times 7) + 2$$

E-71

● Page 72 provides practice in using remainders for building a sequence of equations until the least remainder and the greatest quotient are found. Discuss the example at the top of the page with the class. The equation to be solved is $73 = 4q + r$. First 8 is tried as a replacement for q and the equation $73 = 4 \times 8 + 41$ results. Then the remainder 41 is used to build another equation, $41 = 4q + r$. Then 7 is tried as a replacement for q and this results in the equation $41 = 4 \times 7 + 13$. Ask the class: Where did we get 13? $(28 + 13 = 41)$ Next, the remainder 13 is used to build another equation. Three is used as a replacement for q, resulting in the equation $13 = 4 \times 3 + 1$. Combining all of the equations gives the result:

$$73 = 4 \times (8 + 7 + 3) + 1$$

The following sequence on the chalkboard may help the pupils see how this equation developed:

$73 = (4 \times 8) + 41$	$41 = (4 \times 7) + 13$
$73 = (4 \times 8) + (4 \times 7) + 13$	[substituting $(4 \times 7) + 13$ for the remainder 41]
	$13 = (4 \times 3) + 1$
$73 = (4 \times 8) + (4 \times 7) + (4 \times 3) + 1$	[substituting $(4 \times 3) + 1$ for the remainder 13]
$73 = 4 \times (8 + 7 + 3) + 1$	[using the distributive property]
$73 = (4 \times 18) + 1$	

These equations show that 18 is the greatest q in $73 = 4 \times 18 + 1$.

Then this same equation, $73 = 4q + r$, is solved by first using 9 as the replacement for q. Use the same procedure as discussed previously to discuss this sequence of equations. The pupils should see that the same result is obtained from these equations as was obtained in the first sequence of equations.

$$73 = 4 \times 18 + 1$$

If the teacher feels the pupils are ready, the exercises on page 72 may be assigned for independent work. Otherwise it might be best to complete the page as a class activity.

● Assign some or all of the exercises on page 73 to give the pupils a chance to work on their own. After the children have finished the assignment, let them put their solutions on the chalkboard and discuss their answers.

In the equation $73 = 4q + r$, each remainder may be used to build a new equation.

$73 = 4q + r$	$73 = 4q + r$
$73 = (4 \times 8) + 41$	$73 = (4 \times 9) + 37$
$41 = (4 \times 7) + 13$	$37 = (4 \times 5) + 17$
$13 = (4 \times 3) + 1$	$17 = (4 \times 4) + 1$
$73 = 4 \times (8 + 7 + 3) + 1$	$73 = 4 \times (9 + 5 + 4) + 1$
$73 = (4 \times 18) + 1$	$73 = (4 \times 18) + 1$

Complete each series of equations.

1. $61 = 5q + r$
   - $61 = (5 \times 4) + 41$
   - $41 = (5 \times 5) + 16$
   - $16 = (5 \times 3) + 1$
   - $61 = (5 \times 12) + 1$
2. $180 = 15q + r$
   - $180 = (15 \times 6) + 90$
   - $90 = (15 \times 4) + 30$
   - $30 = (15 \times 2) + 0$
   - $180 = (15 \times 12) + 0$
3. $87 = 6q + r$
   - $87 = (6 \times 9) + 33$
   - $33 = (6 \times 4) + 9$
   - $9 = (6 \times 1) + 3$
   - $87 = (6 \times 14) + 3$
4. $146 = 3q + r$
   - $146 = (3 \times 9) + 119$
   - $119 = (3 \times 8) + 95$
   - $95 = (3 \times 10) + 65$
   - $65 = (3 \times 21) + 2$
   - $146 = (3 \times 48) + 2$
5. $260 = 8q + r$
   - $260 = (8 \times 12) + 164$
   - $164 = (8 \times 15) + 44$
   - $44 = (8 \times 4) + 12$
   - $12 = (8 \times 1) + 4$
   - $260 = (8 \times 32) + 4$
6. $295 = 11q + r$
   - $295 = (11 \times 5) + 240$
   - $240 = (11 \times 10) + 130$
   - $130 = (11 \times 10) + 20$
   - $20 = (11 \times 1) + 9$
   - $295 = (11 \times 26) + 9$

reference page

E-72

Name

Complete.

1. $32 = 2q + r$
   - $32 = (2 \times 2) + 28$
   - $28 = (2 \times 2) + 24$
   - $24 = (2 \times 12) + 0$
   - $32 = (2 \times 16) + 0$
2. $24 = 3q + r$
   - $24 = (3 \times 1) + 21$
   - $21 = (3 \times 2) + 15$
   - $15 = (3 \times 5) + 0$
   - $24 = (3 \times 8) + 0$
3. $39 = 2q + r$
   - $39 = (2 \times 9) + 21$
   - $21 = (2 \times 5) + 11$
   - $11 = (2 \times 5) + 1$
   - $39 = (2 \times 19) + 1$
4. $41 = 2q + r$
   - $41 = (2 \times 20) + 1$
   - $41 = (2 \times 20) + 1$
5. $32 = 2q + r$
   - $32 = (2 \times 16) + 0$
   - $32 = (2 \times 16) + 0$
6. $24 = 3q + r$
   - $24 = (3 \times 8) + 0$
   - $24 = (3 \times 8) + 0$
7. $31 = 2q + r$
   - $31 = (2 \times 10) + 11$
   - $11 = (2 \times 5) + 1$
   - $31 = (2 \times 15) + 1$
8. $53 = 2q + r$
   - $53 = (2 \times 20) + 13$
   - $13 = (2 \times 6) + 1$
   - $53 = (2 \times 26) + 1$
9. $125 = 5q + r$
   - $125 = (5 \times 20) + 25$
   - $25 = (5 \times 5) + 0$
   - $125 = (5 \times 25) + 0$
10. $67 = 7q + r$
    - $67 = (7 \times 5) + 32$
    - $32 = (7 \times 3) + 11$
    - $11 = (7 \times 1) + 4$
    - $67 = (7 \times 9) + 4$
11. $95 = 8q + r$
    - $95 = (8 \times 8) + 31$
    - $31 = (8 \times 2) + 15$
    - $15 = (8 \times 1) + 7$
    - $95 = (8 \times 11) + 7$
12. $378 = 25q + r$
    - $378 = (25 \times 10) + 128$
    - $128 = (25 \times 3) + 53$
    - $53 = (25 \times 2) + 3$
    - $378 = (25 \times 15) + 3$

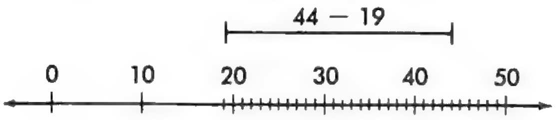
E-73

Supplemental Experiences

■ The following exercises may be written on the chalkboard and used to practice applying the distributive property of multiplication over addition.

$(3 \times 7) + (3 \times 2) + 2 = 3 \times (\_\_ + \_\_) + \_\_$
$(8 \times 1) + (8 \times 5) + 5 = 8 \times (\_\_ + \_\_) + \_\_$
$(9 \times 1) + (9 \times 2) + (9 \times 4) + 6 = 9 \times \_\_ + \_\_$
$(4 \times 5) + (4 \times 2) + (4 \times 1) + 3 = 4 \times \_\_ + \_\_$
$(6 \times 1) + (6 \times 2) + (6 \times 3) + 3 = 6 \times \_\_ + \_\_$
$(7 \times 2) + (7 \times 1) + 5 = 7 \times \_\_ + \_\_$

■ The pupils may enjoy exploring the following addition approach to computing differences. On the chalkboard, write the difference 44 − 19 in vertical form. Above this difference, draw a number line to show numbers from 0 through 50. Mark the distance from 19 to 44, which is the difference 44 − 19.



$$\begin{array}{r} 44 \\ -\ 19 \\ \hline \end{array}$$

Let the pupils suggest ways to cover the distance from 19 to 44. Have these ways pointed out on the number line and write these suggestions beneath the difference as they are given. Someone may say: add 1, add 10, add 10, add 4. By totaling these steps from 19 to 44, this pupil will find the computed difference of 44 and 19.

$$\begin{array}{r} 44 \\ -\ 19 \\ \hline 1 \\ 10 \\ 10 \\ 4 \\ \hline 25 \end{array}$$

Someone else may suggest that another way to cover this distance is to add 20, add 1, and add 4. When he totals his steps, this pupil will also find that he has computed the difference between 44 and 19.

Have other children suggest other ways, point out their steps on the number line, and then let them complete the algorism.

Follow a similar procedure with differences related to sums of three-digit and four-digit addends.

KEY IDEA

There is always a greatest partial quotient.

Scope

To practice finding the greatest partial quotient.

Fundamentals

In building a sequence of equations directed at finding the greatest partial quotient, it is important to notice the freedom allowed in choosing a quotient, q, for each of the equations in the sequence. Consider the equation, $119 = 9q + r$, and the different sequences A and B.

A

$$\begin{aligned} 119 &= (9 \times 7) + 56 \\ 56 &= (9 \times 2) + 38 \\ 38 &= (9 \times 4) + 2 \\ 119 &= 9 \times (7 + 2 + 4) + 2 \\ 119 &= (9 \times 13) + 2 \end{aligned}$$

B

$$\begin{aligned} 119 &= (9 \times 5) + 74 \\ 74 &= (9 \times 5) + 29 \\ 29 &= (9 \times 3) + 2 \\ 119 &= 9 \times (5 + 5 + 3) + 2 \\ 119 &= (9 \times 13) + 2 \end{aligned}$$

Note that the greatest quotient (13) may be found from either sequence.

In building the sequence of equations, the pupil may choose any partial quotients that he wishes. Through experience, he will gradually discover how to make more efficient choices.

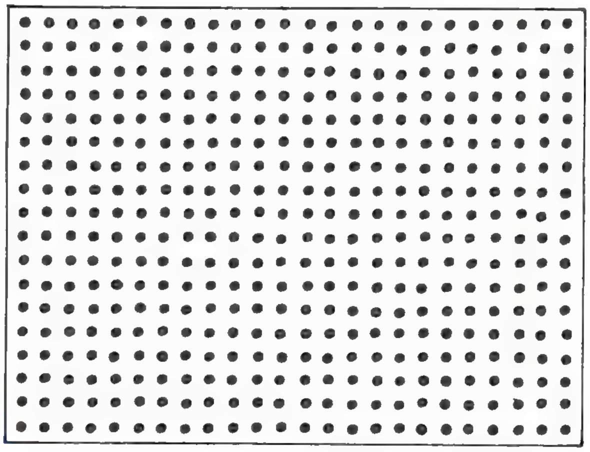
Readiness for Understanding

Knowledge of basic multiplication facts.
Ability to subtract.

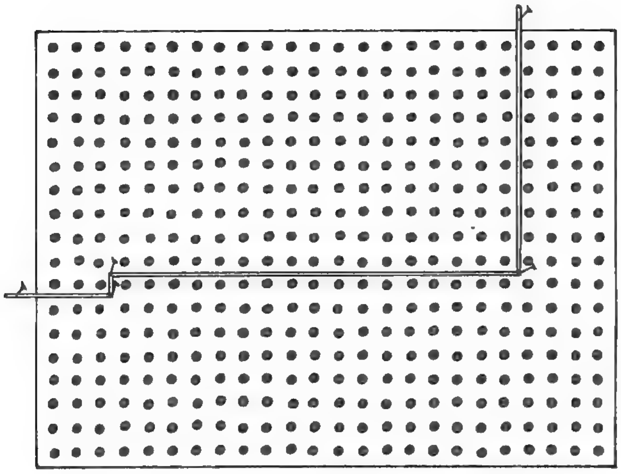
Developmental Experiences

circle stickers
sheet of tagboard (24″ × 30″)
pins
yarn
masking tape
felt-tip pen

► Using circle stickers, make an 18 by 24 array on a 24 by 30 inch sheet of tagboard. Fasten this sheet of tagboard to the bulletin board.



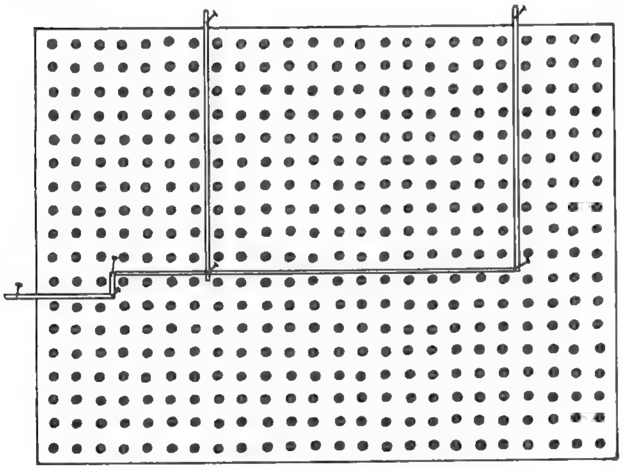
Write the equation $203 = 13q + r$ on the chalkboard. Help one pupil show the meaning of 203, by fastening a piece of yarn to the model to outline 200 + 3.



Ask another pupil to come to the chalkboard and choose a first partial quotient. He could choose 10. Multiplying 13 and 10, he should then compute the difference between this number (130) and 203. He may write the following on the chalkboard as a record of this attempt:

$$\begin{aligned} 203 &= 13q + r \\ 203 &= (13 \times 10) + 73 \end{aligned} \qquad \begin{array}{r} 203 \\ \underline{-\ 130} \\ 73 \end{array}$$

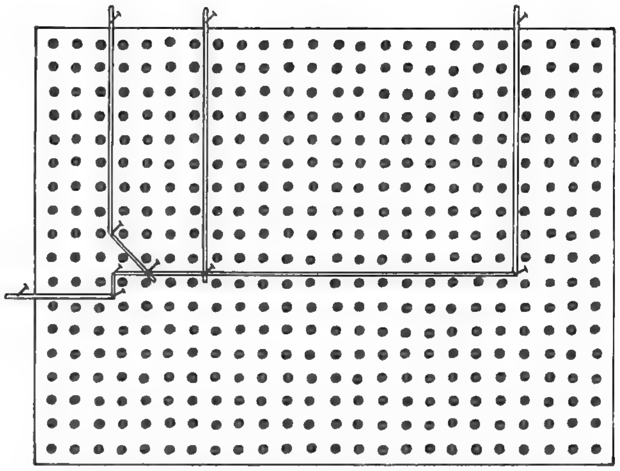
Then ask him to show this attempt at solution on the tagboard model. He would pin a piece of yarn in place like this:



Ask another member of the class to use the remainder in the first pupil's equation, and attempt to come closer to the greatest quotient in the original equation, $203 = 13q + r$. This pupil might use 3 as a partial quotient and then compute the difference between 13×3 and 73. He might write the following as a record of these efforts:

$$\begin{aligned} 203 &= 13q + r \\ 203 &= (13 \times 10) + 73 \\ 73 &= (13 \times 3) + 34 \end{aligned} \qquad \begin{array}{r} 13 \\ \underline{\times\ 3} \\ 9 \\ \underline{30} \\ 39 \end{array} \qquad \begin{array}{r} 73 \\ \underline{-\ 39} \\ 34 \end{array}$$

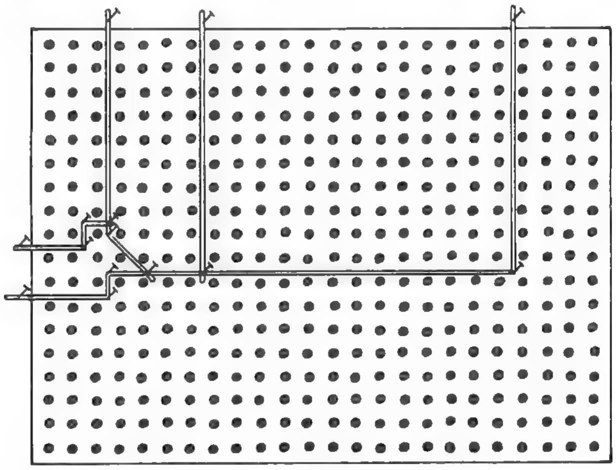
Another member of the class can show this attempt at solution on the tagboard.



Then ask another pupil to use the remainder 34 and attempt to come closer to the greatest quotient, and to complete the search for it if he can. Perhaps this pupil will try 2 as a partial quotient and then compute the difference between 13×2 and 34. He then would write an equation that shows his attempt at solving for the greatest q:

$$\begin{aligned} 203 &= 13q + r \\ 203 &= (13 \times 10) + 73 \\ 73 &= (13 \times 3) + 34 \\ 34 &= (13 \times 2) + 8 \end{aligned} \qquad \begin{array}{r} 13 \\ \underline{\times\ 2} \\ 6 \\ \underline{20} \\ 26 \end{array} \qquad \begin{array}{r} 34 \\ \underline{-\ 26} \\ 8 \end{array}$$

Have this attempt at solution shown on the tagboard.



Ask the class whether the greatest partial quotient has been found. Have them tell what number 13 should be multiplied by in the solution of the equation $203 = 13q + r$ ($10 + 3 + 2$, or 15). Ask them to give the remainder (8). Then call on a pupil to write an equation that shows these replacements for q and r in the equation $203 = 13q + r$.

$$\begin{aligned} 203 &= 13q + r \\ 203 &= (13 \times 10) + 73 \\ 73 &= (13 \times 3) + 34 \\ 34 &= (13 \times 2) + 8 \\ 203 &= (13 \times 15) + 8 \end{aligned}$$

Remove all pieces of yarn from the bulletin board. Continue, in the way suggested in the preceding para-

graphs, to have the pupils solve for the greatest q in equations such as these:

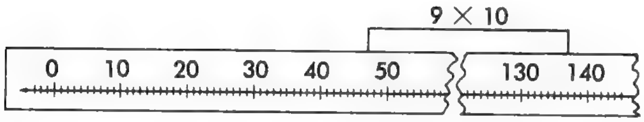
$$196 = 14q + r \qquad 325 = 25q + r$$
$$297 = 18q + r \qquad 352 = 16q + r$$
$$153 = 13q + r \qquad 263 = 19q + r$$

► Fasten a 6-foot strip of 1-inch masking tape to the chalkboard. From left to right through the center of this strip, draw a 6-foot line segment. On this segment, mark points for numbers from 0 through 280. (A felt-tip pen works well for this.) Begin the first point $\frac{1}{2}$-inch in from the left edge of the tape. Use $\frac{1}{4}$-inch intervals between each point and label only the multiples of 10.

On the chalkboard, write the equation $137 = 9q + r$. Have pupils help solve this equation for the greatest quotient. The first child could make the following attempt:

$$137 = (9 \times 10) + 47 \qquad \begin{array}{r} 137 \\ -\ \ 90 \\ \hline 47 \end{array}$$

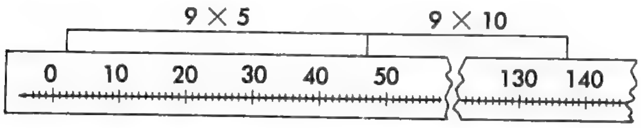
Help a child show this attempt on the number line:



Ask another child to work toward the solution; he could use the remainder 47.

$$137 = (9 \times 10) + 47$$
$$47 = (9 \times 5) + 2$$

Have this attempt shown on the number line:



At this point the class may observe that the greatest quotient has been found. Have them tell what number 9 should be multiplied by in the solution of the equation $137 = 9q + r$ (10 + 5, or 15). Have them tell what the remainder is (2). Call on a pupil to write an equation showing these replacements for q and r.

$$137 = (9 \times 10) + 47$$
$$47 = (9 \times 5) + 2$$
$$137 = (9 \times 15) + 2$$

Erase all drawings from the chalkboard. Continue in the way suggested in the preceding paragraphs to let the pupils show models of their solution of equations such as the following, using both the number line and a sequence of equations:

$$235 = 15q + r \qquad 261 = 21q + r$$
$$177 = 13q + r \qquad 194 = 14q + r$$
$$253 = 23q + r \qquad 279 = 19q + r$$

► The pupils are now ready to solve quotient-remainder equations ($b = aq + r$) in which multiples of 100 are involved. On the chalkboard write the equation $2396 = 19q + r$. Have the pupils take turns in helping to solve this equation for the greatest q. You should suggest that whenever a pupil feels that it is possible to use 100 as a partial quotient, he should do so. After each equation is formed, its remainder is used in building a new equation. Continue having the remainders used to form new equations until r is less than 19.

Perhaps the first child uses 100 as a partial quotient. His computation may look like this:

$$2396 = (19 \times 100) + 496 \qquad \begin{array}{r} 2396 \\ -\ 1900 \\ \hline 496 \end{array}$$

The second pupil works with the remainder, 496. He might choose 10 as his partial quotient:

$$\begin{aligned} 2396 &= (19 \times 100) + 496 \\ 496 &= (19 \times 10) + 306 \end{aligned} \qquad \begin{array}{r} 496 \\ -\ 190 \\ \hline 306 \end{array}$$

A third pupil may make the following attempt:

$$\begin{aligned} 2396 &= (19 \times 100) + 496 \\ 496 &= (19 \times 10) + 306 \\ 306 &= (19 \times 10) + 116 \end{aligned} \qquad \begin{array}{r} 306 \\ -\ 190 \\ \hline 116 \end{array}$$

The attempt to find the greatest q may be completed in two further steps:

$$\begin{aligned} 2396 &= (19 \times 100) + 496 \\ 496 &= (19 \times 10) + 306 \\ 306 &= (19 \times 10) + 116 \\ 116 &= (19 \times 5) + 21 \\ 21 &= (19 \times 1) + 2 \end{aligned}$$

Now the class can give the replacements for q and r in the original equation, $2396 = 19q + r$. Call on a pupil to write an equation showing these replacements:

$$2396 = (19 \times 126) + 2$$

To check this result, the class can compute the product 19×126 and then add 2:

$$\begin{array}{r} 126 \\ \times\ \ 19 \\ \hline 54 \\ 180 \\ 900 \\ 60 \\ 200 \\ \underline{1000} \\ 2394 \\ +\ \ \ 2 \\ \hline 2396 \end{array}$$

Adapt this procedure to other equations such as the following:

$$3797 = 23q + r \qquad 1201 = 11q + r$$
$$4952 = 37q + r \qquad 2050 = 15q + r$$

▶ Write the following story exercises on the chalkboard.

(1) Mrs. Jamison had 126 apples. She gave them to 14 hungry boys. If each eats the same number of apples, and all the apples are eaten, how many will each boy eat?

(2) Jerry has saved 107 pennies. He needs dimes to make phone calls. How many dimes can he get for his pennies?

(3) Mr. Williams is packing eggs into cartons. He has 160 eggs. If each carton holds 12 eggs, how many cartons can he fill?

Ask the children to express an equation of the form $b = aq + r$ for each story. Write these equations on the chalkboard.

$$\begin{array}{ll} (1) & 126 = 14q + r \\ (2) & 107 = 10q + r \\ (3) & 160 = 12q + r \end{array}$$

For each equation, let the class suggest partial quotients that lead to a solution for q and r. Write each suggested solution on the chalkboard and build a sequence of equations until the greatest q has been found. Then have someone write the original equation, showing the appropriate replacements for q and r.

Pages 74 through 80

● Use page 74 as a guide for class discussion. This page is designed to focus the pupils' attention on the fact that there is more than one way of identifying the greatest q and least r in the equation $b = aq + r$. Give the children sufficient time to read and comprehend the situation that is presented. Then discuss Steve's approach to the problem and Marie's approach to the problem. Be sure to let the pupils do all the explaining, even though their language may not be precise. Next, discuss questions 1, 2, and 3 at the bottom of the page. The pupils should realize that Steve's and Marie's solutions are alike in that they both arrived at the same solution to the equation $114 = 9q + r$; $q = 12$, $r = 6$. Their solutions are different in that they used different partial quotients in arriving at the solution.

Finally, ask the pupils to independently prepare a solution for $59 = 14q + r$. Different ways of determining the solutions should be put on the chalkboard and compared.

● Page 75 gives the children practice in finding the quotient. Discuss the examples at the top of the page with the class. Be sure the pupils are aware that the solution to each equation could have been found in other ways.

In the process of building equations to arrive at the quotient, some pupils will be able to write the remainder without showing any computational steps. Other pupils will find it necessary to compute the remainder on paper; this may turn out to be helpful when the division algorism is presented in the next unit. When the algorism is introduced, it can be related to the computation that the pupils may have done in connection with building a sequence of equations.

The exercises on page 75 may be assigned for independent work; the teacher must decide. Perhaps it would be well to assign the easier exercises to the slower pupils since there is no need for every equation to be solved by each pupil. Also, the number of steps taken in the solution will vary from pupil to pupil. A word of caution is necessary. Some pupils may do most of their work on scratch paper and find the solution to an equation such as $119 = 7q + r$ in one step. Some of them have the impression that this, the "answer," is what is best and what the teacher really wants. Explain to your pupils that you are not only interested in the final solution, but also in the steps that led to that solution. As a child continues to build sequences of quotient-remainder equations, he will gradually discover how to reduce the number of steps in his solution.

● Page 76 provides more practice in working with equations of the form $b = aq + r$ and in writing equations of this form for story exercises. For exercises 7 through 10, the pupils are to write only a quotient-remainder equation; they need not solve the equations. It is most important to ascertain whether pupils can see the structure in each story, and the equations will reveal this. When you are sure the pupils understand the procedure, assign the page for independent work.

● Page 77 provides further practice in applying equations of the form $b = aq + r$ to story exercises. First direct the pupils to complete exercises 1 through 6 independently. Check their answers. Then have them read exercise 7. Ask a pupil to tell how he will answer the questions (Mr. Seller can fill 4 boxes; the remaining 5 cans will be placed in another box; 5 boxes will hold all 45 cans). Help the pupils realize how important it is to read the story carefully and to understand the number relationship it describes. Finally, ask the pupils which quotient-remainder equation at the top of the page describes the relationship in the story (the equation in exercise 5). Follow a similar procedure with the rest of the story exercises on page 77.

For Class Discussion

Mrs. Jordan wrote this equation on the chalkboard.

$$114 = 9q + r$$

She asked the children to show how they would find the quotient and the remainder.

This is how Steve worked the exercise.

$114 = (9 \times 2) + 96$
$96 = (9 \times 2) + 78$
$78 = (9 \times 2) + 60$
$60 = (9 \times 2) + 42$
$42 = (9 \times 2) + 24$
$24 = (9 \times 2) + 6$

$114 = 9 \times (2 + 2 + 2 + 2 + 2 + 2) + 6$
$114 = (9 \times 12) + 6$

This is how Marie worked the exercise.

$114 = (9 \times 3) + 87$
$87 = (9 \times 2) + 69$
$69 = (9 \times 4) + 33$
$33 = (9 \times 3) + 6$
$6 = (9 \times 0) + 6$

$114 = 9 \times (3 + 2 + 4 + 3 + 0) + 6$
$114 = (9 \times 12) + 6$

1. How is Marie's solution like Steve's solution? The greatest partial quotients are the same.
2. How is Marie's solution different from Steve's solution? The partial quotients are different.
3. Find the quotient and least remainder for $59 = 14q + r$.

   $59 = (14 \times \underline{4}) + \underline{3}$

Name

We can build new equations until the quotient is found.

$83 = 11q + r$
$83 = (11 \times 3) + 50$
$50 = (11 \times 2) + 28$
$28 = (11 \times 2) + 6$
$6 = (11 \times 0) + 6$

$83 = 11 \times (3 + 2 + 2) + 6$
$83 = (11 \times 7) + 6$

$586 = 14q + r$
$586 = (14 \times 10) + 446$
$446 = (14 \times 20) + 166$
$166 = (14 \times 10) + 26$
$26 = (14 \times 1) + 12$

$586 = 14 \times (10 + 20 + 10 + 1) + 12$
$586 = (14 \times 41) + 12$

Build new equations until the quotient is found. Equations may vary.

1. $102 = 17q + r$
   $102 = (17 \times \underline{3}) + \underline{51}$
   $51 = (17 \times 2) + 17$
   $17 = (17 \times 1) + 0$
   $102 = 17 \times (3 + 2 + 1) + 0$
   $102 = (17 \times 6) + 0$
2. $59 = 13q + r$
   $59 = (13 \times 4) + 7$
3. $81 = 11q + r$
   $81 = (11 \times 7) + 4$
4. $99 = 14q + r$
   $99 = (14 \times 7) + 1$

Copy and build new equations.

5. $98 = 6q + r$ — $98 = (6 \times 16) + 2$
6. $142 = 8q + r$ — $142 = (8 \times 17) + 6$
7. $119 = 7q + r$ — $119 = (7 \times 17) + 0$
8. $165 = 9q + r$ — $165 = (9 \times 18) + 3$
9. $117 = 4q + r$ — $117 = (4 \times 29) + 1$
10. $56 = 3q + r$ — $56 = (3 \times 18) + 2$
11. $101 = 5q + r$ — $101 = (5 \times 20) + 1$
12. $193 = 9q + r$ — $193 = (9 \times 21) + 4$
13. $39 = 2q + r$ — $39 = (2 \times 19) + 1$
14. $897 = 10q + r$ — $897 = (10 \times 89) + 7$
15. $347 = 13q + r$ — $347 = (13 \times 26) + 9$
16. $1893 = 17q + r$ — $1893 = (17 \times 111) + 6$

Solve each equation.

1. $17 = 3q + r$ — $q = \underline{5}$, $r = \underline{2}$
2. $97 = 2q + r$ — $q = \underline{48}$, $r = \underline{1}$
3. $22 = 6q + r$ — $q = \underline{3}$, $r = \underline{4}$
4. $83 = 8q + r$ — $q = \underline{10}$, $r = \underline{3}$
5. $63 = 7q + r$ — $q = \underline{9}$, $r = \underline{0}$
6. $48 = 9q + r$ — $q = \underline{5}$, $r = \underline{3}$

Write a quotient-remainder equation for each exercise.

7. Mr. Barney put 950 fishhooks into small boxes. Each box held 200 fishhooks. How many boxes did he fill and how many fishhooks were left? $950 = 200q + r$
8. He dug up 379 angleworms. He put 17 worms in each can. How many cans of worms did he have and how many worms were left? $379 = 17q + r$
9. He rents 48 fishing poles to fishermen. He puts them in racks that hold 9 poles each. How many racks can he fill and how many poles are left? $48 = 9q + r$
10. He put 980 minnows into containers that held 80 minnows each. How many containers did he use and how many minnows were left? $980 = 80q + r$

Name

Solve each equation.

1. $24 = 6q + r$ — $24 = (6 \times \underline{4}) + \underline{0}$
2. $41 = 15q + r$ — $41 = (15 \times \underline{2}) + \underline{11}$
3. $36 = 8q + r$ — $36 = (8 \times \underline{4}) + \underline{4}$
4. $320 = 60q + r$ — $320 = (60 \times \underline{5}) + \underline{20}$
5. $45 = 10q + r$ — $45 = (10 \times \underline{4}) + \underline{5}$
6. $136 = 8q + r$ — $136 = (8 \times \underline{17}) + \underline{0}$

Answer each question.

7. The Seller family is going camping. Mr. Seller wants to pack 45 cans of food in boxes that hold 10 cans each.
   a. How many boxes can Mr. Seller fill? 4
   b. How many cans will remain unpacked? 5
   c. How many boxes will hold all the cans? 5
8. Mrs. Town fried 24 slices of bacon. She gave each of her 6 children the same number of slices.
   a. What is the most bacon she can give each child? 4 slices
   b. How much bacon will be left? none
9. Thirty-six people want to go to the sixth floor of an apartment building. The elevator cannot carry more than eight people.
   a. How many trips can be made with the elevator full of people? 4
   b. How many people will remain? 4
   c. How many trips are needed to take all of the people to the sixth floor? 5

10. It took 320 minutes to drive from Green Grove to Tampa. (One hour is 60 minutes.)
    It took 5 hours and how many minutes to drive from Green Grove to Tampa? 20

● Page 78 gives the pupils a chance to test their ability to find the greatest partial quotient. Assign the exercises for independent work. Each pupil should feel free to use as many or as few steps as he finds necessary to arrive at this quotient. The pupils may notice that an equation that describes the number relationship in each story exercise is at the top of the page. They will find that the solution of the appropriate equation will give the answers to the story questions.

● On pages 79 and 80, solutions to equations are shown on the number line. The purpose of using the number line is to relate the remainder of one equation in a sequence to the b in the next equation of the sequence. You may want to review the concept of a number line before using these pages.

Explain to the children that the last number line on page 79 pictures an equation that Mrs. Blake had solved, and that each of the other number lines pictured shows an equation in the sequence that she used to solve $112 = 3q + r$. For the first number line on the page, tell the children to start at the right and follow the arrow from 112 to 82. Relate this portion of the number line to the equation $112 = 3 \times 10 + 82$. The children should understand that Mrs. Blake chose 10 as her first partial quotient. For the second number line, point out that the arrow begins at 82, the remainder from the previous equation in the sequence. Follow the procedure outlined above for each of the remaining number lines and equations in the sequence. The children should observe that all of the equations in the sequence are shown on the last number line.

Work the first exercise on page 80 with the children. Some children may need the teacher's help to complete the exercises; others can complete them independently. After the pupils have completed the assignment, let several of them draw on the chalkboard the number lines they used to solve the equation in exercise 5.

What is the quotient and remainder?

1. $33 = 2q + r$ $q =$ 16 ; $r =$ 1	2. $204 = 20q + r$ $q =$ 10 ; $r =$ 4	3. $27 = 4q + r$ $q =$ 6 ; $r =$ 3
4. $53 = 16q + r$ $q =$ 3 ; $r =$ 5	5. $305 = 50q + r$ $q =$ 6 ; $r =$ 5	6. $75 = 25q + r$ $q =$ 3 ; $r =$ 0
7. $35 = 3q + r$ $q =$ 11 ; $r =$ 2	8. $136 = 8q + r$ $q =$ 17 ; $r =$ 0	9. $254 = 50q + r$ $q =$ 5 ; $r =$ 4
10. $47 = 2q + r$ $q =$ 23 ; $r =$ 1	11. $111 = 13q + r$ $q =$ 8 ; $r =$ 7	12. $280 = 9q + r$ $q =$ 31 ; $r =$ 1

Write a sentence to answer each question.

13. The McKinley School picnic was attended by 305 children. They were taken to the picnic in buses. Each bus could take 50 children. How many buses were used to take the children to the picnic?
7 buses were used.

14. Mrs. Brown made sandwiches with 2 slices of ham in each sandwich. How many sandwiches did she make with 33 slices of ham and how many slices were left?
She made 16 sandwiches. 1 slice was left.

15. Miss Frazer's class demonstrated a new dance called "Polka for Three." It is a dance done in groups of 3. How many groups can be made from Miss Frazer's class of 35 children?
11 groups can be made.

16. Twenty-seven children at the picnic wanted to run relay races. Four children were chosen for each relay team. How many teams were formed?
6 teams were formed.

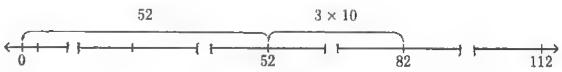
E-78

Name

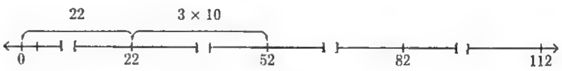
For Class Discussion

Mrs. Blake wanted to solve the equation $112 = 3q + r$. She used the number line and this sequence of equations to show her solution.

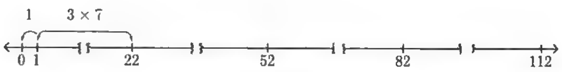
$112 = (3 \times 10) + 82$



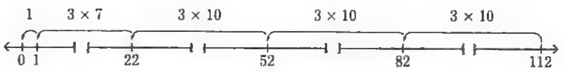
$82 = (3 \times 10) + 52$



$52 = (3 \times 10) + 22$



$22 = (3 \times 7) + 1$

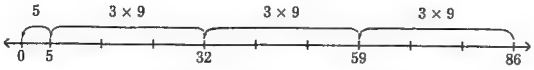


$112 = (3 \times 10) + (3 \times 10) + (3 \times 10) + (3 \times 7) + 1$
$112 = 3 \times (10 + 10 + 10 + 7) + 1$
$112 = (3 \times 37) + 1$

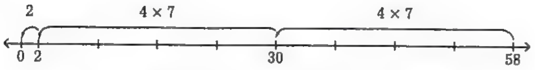
reference page

E-79

Complete the equation shown by the number line.

1. 

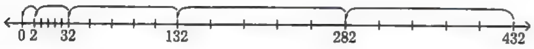
86 = (3 × 9) + (3 × 9) + (3 × 9) + 5

2. 

58 = (4 × 7) + (4 × 7) + 2

3. 

326 = (7 × 30) + (7 × 10) + (7 × 6) + 4

4. 

432 = (5 × 30) + (5 × 30) + (5 × 20) + (5 × 6) + 2

5. On a number line, show the steps you use to solve $113 = 6q + r$.

Diagrams may vary.

reference page

E-80

Supplemental Experiences

■ Write the following sentences on the chalkboard. Have the pupils copy and complete the sentences. Tell them that they are to name the greatest multiple of 10 that will make each sentence true.

____ × 9 is less than 150	(10)
____ × 3 = 60	(20)
____ × 5 is less than 110	(20)
____ × 8 is less than 260	(30)
____ × 10 = 200	(20)
____ × 7 is less than 90	(10)
____ × 10 = 300	(30)
____ × 7 is less than 300	(40)
____ × 20 = 200	(10)
____ × 20 = 400	(20)
____ × 30 = 600	(20)
____ × 6 is less than 340	(50)

■ Separate the class into 2 teams. Assign one panel of the chalkboard to one team and one panel to the other. Call a member of each team forward. In one of the panels of the chalkboard, write $413 = 3q + r$; in the other, write $583 = 4q + r$. Tell the first member of each team to start the process of finding the greatest quotient for his equation. As soon as these two pupils have completed the first step in the solution, they are to return to their seats. The next member of each team should then go to the chalkboard and continue the process. The number of pupils from each team who will be involved in the solution of each equation will vary since the number of steps that each team uses will vary.

Points are earned in the following ways: 1 point for completing the assignment correctly. 1 point for being first to correctly complete the assignment.

Continue in this way to have each team find the greatest quotient for specific equations, until all members of the class have participated in the activity. Then total each team's points and declare a winner.

■ At this time the teacher may wish to have each pupil take a short written quiz in order to ascertain which pupils need additional practice. The following suggested quiz may be used for this purpose.

SUGGESTED QUIZ

1. Complete each sentence.

$67 = 6q + r$
$67 = (6 \times 4) + \underline{43}$
$\underline{43} = (6 \times 4) + \underline{19}$
$\underline{19} = (6 \times 3) + \underline{1}$
This shows that $67 = 6 \times (\underline{4} + \underline{4} + \underline{3}) + \underline{1}$
and $67 = (\underline{6} \times \underline{11}) + \underline{1}$

2. Build new equations until the greatest quotient is found.

(a) $40 = 25q + r$
$\underline{40} = (\underline{25} \times \underline{1}) + \underline{15}$

(b) $209 = 10q + r$
$\underline{209} = (\underline{10} \times \underline{20}) + \underline{9}$

3. Write and solve an equation for the story. Then write a sentence to answer the question.

Mr. Scott is packing cans into boxes. Each box holds 9 cans. How many boxes can he fill if he has 114 cans?

$114 = 9q + r$
$114 = (9 \times 12) + 6$

He can fill 12 boxes.

UNIT 7
THE DIVISION ALGORISM
Pages 81 Through 92

OBJECTIVE

To develop the division algorism.

The child learns that the division algorism is used in solving quotient-remainder equations. He observes that the algorism provides a convenient way to find the greatest quotient and least remainder.

See Key Topics in Mathematics for the Intermediate Teacher: Multiplication and Division of Whole Numbers.

KEY IDEAS

The quotient is a sum of partial quotients.
$242 = 24 \times 10 + 2$.

KEY IDEA

The quotient is a sum of partial quotients.

Scope

To relate quotient-remainder equations to the division algorism.

Fundamentals

The missing-factor equation $ax = b$ does not always have a whole-number solution. When $ax = b$ does not have a solution, a different question is asked. This question is represented by a quotient-remainder equation. To find the greatest quotient, build a sequence of equations, stopping the sequence when r becomes less than a. For example, consider $71 = 4q + r$.

$$71 = (4 \times 10) + 31$$
$$31 = (4 \times 5) + 11$$
$$11 = (4 \times 2) + 3$$
$$3 < 4$$

There are many possible choices when selecting the partial quotient, q, of each equation. Observe that the greatest q for the equation $71 = 4q + r$ is the sum of 10, 5, and 2 [$71 = (4 \times 17) + 3$]. Thus the choices for q could have been any whole numbers whose sum is 17.

The sequence of equations which eventually results in the greatest quotient exhibits the rationale of the procedure. However, the computation involved in solving each equation for the remainder or new "b" is often too difficult to do without completing separately a multiplication and a subtraction algorism.

The division algorism provides for a rearrangement of the parts of the sequence of equations into a computational scheme with the numerals so arranged that the multiplication and subtraction algorisms can be conveniently completed. No new concepts are needed.

Notice the relationship between the quotient-remainder equation and the division algorism.

	4) 71		
	40	10	$(71 = 4 \times 10 + 31)$
(First remainder)	31		
	20	5	$(31 = 4 \times 5 + 11)$
(Second remainder)	11		
	8	2	$(11 = 4 \times 2 + 3)$
(Least remainder)	3	17	(greatest quotient)

Either in building a sequence of equations or in using the division algorism, any partial quotients with the sum of 17 will lead to the greatest partial quotient. In the next example, two different sets of partial quotients are used to obtain the greatest quotient.

33) 892		
330	10	$(892 = 33 \times 10 + 562)$
562		
330	10	$(562 = 33 \times 10 + 232)$
232		
165	5	$(232 = 33 \times 5 + 67)$
67		
66	2	$(67 = 33 \times 2 + 1)$
1	27	(greatest quotient)

33) 892		
660	20	$(892 = 33 \times 20 + 232)$
232		
231	7	$(232 = 33 \times 7 + 1)$
1	27	(greatest quotient)

It is important to let the pupil choose partial quotients freely. As he gains experience, he will become more efficient.

Readiness for Understanding

Knowledge of multiplication and subtraction.

Developmental Experiences

sheet of tagboard (18″ × 24″)
tagboard strips ($\frac{1}{4}$″ × 2″)
felt-tip pen
stapler
pins

► Provide practice for the children in computing products that involve multiplication by tens and by hundreds. Read each exercise aloud to the class and then call on a child to give the result of his computation.

3 tens × 9	6 hundreds × 9
4 tens × 7	7 hundreds × 5
2 tens × 12	9 hundreds × 8
5 tens × 6	4 hundreds × 6
8 tens × 8	8 hundreds × 3

92 × 10
47 × 10
65 × 100
71 × 100

▶ Introduce the class to the division algorism and help them realize that this algorism is a more efficient way to find the greatest quotient and least remainder.

Write $59 = 8q + r$ on the chalkboard. Ask a child to find the quotient and remainder. He may immediately choose 7 or he may begin with a partial quotient such as 2. Ask him to show his computation by writing quotient-remainder equations.

$$\begin{array}{ll} 59 = 8q + r & \\ 59 = (8 \times 7) + 3 & \begin{array}{r} 59 \\ \underline{-\ 56} \\ 3 \end{array} \end{array} \qquad \begin{array}{ll} 59 = 8q + r & \\ 59 = (8 \times 2) + 43 & \begin{array}{r} 59 \\ \underline{-\ 16} \\ 43 \end{array} \end{array}$$

If this child chose a factor less than 7, have him continue until he reaches the greatest quotient. For example, if he chose 2 and then 5:

$$\begin{array}{l} 59 = 8q + r \\ 59 = (8 \times 2) + 43 \\ 43 = (8 \times 5) + 3 \\ 59 = 8 \times (2 + 5) + 3 \\ 59 = (8 \times 7) + 3 \end{array} \qquad \begin{array}{r} 59 \\ \underline{-\ 16} \\ 43 \end{array} \qquad \begin{array}{r} 43 \\ \underline{-\ 40} \\ 3 \end{array}$$

Next to the child's equations, write the division algorism to represent this computation. After you have worked the algorism, ask the child to point to each partial quotient (2, 5), the remainders (43, 3), and the quotient (7).

$$\begin{array}{r|l} 8\overline{)\ 59} & \\ \underline{16} & 2 \\ 43 & \\ \underline{40} & \underline{5} \\ 3 & 7 \end{array} \qquad \begin{array}{l} 59 = 8q + r \\ 59 = (8 \times 2) + 43 \\ 43 = (8 \times 5) + 3 \\ 59 = (8 \times 7) + 3 \end{array}$$

Continue this activity using other examples such as: $47 = 6q + r$, $33 = 4q + r$, $21 = 7q + r$, $85 = 9q + r$, $105 = 10q + r$.

▶ When the children have achieved some facility finding quotients and using the division algorism, write $226 = 7q + r$ on the chalkboard. Suggest that some child find the quotient by testing a partial quotient of 10. He may write the following equations.

$$\begin{array}{l} 226 = 7q + r \\ 226 = (7 \times 10) + 156 \end{array} \qquad \begin{array}{r} 226 \\ \underline{-\ \ 70} \\ 156 \end{array}$$

Next to these equations, ask him to write the division algorism to represent the same computations.

$$\begin{array}{l} 226 = 7q + r \\ \\ 226 = (7 \times 10) + 156 \end{array} \qquad \begin{array}{r} 226 \\ \underline{-\ \ 70} \\ 156 \end{array} \qquad \begin{array}{r|l} 7\overline{)\ 226} & \\ \underline{70} & 10 \\ 156 & \end{array}$$

Call on another child to find another partial quotient using the remainder 156. He may also suggest 10; ask him to show his equation and to then add to the division algorism.

$$156 = (7 \times 10) + 86 \qquad \begin{array}{r} 156 \\ \underline{-\ \ 70} \\ 86 \end{array} \qquad \begin{array}{r|l} 7\overline{)\ 226} & \\ \underline{70} & 10 \\ 156 & \\ \underline{70} & 10 \\ 86 & \end{array}$$

Suppose that the next volunteer suggests 10 again; have him solve an equation and continue the division algorism also.

$$86 = (7 \times 10) + 16 \qquad \begin{array}{r} 86 \\ \underline{-\ 70} \\ 16 \end{array} \qquad \begin{array}{r|l} 7\overline{)\ 226} & \\ \underline{70} & 10 \\ 156 & \\ \underline{70} & 10 \\ 86 & \\ \underline{70} & 10 \\ 16 & \end{array}$$

The final volunteer will notice that the remainder 16 is $(7 \times 2) + 2$. Ask him to show this observation in both ways.

$$16 = (7 \times 2) + 2 \qquad \begin{array}{r} 16 \\ \underline{-\ 14} \\ 2 \end{array} \qquad \begin{array}{r|l} 7\overline{)\ 226} & \\ \underline{70} & 10 \\ 156 & \\ \underline{70} & 10 \\ 86 & \\ \underline{70} & 10 \\ 16 & \\ \underline{14} & \underline{\ 2\ } \\ 2 & \end{array}$$

Finally, add the partial quotients to find the quotient. Show this both as equations and in the algorism.

$$\begin{array}{l} 226 = 7q + r \\ \\ 226 = (7 \times 10) + 156 \\ \\ 156 = (7 \times 10) + 86 \\ \\ 86 = (7 \times 10) + 16 \\ \\ 16 = (7 \times 2) + 2 \\ \\ 226 = 7 \times (10 + 10 + 10 + 2) + 2 \\ \\ 226 = (7 \times 32) + 2 \end{array} \qquad \begin{array}{r|l} 7\overline{)\ 226} & \\ \underline{70} & 10 \\ 156 & \\ \underline{70} & 10 \\ 86 & \\ \underline{70} & 10 \\ 16 & \\ \underline{14} & \underline{\ 2\ } \\ 2 & 32 \end{array}$$

Pages 81 through 86

● Page 81 gives the children an opportunity to review the procedure for building a sequence of equations to find the greatest q, or quotient, for the equation $b = aq + r$. Remind the pupils that they must first choose a partial quotient (q) and compute. Then they must use the remainder (r) of this solution to build a new equation and compute again. Although partial quotients that the children choose for an exercise will

vary, the quotient will be the same in each case.

Discuss the example $58 = 8q + r$ on page 81. Point out that the first equation $[58 = (8 \times 3) + 34]$ results when 3 is chosen as a replacement for q. The remainder (34) is then used to build a new equation $(34 = 8q + r)$. In this equation, 4 is used as a replacement for q, giving $34 = (8 \times 4) + 2$. By combining the equations, we see that $58 = (8 \times 7) + 2$; so 7 is the greatest q for the original equation, $58 = 8q + r$. The pupils should note that they may go directly to the equation $58 = (8 \times 7) + 2$; it is not necessary that the equations $58 = (8 \times 3) + (8 \times 4) + 2$ and $58 = 8 \times (3 + 4) + 2$ be written out. Follow a similar procedure in discussing the other approach for solving the equation $58 = 8q + r$.

Work a few exercises with the class, if necessary, to clarify the procedure to be used; then assign the rest of the page as independent work. When the assignment has been completed, discuss the results with the class. The discussion should focus upon the quotients and remainders rather than upon the individual sequences of equations that were used.

● On page 82, the pupils begin to use the division algorism. Discuss the example on this page with the class. Help the pupils discover that the thinking involved in using the division algorism is the same as that used when they worked with quotient-remainder problems.

Compare the positions of the numbers in the equations with their corresponding positions in the algorism. For example, the 3 in the algorism is recorded opposite the 24. Ask the children if they have any idea why the 3 is recorded here and not opposite the 58 or the 34. Note that 24 is the result of computing the product 8×3, which occurs in the equation. Discuss also the importance of keeping columns straight; this will help in computing differences correctly.

After discussing each step in the algorism at the top of the page, have the pupils relate these steps to the example in the middle of the page where the steps are combined in the completed algorism. Again, the pupils should do most of the explaining. They should tell where each number in the series of equations is written in the algorism. Discuss the placement of the 4 in relation to the 3; both specify a number of ones so they are placed in the same column.

Do not make the mistake of assuming that the children automatically see each of these steps and intuitively know where to draw lines and write numerals. These are all minor details but they do help with the understanding of the algorism; the pupils can see each step in the algorism better if they are asked to give an explanation of why they think each detail occurs as it does. Time spent on details at the beginning will prevent the pupils from repeating mistakes when they later use the algorism. Therefore, it might be best to complete exercises 1 and 2 with the class. Assign the remaining exercises as independent work, providing help for pupils who may need it. When the children have completed the assignment, have various children write on the chalkboard their solutions for the equations.

Name

UNIT 7 THE DIVISION ALGORISM

See pupil page suggestions.

We have used equations of the form $b = aq + r$ to find the greatest partial quotient (q) and the least remainder (r).

$58 = 8q + r$
$58 = (8 \times 3) + 34$
$34 = (8 \times 4) + 2$
$58 = (8 \times 3) + (8 \times 4) + 2$
$58 = 8 \times (3 + 4) + 2$
$58 = (8 \times 7) + 2$

$58 = 8q + r$
$58 = (8 \times 5) + 18$
$18 = (8 \times 2) + 2$
$58 = (8 \times 5) + (8 \times 2) + 2$
$58 = 8 \times (5 + 2) + 2$
$58 = (8 \times 7) + 2$

Copy and find the quotient (q) and the least remainder (r).

1. $65 = 9q + r$ — $65=(9\times 7)+2$
2. $320 = 4q + r$ — $320=(4\times 80)+0$
3. $65 = 8q + r$ — $65=(8\times 8)+1$
4. $53 = 8q + r$ — $53=(8\times 6)+5$
5. $91 = 8q + r$ — $91=(8\times 11)+3$
6. $430 = 70q + r$ — $430=(70\times 6)+10$
7. $270 = 3q + r$ — $270=(3\times 90)+0$
8. $277 = 9q + r$ — $277=(9\times 30)+7$
9. $575 = 50q + r$ — $575=(50\times 11)+25$
10. $177 = 7q + r$ — $177=(7\times 25)+2$
11. $113 = 7q + r$ — $113=(7\times 16)+1$
12. $313 = 6q + r$ — $313=(6\times 52)+1$
13. $525 = 100q + r$ — $525=(100\times 5)+25$
14. $600 = 80q + r$ — $600=(80\times 7)+40$
15. $87 = 3q + r$ — $87=(3\times 29)+0$

E-81

For Class Discussion *See pupil page suggestions.*

We can use the division algorism to record the steps used in computing the greatest quotient (q) and the least remainder (r) for $58 = 8q + r$.

$58 = (8 \times 3) + 34$

```
8|58
  24 3
  34
```

$34 = (8 \times 4) + 2$

```
8|34
  32 4
   2
```

$58 = (8 \times 3) + (8 \times 4) + 2$
$58 = 8 \times (3 + 4) + 2$
$58 = (8 \times 7) + 2$

The two steps may be combined.

$58 = 8q + r$

```
8|58
  24 3
  34
  32 4
   2 7
```

$58 = (8 \times 3) + 34$
$34 = (8 \times 4) + 2$
$58 = (8 \times 7) + 2$

Use the division algorism as you solve these equations.
Steps in division algorism may vary.

1. $76 = 6q + r$ — $76=(6\times 12)+4$

```
6|76
  36  6
  40
  36  6
   4 12
```

2. $81 = 9q + r$ — $81=(9\times 9)+0$

```
9|81
  36 4
  45
  45 5
   0 9
```

3. $57 = 8q + r$ — $57=(8\times 7)+1$
4. $87 = 7q + r$ — $87(7\times 12)+3$
5. $47 = 4q + r$ — $47=(4\times 11)+3$
6. $78 = 9q + r$ — $78=(9\times 8)+6$

reference page

E-82

● The examples on page 83 serve to emphasize two things: the steps used to arrive at the greatest q and least r may vary, and the number of steps can be decreased by using 10 and multiples of 10 for the partial quotients. Give the pupils time to study the examples carefully before asking them to complete the equations that correspond to each step. Follow a discussion similar to that suggested for the preceding page as to the location of numerals, lines, and so on.

● Page 84 provides practice in relating equations of the form $b = aq + r$ to the division algorism. Refer to the first example at the top of the page and help the pupils understand that 85, the number inside the $\overline{)\quad}$ is the b of the equation $b = aq + r$; it is the number to be divided by 6. Observe also that in this example, 6, the a of the equation $b = aq + r$ is placed outside the $\overline{)\quad}$. Then work with the class in completing the equations that correspond to the other two examples at the top of the page. When you are sure that the pupils understand the procedure to be followed, assign exercises 1 through 9 for independent work. Note that the pupils are not asked to compute in these exercises. The teacher may return to these exercises for additional practice in computation at a later time.

The exercises at the bottom of page 84 are designed to help the pupils relate the equations to the division algorism. The equations are provided for the pupils and they are to translate each sequence of equations into an algorism. This enables each pupil to concentrate on the form of the algorism, since the computations have already been done for him. After the children have put the equations into algorismic form, they are to complete the equation that shows the final result. It is not only important for the pupils to be able to use the algorism but also to be able to interpret the result; they should know which number is the q, or quotient, and which is the r, or least remainder.

On the chalkboard, draw the frame $\overline{)\quad}$. To the right of the frame, copy the series of equations given in exercise 1 at the bottom of the page. Explain to the class that you want them to show, in the algorism, the information given in the equations. Let the pupils tell where the numbers given in the equations should be placed in the algorism. Then let them complete the last equation to show the quotient and remainder. Follow a similar procedure for the remaining exercises.

Name

For Class Discussion See pupil page suggestions.

Sandra and Mary used the division algorism to find the greatest partial quotient and least remainder in this exercise:

$85 = 3q + r$

Complete Mary's solution.

3)85		
27	9	$85 = (3 \times 9) + 58$
58		
24	8	$58 = (3 \times 8) + 34$
34		
27	9	$34 = (3 \times 9) + 7$
7		
6	2	$7 = (3 \times 2) + 1$
1	28	$85 = (3 \times 28) + 1$

Complete Sandra's solution.

3)85		
30	10	$85 = (3 \times 10) + 55$
55		
30	10	$55 = (3 \times 10) + 25$
25		
24	8	$25 = (3 \times 8) + 1$
1	28	$85 = (3 \times 23) + 1$

How are the two solutions alike? How are they different?

John and Fred found the quotient and remainder for $71 = 4q + r$.

Complete John's solution.

	4)71	
$71 = 4q + r$		
$71 = (4 \times 9) + 35$	36	9
	35	
$35 = (4 \times 4) + 19$	16	4
	19	
$19 = (4 \times 4) + 3$	16	4
$71 = (4 \times 17) + 3$	3	17

Complete Fred's solution.

	4)71	
$71 = 4q + r$		
$71 = (4 \times 10) + 31$	40	10
	31	
$31 = (4 \times 5) + 11$	20	5
	11	
$11 = (4 \times 2) + 3$	8	2
$71 = (4 \times 17) + 3$	3	17

How are the two solutions alike? How are they different? They have the same quotient and remainder but different steps.

E-83

What equation can be solved with each algorism? See pupil page suggestions.

6)85 $85 = 6q + r$

19)92 $92 = \boxed{19}q + r$

63)541 $\boxed{541} = \boxed{63}q + r$

1. 12)63 $63 = 12q + r$
2. 32)75 $75 = 32q + r$
3. 8)500 $500 = 8q + r$
4. 300)981 $981 = 300q + r$
5. 248)888 $888 = 248q + r$
6. 65)370 $370 = 65q + r$
7. 7)38692 $38692 = 7q + r$
8. 6429)841962 $841962 = 6429q + r$
9. 326)89999 $89999 = 326q + r$

Write the division algorism for each set of equations. Then complete the last equation.

1.
$98 = (5 \times 4) + 78$
$78 = (5 \times 10) + 28$
$28 = (5 \times 5) + 3$
$98 = (5 \times 19) + 3$

5)98	
20	4
78	
50	10
28	
25	5
3	19

2.
$168 = (20 \times 3) + 108$
$108 = (20 \times 4) + 28$
$28 = (20 \times 1) + 8$
$168 = 20 \times 8) + 8$

20)168	
60	3
108	
80	4
28	
20	1
8	8

3.
$342 = (40 \times 3) + 222$
$222 = (40 \times 2) + 142$
$142 = (40 \times 2) + 62$
$62 = (40 \times 1) + 22$
$342 = (40 \times 8) + 22$

4.
$392 = (7 \times 10) + 322$
$322 = (7 \times 20) + 182$
$182 = (7 \times 20) + 42$
$42 = (7 \times 6) + 0$
$392 = (7 \times 56) + 0$

E-84

● Page 85 provides practice in the use of the division algorism. Before you assign any exercises, the class should work an example on a piece of tagboard and post the example in the room for reference. Have a pupil suggest an equation to be solved and let various children contribute to the solution by writing the steps on the tagboard. The chart then becomes a class project and not a teacher project.

The teacher should assign selected exercises from this page. Those not used at this time can be assigned at a later date as review exercises. It is suggested that, if some pupils are not ready to work independently with the algorism, they may work in pairs with each pupil supplying every other step in the algorism. In this way, each pupil may help the other to master particular parts of the procedure.

Name

Compute. Use the division algorism. Then write an equation to show your solution.

```
4|49           11|781
  40 10          770 70
   9              11
   8  2           11  1
   1 12            0 71
```

49 = (4 × 12) + 1   781 = (11 × 71) + 0

Compute. *Steps in division algorism may vary.*

1. 9|61   2. 20|156   3. 3|100

61 = (9 × 6) + 7   156 = (20 × 7) + 16   100 = (3 × 33) + 1

4. 8|97   5. 7|105   6. 2|39

97 = (8×12)+1   105 = (7×15)+0   39 = (2×19)+1

Copy and compute.

7. 30|130   8. 12|258   9. 50|365   10. 5|213

130 = (30×4)+10   258 = (12×21)+6   365 = (50×7)+15   213 = (5×42)+3

11. 21|650   12. 32|329   13. 60|4000   14. 33|693

650 = (21×30)+20   329 = (32×10)+9   4000 = (60×66)+40   693 = (33×21)+0

E-85

● The examples on page 86 show that the steps used to obtain the greatest q and least r may vary. These examples are also designed to encourage the pupils to use multiples of 10 or 100 as their partial quotients. But this should not, as yet, be emphasized. Some children will use the idea immediately; others will not.

Let the children work together to complete the equations that relate to each step in the algorism. As one pupil explains a step in the solution, the other pupils may write the information in the appropriate equation. After both examples have been completed, discuss the question at the bottom of the page. The following comments on the two solutions could be made.

Both solutions are the same:

$$397 = 3 \times 132 + 1$$

Each boy used the same number of steps.

While Marty's partial quotients are 100, 20, 10, and 2, and Andy's partial quotients are 50, 50, 30, and 2, the sum of the partial quotients is the same number.

Assign the exercises at the bottom of the page for independent work. When the class has completed these exercises, let several pupils write their solutions on the chalkboard. Discuss the steps taken by various pupils in using the division algorism for the same equation. Do not suggest that one method for finding the solution is better than another method.

For Class Discussion

Mrs. Blake asked her class to use the division algorism to find the quotient and remainder in this exercise.

397 = 3q + r

Marty explained his solution by writing an equation for each step. Complete each equation.

```
3|397
  300 100
   97
   60  20
   37
   30  10
    7
    6   2
    1 132
```

397 = (3 × 100) + 97
97 = (3 × 20) + 37
37 = (3 × 10) + 7
7 = (3 × 2) + 1

Therefore, 397 = (3 × 132) + 1

Andy explained his solution by writing an equation for each step. Complete each equation.

```
3|397
  150  50
  247
  150  50
   97
   90  30
    7
    6   2
    1 132
```

397 = (3 × 50) + 247
247 = (3 × 50) + 97
97 = (3 × 30) + 7
7 = (3 × 2) + 1

Therefore, 397 = (3 × 132) + 1

How does Marty's solution compare with Andy's?

Use the division algorism to find the quotient and remainder.

1. 3140 = 6q + r 2. 2436 = 12q + r

3140 = (6×523)+2   2436 = (12×203)+0

reference page

E-86

Supplemental Experiences

■ Copy this table on the chalkboard.

4 × 10 is greater than □	20, 30, 40, 50, 60
12 × □ is greater than 48	2, 3, 4, 5, 6
6 × □ is greater than 18	1, 2, 3, 4, 5
9 × 9 is less than □	80, 81, 82, 83, 84
11 × □ is less than 22	1, 2, 3, 4, 5
10 × □ is less than 70	5, 6, 7, 8

Instruct the pupils to copy the table and to circle the numerals in each row that will make the sentence true.

■ Write equations of the form $b = aq + r$ on several cards.

$\boxed{347 = 5q + r}$ $\boxed{439 = 7q + r}$

$\boxed{654 = 17q + r}$ $\boxed{582 = 21q + r}$

Place these cards in a box; then separate the class into two teams. Let the first member of each team select a card from the box and then write on the chalkboard a division algorism for the equation appearing on the card.

Team I

$$5\overline{)\,347}\,|$$

Team II

$$7\overline{)\,439}\,|$$

A second member of each team should then go to the chalkboard and complete the first step in the division algorism. Remind these pupils that they may use any number they wish as the first partial quotient.

Team I

$$\begin{array}{r|l} 5\overline{)\,347} & \\ \underline{100} & 20 \\ 247 & \end{array}$$

Team II

$$\begin{array}{r|l} 7\overline{)\,439} & \\ \underline{70} & 10 \\ 369 & \end{array}$$

A third member of each team may then go to the chalkboard and complete the next step in the division algorism. Each team should continue in this way until it has found the greatest q and least r for its equation. Let one team decide if the other team has completed its assignment correctly. Score 1 point for correctly completing the assignment. Score an additional point for the team that finished first—provided the computations were correct. Continue until all members of the class have participated at least once in the activity. Then have each team total its points and declare a winner.

■ Play the game "Climb the Ladder" to review multiplication involving multiples of 10 and 100.

Draw pictures of six or more ladders on the chalkboard. Write the same set of factors on the rungs of each ladder, but vary the order in which they appear. Send a pupil to each ladder pictured on the chalkboard and provide each of these pupils with a piece of chalk and an eraser. In this game, the teacher calls out a number such as "six" or "thirteen." Starting at the bottom rung of his ladder and continuing upward, each pupil writes the standard numeral for the product of the number on each rung and the one called by the teacher. The winner is the one who reaches the top with all products correct. Continue the game by selecting another group of children and assigning each of them to a ladder pictured on the board. Call out another number and let these children "climb the ladder." Let every child have an opportunity to play this game.

20	4
100	10
60	20
4	80
500	3
80	60
3	100
10	500

KEY IDEA

$242 = 24 \times 10 + 2.$

Scope

To use the division algorism to find the quotient and the remainder.

Fundamentals

One advantage of this division algorism is that the pupil is free to choose his own partial quotients. Some choices may lead to extensive multiplication and subtraction computations, others may lead directly to the final result with a minimum of computation.

Since our numeration system provides relatively easy computations when multiplying by 10 or a multiple of 10, the use of these multiples as partial quotients can simplify and shorten the number of steps. This is illustrated in the solution shown for the equation $706 = 23q + r$.

$$\begin{array}{r|l} 23\overline{)\,706} & \\ \underline{460} & 20 \\ 246 & \\ \underline{230} & \underline{10} \\ 16 & 30 \end{array}$$

Readiness for Understanding

Knowledge of numeration.

Ability to subtract and multiply.

Developmental Experiences

sheet of tagboard (18″ × 24″)

tagboard strips ($\frac{1}{4}$″ × 2″)

pins

U.S. coins and bills

► Write both the equation and the division algorism for $2355 = 21q + r$ on the chalkboard.

$$21\overline{)\,2355} \qquad 2355 = 21q + r$$

Ask someone to come to the board and begin to find partial quotients using multiples of 10. The child could begin with 90. Have him show his work for this step.

$$\begin{array}{r|l l}
21\overline{)\,2355} & & 2355 = 21q + r \\
\underline{1890} & 90 & \\
465 & & 2355 = (21 \times 90) + 465
\end{array}$$

Ask a second child to suggest another partial quotient. Have him continue the algorism and write the appropriate equation.

$$\begin{array}{r|l l}
21\overline{)\,2355} & & 2355 = 21q + r \\
\underline{1890} & 90 & \\
465 & & 2355 = (21 \times 90) + 465 \\
\underline{420} & 20 & \\
45 & & 465 = (21 \times 20) + 45
\end{array}$$

Ask a third child to complete the algorism and write the equation for this step.

$$\begin{array}{r|l l}
21\overline{)\,2355} & & 2355 = 21q + r \\
\underline{1890} & 90 & \\
465 & & 2355 = (21 \times 90) + 465 \\
\underline{420} & 20 & \\
45 & & 465 = (21 \times 20) + 45 \\
\underline{42} & \underline{2} & \\
3 & & 45 = (21 \times 2) + 3
\end{array}$$

Now ask a volunteer to add the partial quotients and write the equation representing the quotient and remainder.

$$\begin{array}{r|l l}
21\overline{)\,2355} & & 2355 = 21q + r \\
\underline{1890} & 90 & \\
465 & & 2355 = (21 \times 90) + 465 \\
\underline{420} & 20 & \\
45 & & 465 = (21 \times 20) + 45 \\
\underline{42} & \underline{2} & \\
3 & 112 & 45 = (21 \times 2) + 3 \\
 & & 2355 = (21 \times 112) + 3
\end{array}$$

Suggest that since the quotient is greater than 100, 100 could be used as the first partial quotient. If a child begins with the partial quotient of 100 followed by factors of 10 and less, his computation will probably look like this.

$$\begin{array}{r|l l}
21\overline{)\,2355} & & 2355 = 21q + r \\
\underline{2100} & 100 & \\
255 & & 2355 = (21 \times 100) + 255 \\
\underline{210} & 10 & \\
45 & & 255 = (21 \times 10) + 45 \\
\underline{42} & \underline{2} & \\
3 & 112 & 45 = (21 \times 2) + 3 \\
 & & 2355 = (21 \times 112) + 3
\end{array}$$

Have the class check the accuracy of the quotient and remainder by computing the product 21 × 112. Then have them add this product to the remainder 3.

► On the chalkboard, write the equation $9712 = 31q + r$. Let the pupils work in pairs at the chalkboard to find the partial quotient for this equation. One member of the first pair should write a step in the division algorism while the other writes an equation to correspond to this step. Then let another pair work a second step in the process, and so on until the algorism is completed. The last pair of pupils should compute the sum of the partial quotients and write an equation using this sum as the replacement for q. Here is a possible solution.

$$\begin{array}{r|l l}
31\overline{)\,9712} & & \\
\underline{9300} & 300 & \\
412 & & 9712 = (31 \times 300) + 412 \\
\underline{310} & 10 & \\
102 & & 412 = (31 \times 10) + 102 \\
\underline{93} & \underline{3} & \\
9 & 313 & 102 = (31 \times 3) + 9 \\
 & & 9712 = (31 \times 313) + 9
\end{array}$$

Let the class help check the correctness of the replacements for q and r. On the chalkboard, write the product 31 × 313 in the vertical form of the multiplication algorism. Let the class compute the partial products and the sum of these products. Write the results on the chalkboard. Finally, add the remainder 9.

$$\begin{array}{r}
313 \\
\underline{\times\ 31} \\
3 \\
10 \\
300 \\
90 \\
300 \\
\underline{9000} \\
9703 \\
\underline{+\quad 9} \\
9712
\end{array}$$

Continue in this way to have the class compute the quotient and remainder in the following equations.

$9898 = 38q + r$ $\qquad$ $6975 = 25q + r$
$7604 = 42q + r$ $\qquad$ $3421 = 97q + r$

► Briefly review the different ways of writing amounts of money. Show to the class each of the five U. S. coins less than 1 dollar. Ask several pupils to go to the chalkboard and, as each coin is shown, to write the value of it in 2 ways: using the cent sign (¢), and using the dollar sign ($) and the decimal point (.).

1¢	5¢	10¢	25¢	50¢
$.01	$.05	$.10	$.25	$.50

Hold up several examples of U. S. paper currency, and ask some pupils to write their value.

$1.00 $\qquad$ $5.00 $\qquad$ $10.00 $\qquad$ $20.00

When these values have been put on the chalkboard, the following facts may be discussed.

Quantities less than one dollar may be written either as a whole number of cents, using the cent sign (¢), or as a decimal fraction of a dollar, using the dollar sign ($) and a decimal point. Either 99¢ or $.99 is correct. Both written forms are usually read as a whole number of cents (99).

Quantities greater than one dollar are usually read or spoken in terms of both dollars and cents.

The number of dollars is named by the digits to the left of the decimal point, and the number of cents is named by two digits to the right of the decimal point. For example, \$3.65 is read, "three dollars and sixty-five cents," and \$2.05 is read "two dollars and five cents." However, each of these amounts is equal to a whole number of cents: 365 cents and 205 cents, respectively.

When using a dollar sign and a decimal point, two digits are always written to the right of the decimal point: 1¢ is \$.01; 2¢ is \$.02; 3¢ is \$.03; and so on.

Then discuss the idea that the product of a monetary value and a number is another monetary value. Place the product \$1.11 × 5 on the chalkboard and have a child do the computation. Call on another pupil to read the result to the class.

$$\begin{array}{r} \$1.11 \\ \times \quad 5 \\ \hline \$5.55 \end{array}$$

Write the following story on the chalkboard.

Twelve boys wanted to buy a radio for a friend. They chose one that cost \$11.64. Decide whether the cost of the radio can be divided equally among the 12 boys. Decide how much money each boy must contribute as his share of the cost.

Ask a pupil to write on the chalkboard an equation for this story, using 1164¢ for \$11.64 ($1164 = 12q + r$). Let various children take turns at the chalkboard using the division algorism to solve this equation and answer the questions in the story. Point out that a whole number of cents (expressed in the form 1164¢) is being divided by a whole number: the result will be a whole number of cents. Therefore, the last pupil should write the sum of the partial quotients in terms of dollars and cents. The following steps could be used by the pupils.

$$\begin{array}{r|r} 12\overline{)\,1164} & \\ \underline{600} & 50 \\ 564 & \\ \underline{480} & 40 \\ 84 & \\ \underline{84} & \underline{7} \\ 0 & 97 \end{array}$$

Ask someone to write an equation that shows the replacements for q and r in the original equation.

$$1164 = 12 \times 97$$

Each boy must contribute 97¢ (\$.97).

Continue in this same way with two or three similar story exercises. Before any computation is done, decide whether the answer is a number, or a number that refers to an amount of money.

Pages 87 through 92

● Pages 87 and 88 give the pupils an opportunity to test their ability to use the division algorism. Work the example at the top of page 87 with the class. The example is designed to encourage the pupils to use 10 as a partial quotient. The children may also be encouraged to use multiples of 10 or 100 if they are able to do so. For example, in the exercise $409 = 12q + r$, a pupil could use 20 as his first partial quotient. He can compute 12×20 without pencil and paper by considering the product as 12×2 tens, and concluding that the product is less than 409. If a pupil is unable to compute 12×20 mentally, allow him to use 10 or a smaller number as his first partial quotient.

Work several exercises from page 87 with the class. Encourage the pupils to try 10, 100, and 1000 as partial quotients mentally before proceeding to write a step in an algorism. For example, in an exercise such as $437 = 6q + r$, the pupils may first try 10 mentally: $6 \times 10 = 60$, and 60 is less than 437. Thus, 10 can be used as a partial quotient. But before settling for 10 as the partial quotient, 100 should also be tried: $6 \times 100 = 600$, and 600 is greater than 437. Since 100 won't work as a partial quotient, neither will 1000. So the first partial quotient can be 10 or a multiple of 10 that is less than 100. The pupils who have a very good understanding of multiplication combinations will probably use 70 as the first partial quotient. Others may need guidance from you in the form of a suggestion to try 50, the multiple of 10 halfway between 0 and 100. Continue to encourage all of the pupils to decrease the number of steps needed to find the greatest q and least r, but do not insist that they use the fewest steps possible. In general, the pupils should not shorten the procedure so much that they need to make separate pencil and paper computations. Also, some pupils may not be ready to use large partial quotients. A few of the pupils may need to use 10 as a partial quotient six or seven times in a single exercise. As they become more confident, they will attempt to use 20 or 30 as their first partial quotients.

In assigning the exercises on pages 87 and 88, give the slower pupils fewer exercises so that they will be able to complete the assignment.

You may expect a variety of acceptable answers for each exercise on this page. The following are some possible solutions for exercise 2 on page 87. The pupil who computes as in example A should receive as much praise as the pupil who computes as in example B or C.

A

$$\begin{array}{r|r} 6\overline{)\,437} & \\ \underline{60} & 10 \\ 377 & \\ \underline{60} & 10 \\ 317 & \\ \underline{60} & 10 \\ 257 & \\ \underline{60} & 10 \\ 197 & \\ \underline{60} & 10 \\ 137 & \\ \underline{60} & 10 \\ 77 & \\ \underline{60} & 10 \\ 17 & \\ \underline{12} & \underline{2} \\ 5 & 72 \end{array}$$

B

$$\begin{array}{r|r} 6\overline{)\,437} & \\ \underline{420} & 70 \\ 17 & \\ \underline{12} & \underline{2} \\ 5 & 72 \end{array}$$

C

$$\begin{array}{r|r} 6\overline{)\,437} & \\ \underline{120} & 20 \\ 317 & \\ \underline{180} & 30 \\ 137 & \\ \underline{120} & 20 \\ 17 & \\ \underline{12} & \underline{2} \\ 5 & 72 \end{array}$$

Name

Complete the algorism to find the quotient (q) and the remainder (r). Then complete the equation.

$409 = 12q + r$

12)409		
120	10	$409 = (12 \times 10) + 289$
289		
120	10	$289 = (12 \times 10) + 169$
169		
120	10	$169 = (12 \times 10) + 49$
49		
48	4	$49 = (12 \times 4) + 1$
1	34	

$409 = (12 \times 34) + 1$

Complete each algorism. Write an equation to show your results.

1. 5)720 $720 = (5 \times 144) + 0$
2. 6)437 $437 = (6 \times 72) + 5$
3. 3)191 $191 = (3 \times 63) + 2$
4. 8)431 $431 = (8 \times 53) + 7$
5. 12)329 $329 = (12 \times 27) + 5$
6. 9)735 $735 = (9 \times 81) + 6$
7. 5)427 $427 = (5 \times 85) + 2$
8. 3)244 $244 = (3 \times 81) + 1$

E-87

Complete each algorism. Write an equation to show your results.

1. 14)725 $725 = (14 \times 51) + 11$
2. 4)371 $371 = (4 \times 92) + 3$
3. 7)656 $656 = (7 \times 93) + 5$
4. 5)789 $789 = (5 \times 157) + 4$
5. 9)342 $342 = (9 \times 38) + 0$
6. 6)519 $519 = (6 \times 86) + 3$
7. 9)347 $347 = (9 \times 38) + 5$
8. 13)334 $334 = (13 \times 25) + 9$
9. 14)154 $154 = (14 \times 11) + 0$
10. 30)162 $162 = (30 \times 5) + 12$
11. 15)389 $389 = (15 \times 25) + 14$
12. 21)4221 $4221 = (21 \times 201) + 0$

E-88

● Pages 89 and 90 introduce the pupils to the use of multiplication as a check for division. When working a division algorism, the pupils also get practice in multiplication, addition, and subtraction. The procedure of checking contributes to this practice. Work the example at the top of page 89 with the class. Let a group of children work a similar example on tagboard and post it in the classroom.

Be selective in assigning the exercises on these pages. Also remind the children to try 10, 100, and 1000 mentally as partial quotients before they actually write a partial quotient.

Name

30)824		Check: $824 = (30 \times 27) + 14$
180	6	
644		30
300	10	× 27
344		210
300	10	600
44		810
30	1	+ 14
14	27	824

$824 = (30 \times 27) + 14$

Compute. Write an equation to show your results. Check by multiplying. *Steps in algorism may vary.*

1. 9)191 — 90 | 10; 101; 90 | 10; 11; 9 | 1; 2 | 21. Check: 21 × 9 = 9, 180, 189, + 2, 191. $191 = (9 \times 21) + 2$

2. 13)334 — 130 | 10; 204; 130 | 10; 74; 65 | 5; 9 | 25. Check: 25 × 13 = 15, 60, 50, 200, 325, + 9, 334. $334 = (13 \times 25) + 9$

3. 20)176 — 80 | 4; 96; 80 | 4; 16 | 8. Check: 8 × 20 = 160, + 16, 176. $176 = (20 \times 8) + 16$

4. 55)180 — 110 | 2; 70; 55 | 1; 15 | 3. Check: 3 × 55 = 15, 150, 165, 15, 180. $180 = (55 \times 3) + 15$

E-89

Steps in algorism may vary.

Compute. Write an equation to show your results. Check by multiplying.

1. $8\overline{)524}$ Check: 65 × 8 = 40 + 480 = 520 + 4 = 524

   524 = (8 × 65) + 4

2. $6\overline{)777}$ Check: 129 × 6 = 54 + 120 + 600 = 774 + 3 = 777

   777 = (6 × 129) + 3

3. $7\overline{)987}$ Check: 141 × 7 = 7 + 280 + 700 = 987

   987 = (7 × 141) + 0

4. $90\overline{)1122}$ Check: 12 × 90 = 180 + 900 = 1080 + 42 = 1122

   1122 = (90 × 12) + 42

5. $38\overline{)4938}$ Check: 129 × 38 = 72 + 160 + 800 + 270 + 600 + 3000 = 4902 + 36 = 4938

   4938 = (38 × 129) + 36

6. $40\overline{)4306}$ Check: 107 × 40 = 280 + 4000 = 4280 + 26 = 4306

   4306 = (40 × 107) + 26

7. $3\overline{)224}$ Check: 74 × 3 = 12 + 210 = 222 + 2 = 224

   224 = (3 × 74) + 2

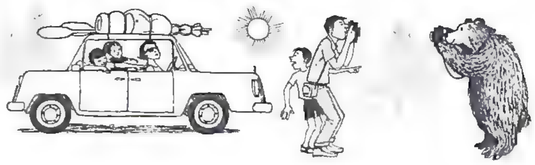
8. $4\overline{)444}$ Check: 111 × 4 = 4 + 40 + 400 = 444

   444 = (4 × 111) + 0

reference page

E-90

Name



Write an equation for each story. Solve the equation. Then write a sentence to answer each question.

Equations may vary.

1. On their vacation trip the Holmes family traveled 735 miles at an average rate of 35 miles an hour. How many hours did it take them to drive 735 miles?
   735 = (35 × □) □ = 21
   It took them 21 hours.
2. Mr. Holmes bought 18 gallons of gas. The car went 216 miles on that amount. How far did the car go on each gallon of gas? 216 = (18 × □) □ = 12
   The car went 12 miles.
3. Jim swam out to the raft 18 times each day for 12 days. How many times did he swim out to the raft?
   18 × 12 = □ □ = 216
   He swam out to the raft 216 times.
4. Tom had $5.00. He bought 6 Indian arrowheads. Each arrowhead cost $.39. How much money did he have left?
   $5.00 = ($.39 × 6) + r r = $2.66
   He had $2.66 left.
5. Sue spent $3.84 for a bathing suit and $1.92 for a sun hat. How much did Sue spend? $3.84 + $1.92 = □
   □ = $5.76 *Sue spent $5.76.*
6. During the trip Mrs. Holmes took 84 slides. She took the same number of slides each day for 6 days. How many slides did she take each day? 84 = (6 × □)
   □ = 14 *She took 14 slides a day.*
7. When they got back home, Mr. Holmes figured out that the trip cost $97.00 for his family of 5 persons. How much did it cost for each person? $97.00 = (5 × □)
   □ = $19.40 *It cost $19.40 for each person.*

E-91

● Page 91 provides practice for the children in writing equations for story exercises. Have each story read aloud. Encourage the pupils to ask questions if they do not understand a particular situation. Then assign the exercises for independent work. When the class has completed the page, call on different pupils to explain their equations and to give their answers to each story question.

● Practice in using the division algorism is provided on page 92. Be selective in assigning the exercises for independent work; the assignment should be structured so that each pupil will experience a degree of success. Remind the pupils of the value of trying 10, 100, and 1000 as partial quotients.

Copy and compute.

1. $4\overline{)365}$ 365 = (4 × 91) + 1
2. $9\overline{)813}$ 813 = (9 × 90) + 3
3. $3\overline{)339}$ 339 = (3 × 113) + 0
4. $7\overline{)284}$ 284 = (7 × 40) + 4
5. $5\overline{)252}$ 252 = (5 × 50) + 2
6. $4\overline{)163}$ 163 = (4 × 40) + 3
7. $9\overline{)631}$ 631 = (9 × 70) + 1
8. $6\overline{)547}$ 547 = (6 × 91) + 1
9. $8\overline{)723}$ 723 = (8 × 90) + 3
10. $12\overline{)481}$ 481 = (12 × 40) + 1
11. $11\overline{)776}$ 776 = (11 × 70) + 6
12. $23\overline{)693}$ 693 = (23 × 30) + 3
13. $14\overline{)287}$ 287 = (14 × 20) + 7
14. $50\overline{)306}$ 306 = (50 × 6) + 6
15. $24\overline{)483}$ 483 = (24 × 20) + 3
16. $16\overline{)327}$ 327 = (16 × 20) + 7
17. $19\overline{)389}$ 389 = (19 × 20) + 9
18. $15\overline{)3012}$ 3012 = (15 × 200) + 12
19. $21\overline{)543}$ 543 = (21 × 25) + 18
20. $14\overline{)735}$ 735 = (14 × 52) + 7
21. $19\overline{)610}$ 610 = (19 × 32) + 2
22. $27\overline{)1029}$ 1029 = (27 × 38) + 3
23. $13\overline{)536}$ 536 = (13 × 41) + 3
24. $28\overline{)870}$ 870 = (28 × 31) + 2

E-92

Supplemental Experiences

■ Review the concept of one million. Ask a pupil to write on the chalkboard the standard numeral for one thousand. Then ask several pupils to express 1000 as a number of ones, as a number of tens, as a number of hundreds, and so on. Ask them to write these expressions both in English sentences and in equations.

1000

One thousand is one thousand ones.

$1000 = 1000 \times 1$

One thousand is one one thousand.

$1000 = 1 \times 1000$

One thousand is one hundred tens.

$1000 = 100 \times 10$

One thousand is ten one hundreds.

$1000 = 10 \times 100$

Remind the pupils that numbers which are multiples of 1000 are written with three zeros to the right of the numeral that tells how many thousands are named. We can thus express the product of any whole number and 1000 by writing the standard numeral for this number followed by three zeros. Have several pupils write on the chalkboard the standard numerals for the numbers you read aloud, such as seven thousands, thirteen thousands, and fifty-seven thousands. Then ask the pupils if they could write the standard numeral for 1000 thousands. Help them understand that this numeral would be the standard numeral for 1000 followed by three zeros. Write the numeral on the chalkboard and review the use of the word million as another name for 1000 thousands.

One million is a thousand thousands.

$1{,}000{,}000 = 1000 \times 1000$

Also review the use of the comma in numerals containing five or more digits. Starting at the right end of the numeral, the comma marks off each group of three digits. The use of the comma is optional with a four-digit numeral.

Finally, ask the pupils to express the number one million in ways other than a thousand thousands. List their suggestions on the chalkboard. Possible suggestions the children might give are included in the following list.

One million is a thousand thousands.

$1{,}000{,}000 = 1000 \times 1000$

One million is a hundred ten thousands.

$1{,}000{,}000 = 100 \times 10{,}000$

One million is ten thousand hundreds.

$1{,}000{,}000 = 10{,}000 \times 100$

One million is one million ones.

$1{,}000{,}000 = 1{,}000{,}000 \times 1$

■ Review the grouping of digits in periods of 3 as an aid in interpreting numerals for large numbers. Write the following numeral on the chalkboard.

7643821

Have a pupil place commas in the appropriate places in the numeral.

7,643,821

Tell the pupils that starting from the right, each group of 3 digits is called a period. Explain that just as each place in the numeral has a value, so each period in the numeral has a value. Point out that the first group of 3 digits on the right is the ones period, the next group to the left is the thousands period, and the next group to the left of the thousands period is the thousand thousands, or millions, period.

Draw on the chalkboard a large model of the chart shown.

	periods	
millions	thousands	ones

Ask a pupil to arrange the numeral 7,643,821 by periods in the frame.

	periods	
millions	thousands	ones
7	643	821

Explain to the class that thinking of a numeral in terms of periods will help them to read the numeral and to understand its meaning. Tell the pupils that in reading a numeral, the word "million" is used to name the millions period, and the word "thousand" is used to name the thousands period. It is not necessary to name the ones period in this way. As you read the numeral to the class, point to the appropriate parts of the numeral on the chalkboard: "Seven million, six hundred forty-three thousand, eight hundred twenty-one." Then have the pupils read the numeral aloud.

Follow a similar procedure with other numerals of four digits through nine digits. In each instance, one pupil should place commas in the appropriate places (commas are optional with four digits), a second pupil should arrange the numeral in periods in the frame, and a third pupil should read the numeral aloud.

■ For the following game, you will need a pack of cards on each of which is written a sum of 3 three-digit or 3 four-digit numbers.

322 + 419 + 276	7248 + 3846 + 6589

Separate the class into two teams. In this game, each of two pupils from each team will take a card from the pack and compute his sum on the chalkboard. The two pupils in each pair must then compare their sums to see which is greater. Ask them to write a sentence using the words "is greater than" to compare their two computed sums. For example, if the sums computed by the members of one team are 1017 and 1683, the members of this team would write their sums in the following way:

1683 is greater than 1017

If both pupils in a pair have computed correctly, they earn a point for their team. If they also compare their sums correctly, they earn another point for their team. After all of the pupils in the class have participated in the game, total the points earned and declare a winner.

UNIT 8
MULTIPLES AND THE DIVISION ALGORISM
Pages 93 Through 106

OBJECTIVE

To develop computational skill in using the division algorism.

The pupil reviews the use of multiples of 10, 100, and 1000 and uses these multiples as partial quotients in the division algorism. He reviews the subtraction algorism and learns that computational skill in subtraction is essential to proficiency in division.

See Key Topics in Mathematics for the Intermediate Teacher: Multiplication and Division of Whole Numbers.

KEY IDEAS

600 is 6 hundreds; 4 × 600 is (4 × 6) hundreds.
Not enough ones, subtract from a ten.
Not enough tens, subtract from a hundred.

KEY IDEA

600 is 6 hundreds; 4 × 600 is (4 × 6) hundreds.

Scope

To relate decimal numeration to the division algorism.
To practice the division algorism.

Fundamentals

Products such as 20 × 36 and 300 × 83 may be easily computed. It is often convenient to use multiples of 10, 100, or 1000 as partial quotients in the division algorism. For example, consider $25{,}267 = 83q + r$. It is clear that 83 × 100 is less than 25,267. The desirable choices of a first partial quotient for this example are:

```
83) 25,267 |
     8 300 |  100
```

```
83) 25,267 |
    16 600 |  200
```

```
83) 25,267 |
    24 900 |  300
```

Any of the multiples, 100, 200, or 300, are convenient choices for the pupil as he begins to use the algorism. With experience he will learn to choose the larger multiples, and as he gains skill in multiplying by them, he will soon be able to find the greatest quotient more efficiently.

Readiness for Understanding

Understanding of numeration.
Ability to subtract and multiply.

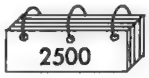
Developmental Experiences

tagboard cards (3″ × 9″)
felt-tip pen
hand punch
notebook rings

▶ Provide pupils with practice in deciding how many tens or how many hundreds are in given multiples of ten. On tagboard cards (3 inches by 9 inches), write numbers that are multiples of 10 or of 100:

4600	23900	1090
3920	45210	6220
5000	87000	71000

Make one card for each pupil in the room. Punch three holes along the top edge of each of these cards and insert three notebook rings in the holes. The cards swing freely on these rings.



Explain to the class that you will show them multiples of 10 and multiples of 100. Show the first pupil a card and ask him whether the number on the card is a multiple of 100 or of 10 only. (If it is a multiple of 100, it is also a multiple of 10.) If the number is a multiple of 10 only, ask the pupil to express it as a number of tens. For example, 3920 is 392 tens. If the number on the card is a multiple of 100, ask the pupil to express it first as a number of hundreds and then as a number of tens. For example, 23,900 is 239 hundreds or 2390 tens. Continue this activity until each pupil has examined at least two numbers.

▶ In vertical form on the chalkboard, write the product 57 × 3. Ask the class to help you compute this product, and tell them that you will use a shortcut in writing the results. As the class computes the product 7 ones × 3, explain that you are going to write only the 1 one of the partial product 21; they are to remember the 2 tens.

```
  57
× 3
   1
```

Ask someone to compute the product 5 tens × 3 and then add to it the 2 tens. Write the result of these computations (17 tens).

```
  57
× 3
 171
```

Have two pupils come to the chalkboard in turn

and describe the steps that led to writing 1 one and 17 tens. On the chalkboard, make notes of the pupil's comments.

```
 57
× 3
171←——(3 × 7 = 21; write 1, remember 2 tens).
 └———(3 × 5 tens is 15 tens;
       15 tens + 2 tens (remembered) is
       17 tens; write 17 tens).
```

In this same way, let the class compute such products as 14×6, 23×8, and 15×6 and have them write the appropriate digits in the product after each step.

You can adapt the suggested procedure to compute such products as 19×50, 24×80, and 32×70. Because there are 0 ones in the factor 50, the computation may begin with the product of tens (5 tens $\times$ 9). Ask the class to compute this product (45 tens). Call on a pupil to write the 5 tens and have the class remember the 4 hundreds.

$$\begin{array}{r} 19 \\ \times\, 50 \\ \hline 50 \end{array}$$

Let the class compute the product 5 tens $\times$ 1 ten. The result is 5 ten tens, or 5 hundreds. To this product, let the pupils add the remembered 4 hundreds (5 hundreds + 4 hundreds = 9 hundreds) and then name the computed sum. Call on a second pupil to write the hundreds.

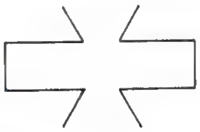
$$\begin{array}{r} 19 \\ \times\, 50 \\ \hline 950 \end{array}$$

Have two pupils go to the chalkboard and describe the steps that led to writing 5 tens and 9 hundreds in 950. On the chalkboard, make notes of the pupils' comments.

Direct the class to compute such products as 45×30, 29×70, and 33×60.

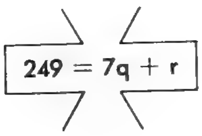
► Draw 5 or 6 quotient machines on the chalkboard.

Quotient Machine
$b = aq + r$

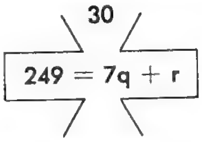


Tell the class that these machines can be used to find quotients in equations of the form $b = aq + r$. Demonstrate the quotient machines by working the example, $249 = 7q + r$, with the class.

Begin work with the machines by writing the equation $249 = 7q + r$ in the center of the first drawing of the machine on the chalkboard.



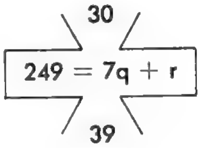
Ask the class to suggest a q to feed into the machine. Write this number in the input funnel of the machine.



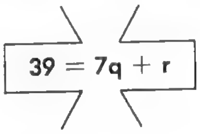
Write on the chalkboard the division algorism that shows the machine's operations. Let the pupils do the computation using the algorism.

$$\begin{array}{r|l} 7\overline{)\,249} & \\ \underline{210} & 30 \\ 39 & \end{array}$$

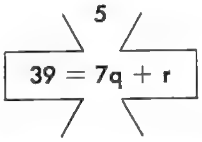
When the remainder r has been computed, write it in the output funnel of the machine.



With this r, a pupil can now write a new equation in the second machine on the chalkboard.



Have the class suggest a q to feed into the machine and write this number in the input funnel of the machine.



A pupil should come to the chalkboard and continue the division algorism to show the new operations of the machine.

$$\begin{array}{r|l} 7\overline{)\,249} & \\ \underline{210} & 30 \\ 39 & \\ \underline{35} & 5 \\ 4 & \end{array}$$

Help the class decide that this r should not be used to form a new equation for the machine. Add the partial quotients to find the quotient.

Erase all the numbers and equations and begin the activity again with the equation $568 = 18q + r$. Use the machines in the same way to find the quotient for this equation. Let the pupils do all or most of the work.

Continue the activity with equations such as the following.

$1333 = 3q + r$ $\quad$ $1672 = 9q + r$
$3295 = 45q + r$ $\quad$ $6699 = 36q + r$
$2102 = 32q + r$ $\quad$ $4716 = 13q + r$

Pages 93 through 99

● Pages 93 and 94 review multiplication by 10's, 100's, and 1000's and provide practice in recognizing multiples of 10, 100, and 1000. The ability to compute with 10, 100, and 1000 should help the pupils to use the division algorism more efficiently.

Use the example at the top of page 93 to review the concept of a multiple of 10. With the class, work exercises 1 and 2. Then assign exercises 3 through 10 for independent work. When you are sure that the pupils understand this concept, discuss the example at the bottom of the page, which involves the concept of a multiple of 100. Then assign exercises 1 through 10 at the bottom of the page for independent work. When the pupils have completed this assignment, discuss any exercise that may have caused difficulty. The answers to the question at the bottom of the page are:

1. 270 tens	2. 90 tens
3. 620 tens	4. 490 tens
5. 1270 tens	6. 380 tens
7. 4620 tens	8. 3820 tens
9. 6290 tens	10. 97,610 tens

Use the example at the top of page 94 to review the concept of a multiple of 1000. Then assign exercises 1 through 10 for independent work. The remaining exercises on page 94 provide practice on multiples of 10, 100, and 1000. The children should complete these exercises independently.

Name

UNIT 8 MULTIPLES AND THE DIVISION ALGORISM

For Class Discussion *See pupil page suggestions.*

The product of a whole number and 10 is a *multiple* of 10.

50 is 5 × 10, or 5 tens.
Therefore, 50 is a multiple of 10.

110 = 11 × 10 = 11 tens
Is 110 a multiple of 10?

Complete.

1. 70 = 7 × 10 = 7 tens	2. 30 = 3 × 10 = 3 tens
3. 120 = 12 × 10 = 12 tens	4. 140 = 14 × 10 = 14 tens
5. 60 = 6 × 10 = 6 tens	6. 980 = 98 × 10 = 98 tens
7. 250 = 25 × 10 = 25 tens	8. 2380 = 238 × 10 = 238 tens
9. 800 = 80 × 10 = 80 tens	10. 7000 = 700 × 10 = 700 tens

Multiples of 100 are also multiples of 10.

1300 = 13 × 100 = 13 hundreds
1300 = 130 × 10 = 130 tens
Is 1300 a multiple of 10? Of 100? Of 1000?

Complete. *1300 is a multiple of 10 and of 100.*

1. 2700 = 27 × 100 = 27 hundreds	2. 900 is 9 hundreds.
3. 6200 = 62 × 100 = 62 hundreds	4. 4900 is 49 hundreds.
5. 12,700 = 127 × 100 = 127 hundreds	6. 3800 is 38 hundreds.
7. 46,200 = 462 × 100 = ~~462~~ hundreds	8. 38,200 is 382 hundreds.
9. 62,900 = 629 × 100 = 629 hundreds	10. 976,100 is 9761 hundreds.

How many tens in each exercise above? *
2700 = □ × 10 = □ tens

reference page

F-93

* *For answers, see pupil page suggestions.*

Multiples of 1000 are also multiples of 100 and of 10.

23,000 = 23 × 1000 = 23 thousands
23,000 = 230 × 100 = 230 hundreds
23,000 = 2300 × 10 = 2300 tens

Is 23,000 a multiple of 10? Of 100? Of 1000? *Yes*

Complete.

1. 87,000 = 87 × 1000 = 87 thousands	2. 63,000 is 63 thousands.
3. 92,000 = 92 × 1000 = 92 thousands	4. 17,000 is 17 thousands.
5. 129,000 = 129 × 1000 = 129 thousands	6. 683,000 is 683 thousands.
7. 721,000 = 721 × 1000 = 721 thousands	8. 396,000 is 396 thousands.
9. 246,000 = 246 × 1000 = 246 thousands	10. 861,000 is 861 thousands.
11. 240 is 24 tens.	12. 3010 is 301 tens.
13. 1500 is 15 hundreds. 150 tens.	14. 700 is 7 hundreds. 70 tens.
15. 2500 is 250 tens. 25 hundreds.	16. 5700 is 570 tens. 57 hundreds.
17. 32,000 is 3200 tens. 320 hundreds. 32 thousands.	18. 965,000 is 96,500 tens. 9650 hundreds. 965 thousands.
19. 1,628,000 is 162,800 tens. 16,280 hundreds. 1628 thousands.	20. 7,184,000 is 718,400 tens. 71,840 hundreds. 7184 thousands.

reference page

F-94

● On page 95 there is a review of the basic multiplication combinations. Many of the exercises involve multiples of 10, 100, and 1000. Some of the exercises are in the form of missing-factor problems. Most of the pupils' earlier work with these multiples involved finding the product when given both factors.

Some pupils find it easier to think in terms of multiplication rather than division. When given an exercise such as ____ × 7 tens = 63 hundreds, the children may think of it as "What number is 63 hundreds divided by 7 tens?", or as "What number times 7 tens equals 63 hundreds?" On the basis of their knowledge of multiplication, they will know that the missing factor is 9 tens.

With the class, work the first row in the table at the top of page 95. Then have the pupils complete the table independently. Before assigning the other exercises, discuss several of them. Ask the pupils to look at an equation in one of the exercises and tell whether the missing factor is a number of ones, tens, or hundreds. In working exercises 15 through 26, they should first express each example in terms of a multiplication or division equation. Each number in the equation should be expressed as a number of ones, tens, or hundreds. For example, exercise 22 may be written in either of these two ways:

___ × 8 tens = 56 hundreds

56 hundreds ÷ 8 tens = ___

Since the pupils know that the product of a number of tens and a number of tens is a number of hundreds, they should identify the missing factor as a certain number of tens. After the pupils have completed all of the exercises, discuss their responses to the specific exercises.

● The example on page 96 shows that fewer steps are necessary in the division algorism when multiples of 10 and 100 are used as partial quotients. Use the page for discussion. On the chalkboard, copy a model of Polly's and Jane's solutions to the equation $1312 = 4q + r$. As each question is discussed, have several pupils point to the part of the algorism that each question concerns. The pupils should be encouraged, but not forced, to use multiples of 10 and 100 as partial quotients. Therefore, in response to exercise 7, some of the children will like Polly's solution while others will prefer Jane's solution.

Name

Complete. *See pupil page suggestions.*

	×	8	80	800	8000
1.	3	24	240	2400	24,000
2.	5	40	400	4000	40,000
3.	7	56	560	5600	56,000
4.	9	72	720	7200	72,000
5.	8	64	640	6400	64,000
6.	6	48	480	4800	48,000
7.	4	32	320	3200	32,000
8.	2	16	160	1600	16,000

9. 8 × 7 = 56; 70 × 8 = 560; 80 × 7 = 560; 70 × 80 = 5600

10. 9 × 8 = 72; 80 × 9 = 720; 90 × 8 = 720; 80 × 90 = 7200

11. 7 × 9 = 63; 7 × 90 = 630; 9 × 70 = 630; 70 × 90 = 6300

12. 48 = 6 × 8; 480 = 8 × 60; 480 = 6 × 80; 4800 = 60 × 80

13. 54 = 6 × 9; 540 = 6 × 90; 540 = 9 × 60; 5400 = 60 × 90

14. 8 × 8 = 64; 8 × 80 = 640; 8 × 80 = 640; 80 × 80 = 6400

15. 560 ÷ 70 = 8

16. 720 ÷ 8 = 90

17. 4800 ÷ 80 = 60

18. 630 ÷ 90 = 7

19. 3500 ÷ 700 = 5

20. 4200 ÷ 60 = 70

21. 5600 ÷ 70 = 80

22. 5600 ÷ 80 = 70

23. 720 ÷ 80 = 9

24. 240 ÷ 80 = 3

25. 1800 ÷ 200 = 9

26. 2700 ÷ 30 = 90

F-95

For Class Discussion

Polly and Jane worked the same division exercise.

$1312 = 4q + r$

Polly's solution

4)1312	
400	100
912	
400	100
512	
400	100
112	
40	10
72	
40	10
32	
32	8
0	328

Jane's solution

4)1312	
800	200
512	
400	100
112	
80	20
32	
32	8
0	328

1. What partial quotient did Polly use in each of the first three steps? 100; 100; 100
2. What partial quotients did Jane use in the first two steps? 200; 100
3. What partial quotients did Polly use in the next two steps? 10; 10
4. What partial quotient did Jane use in the third step? 20
5. What partial quotient did Polly use in the last step? 8
6. What partial quotient did Jane use in the last step? 8
7. Which solution do you like better? answers
   Why? will vary.
8. Write an equation to show the quotient and remainder. 1312 = (4 × 328) + 0

reference page

F-96

● Page 97 provides practice in using the division algorism and emphasizes the use of multiples of 10, 100, and 1000 as partial quotients. Help the class work the examples at the top of the page. (Note that the emphasis is not on finding the greatest multiple of 10, 100, or 1000 that can be used as the first partial quotient. To do this, the pupils would need a sophisticated ability to estimate.) Work exercises 1 and 2 with the class and ask questions similar to those at the top of the page. Assign the remaining exercises for independent work. Remind the children that the directions are that they use multiples of 10, 100, or 1000 where they can.

● Page 98 provides additional practice in using the division algorism. Assign the exercises to be completed independently. Encourage the pupils to use multiples of 10, 100, or 1000 as partial quotients. When they have finished the assignment, discuss their answers. Then work the story exercises with the class. Ask a pupil to explain what q and r represent in each equation (in exercise 17, q is the number of boxes filled and r is the number of donuts that remain; in exercise 18, q is the number of pages Jane filled and r is the number of postcards that remain).

● The exercises on page 99 emphasize accurate subtraction in the division algorism. Work several exercises with the class to be sure the pupils understand the procedure to be followed. They need only complete the first step in each algorism. At a later date, you may have them complete these algorisms as a review.

Name

Which multiples of 10, 100, or 1000 could be used as the first partial quotient?

$7\overline{)3748}$ | □

$7 \times □$ is less than 3748.

Could □ be 200? *Yes*

Could □ be 400? *Yes*

$35\overline{)2963}$ | □

$35 \times □$ is less than 2963.

Could □ be 20? *Yes*

Could □ be 60? *Yes*

Compute. Choose a multiple of 10, 100, or 1000 as the first partial quotient.

1. $5\overline{)137}$ 2. $8\overline{)4788}$ 3. $6\overline{)5246}$ 4. $3\overline{)7840}$

1. 137 = (5 × 27) + 2
2. 4788 = (8 × 598) + 4
3. 5246 = (6 × 874) + 2
4. 7840 = (3 × 2613) + 1

5. $9\overline{)613}$ 6. $7\overline{)6954}$ 7. $11\overline{)458}$ 8. $16\overline{)6700}$

5. 613 = (9 × 68) + 1
6. 6954 = (7 × 993) + 3
7. 458 = (11 × 41) + 7
8. 6700 = (16 × 418) + 12

9. A bookstore ordered 720 books. They came packed 24 in a case. How many cases?
There were 30 cases.


reference page

F-97

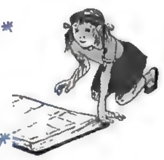
Copy and compute.

1. $3\overline{)108}$ *108 = (3 × 36) + 0*
2. $7\overline{)229}$ *229 = (7 × 32) + 5*
3. $4\overline{)376}$ *376 = (4 × 94) + 0*
4. $6\overline{)424}$ *424 = (6 × 70) + 4*
5. $9\overline{)558}$ *558 = (9 × 62) + 0*
6. $6\overline{)443}$ *443 = (6 × 73) + 5*
7. $8\overline{)7643}$ *7643 = (8 × 955) + 3*
8. $5\overline{)350}$ *350 = (5 × 70) + 0*
9. $7\overline{)5027}$ *5027 = (7 × 718) + 1*
10. $15\overline{)395}$ *395 = (15 × 26) + 5*
11. $17\overline{)3624}$ *3624 = (17 × 213) + 3*
12. $12\overline{)502}$ *502 = (12 × 41) + 10*
13. $11\overline{)5729}$ *5729 = (11 × 520) + 9*
14. $18\overline{)56236}$ *56236 = (18 × 3124) + 4*
15. $13\overline{)286}$ *286 = (13 × 22) + 0*
16. $16\overline{)33816}$ *33816 = (16 × 2113) + 8*

Solve each equation. Then answer the question.

17. Beverly packed 105 donuts in boxes. Each box held 9 donuts. How many boxes did she fill with donuts?
Equation: $105 = 9q + r$ $q =$ *11* $r =$ *6*
*She filled 11 boxes and had 6 extra donuts.**

18. Jane put her collection of 351 postcards into a scrapbook. Each page holds 16 cards. How many pages did she fill?
Equation: $351 = 16q + r$ $q =$ *21* $r =$ *15*
*She filled 21 pages and had 15 on another page.**

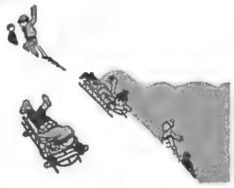


**Answers will vary.*

F-98

Name

Accurate subtraction is important in the division algorism.



$6\overline{)64}$
60 | 10
4

Complete the first subtraction in each exercise.

1. $9\overline{)76}$; 72 | 8; *4*
2. $8\overline{)97}$; 80 | 10; *17*
3. $9\overline{)80}$; 72 | 8; *8*
4. $6\overline{)217}$; 180 | 30; *37*
5. $7\overline{)139}$; 70 | 10; *69*
6. $8\overline{)503}$; 480 | 60; *23*
7. $19\overline{)3986}$; 3800 | 200; *186*
8. $37\overline{)823}$; 740 | 20; *83*
9. $86\overline{)412}$; 344 | 4; *68*
10. $25\overline{)1906}$; 1750 | 70; *156*
11. $11\overline{)1822}$; 110 | 10; *1712*
12. $99\overline{)1000}$; 990 | 10; *10*
13. $777\overline{)7100}$; 6993 | 9; *107*
14. $548\overline{)5197}$; 3288 | 6; *1909*
15. $930\overline{)8700}$; 6510 | 7; *2190*

F-99

Supplemental Experiences

■ The pupils will become more efficient in the use of the division algorism if they are able to compute multiples of 10 or 100 without writing down partial products. The pupils will benefit from practicing this type of multiplication without seeing the exercise in the forms:

$$20 \times 59 \quad \text{or} \quad \begin{array}{r} 20 \\ \times 59 \\ \hline \end{array}$$

The reason for this is that they don't see the multiplication in these forms in the division algorism. The 20 appears on one side of the "box" and the 59 appears on the other side. One way to help the pupils with this multiplication is to write a multiple of 10 or 100 in the center of the chalkboard and the numerals for various numbers less than 100 on pieces of tagboard. Then show one of the cards to the class and direct the pupils to multiply the number named on the card by the multiple of 10 or 100 named on the chalkboard. They are to state the final product, or write it on the chalkboard.

Vary the activity by having each row of children serve as a team. Show five cards to the teams, have each pupil write his products on a sheet of paper, and then have the pupils correct their papers. Each correct answer wins one point and the team's total number of correct responses is its score for the first round. Change the multiple of 10 or 100 on the chalkboard and flash five different cards. The team with the greatest number of points after three or four rounds is the winner.

■ Prepare a set of cards showing quotients equal to numbers 0 through 9:

$8 \div 8$	$12 \div 6$	$49 \div 7$
$32 \div 8$	$63 \div 7$	$27 \div 9$
$0 \div 7$	$48 \div 6$	$30 \div 5$
$20 \div 4$	$54 \div 6$	$56 \div 7$

These cards should be distributed to the pupils and used for such activities as the following:

Name a number from 0 to 9 and ask all the pupils whose cards show quotients equal to it to stand and display their cards.

Call on two pupils to stand and compare the quotients written on their cards. Have one of them write a sentence comparing the two numbers. For example, "$54 \div 6$ is greater than $49 \div 7$." Have the other pupil write a subtraction equation showing the computed difference: $(54 \div 6) - (49 \div 7) = 2$.

Call on two pupils to display their cards. Have one pupil compute the sum of the quotients, the other, the product of the quotients. For example, $(20 \div 4) + (63 \div 7) = 14$ and $(20 \div 4) \times (63 \div 7) = 45$.

KEY IDEA

Not enough ones, subtract from a ten.
Not enough tens, subtract from a hundred.

Scope

To strengthen the pupil's understanding of place value.

To review the subtraction algorism.

Fundamentals

Combinations of 10 are most important in addition and subtraction. Whenever a pupil adds 1 to a number whose last digit is 9, he notices that the ones digit changes from 9 to 0. In effect, he adds 1 ten and subtracts 9 ones. For example:

$$\begin{array}{rcl} 9 & = & 0 \text{ tens} + 9 \text{ ones} \\ \underline{+\ 1} & = & \underline{1 \text{ ten} \ - 9 \text{ ones}} \\ & & 1 \text{ ten} \ + 0 \text{ ones} = 10 \end{array}$$

In the exercise $7 + 6 = \square$, the pupil adds 10 and subtracts 4. This is the same as adding 6.

$$\begin{array}{rcl} 7 & = & 0 \text{ tens} + 7 \text{ ones} \\ \underline{+\ 6} & = & \underline{1 \text{ ten} \ - 4 \text{ ones}} \\ & & 1 \text{ ten} \ + 3 \text{ ones} = 13 \end{array}$$

The most important subtraction facts are the computed differences of 10 and the one-digit numbers. Examples are $10 - 1 = 9$, $10 - 2 = 8$, and so on. These facts are used in the subtraction algorism.

$$\begin{array}{rcl} 10 & = & 1 \text{ ten} \ + 0 \text{ ones} \\ \underline{-\ \ 1} & = & \underline{-\ 1 \text{ ten} \ + 9 \text{ ones}} \\ & & 0 \text{ tens} + 9 \text{ ones} = 9 \end{array}$$

In computing differences such as $13 - 6$, the pupil has learned to think: subtract 10, add 4. This is the same as subtracting 6.

$$\begin{array}{rcl} 13 & = & 1 \text{ ten} \ + 3 \text{ ones} \\ \underline{-\ \ 6} & = & \underline{-\ 1 \text{ ten} \ + 4 \text{ ones}} \\ & & 0 \text{ tens} + 7 \text{ ones} = 7 \end{array}$$

The teacher will see the importance of knowing the differences of 10 and the one-digit numbers. The digit in the ones place of the number which is being subtracted may be greater than the digit in the ones place of the first number. The same may be true of the digits in the tens and hundreds places. Consider $6257 - 3469$. It is impossible to subtract the 9 ones of 3469 from the 7 ones of 6257 and get a whole number. Instead, subtract the 9 ones of 3469 from ten ones in the tens place of 6257. Then add the difference $(10 - 9$, or $1)$ to the 7 ones of 6257 to get 8. This leaves 4 tens in the tens place.

$$\begin{array}{r} 4 \\ 62\cancel{5}7 \\ -\ 3469 \\ \hline 8 \end{array}$$

Similarly, since it is impossible to subtract 6 tens from 4 tens, subtract the 6 tens of 3469 from 10 tens in

the hundreds place of the first number. Add the difference (10 tens − 6 tens, or 4 tens) to 4 tens to get 8 tens.

$$\begin{array}{r} 14 \\ 6\cancel{2}\cancel{5}7 \\ -\ 3469 \\ \hline 88 \end{array}$$

In the same way, subtract the 4 hundreds in 3469 from 10 hundreds in the thousands place of the first number. Add the difference (10 hundreds − 4 hundreds, or 6 hundreds) to 1 hundred to get 7 hundreds.

$$\begin{array}{r} 514 \\ \cancel{6}\cancel{2}\cancel{5}7 \\ -\ 3469 \\ \hline 788 \end{array}$$

The thousands are easily subtracted to complete the algorism.

$$\begin{array}{r} 514 \\ \cancel{6}\cancel{2}\cancel{5}7 \\ -\ 3469 \\ \hline 2788 \end{array}$$

Readiness for Understanding

Knowledge of basic subtraction facts.
Knowledge of place value.

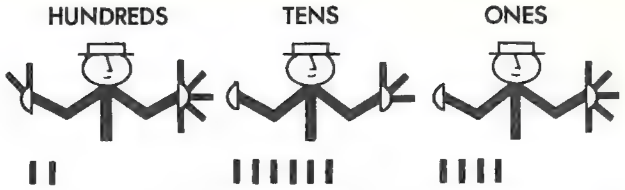
Developmental Experiences

Countingmen
felt-tip pen
tagboard disks
yarn
pins

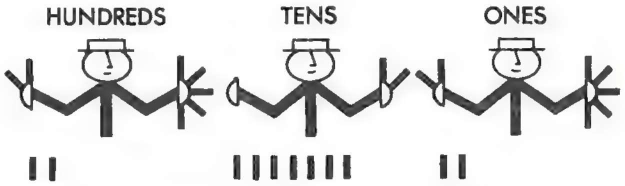
► Since the subtraction algorism is used in the division algorism, the pupils would benefit from a review of the computation of differences. The following activities provide several types of review and it is not necessary to use all the activities. On the chalkboard, have one child write the computation of 735 − 268 while another pupil shows it on Countingmen. (If Countingmen are not available, the teacher can draw and label the hands on the chalkboard.)

Remind the pupils that no number in a computation may have more than 9 ones, 9 tens, 9 hundreds, and so on. Thus, no Countingman has more than 9 fingers. The pupils may explain and work the computation in the following way:

Place 5 fingers on the Ones-man, 3 fingers on the Tens-man, and 7 fingers on the Hundreds-man to show the number 735.



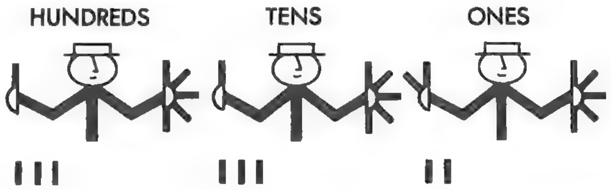
To subtract 8 ones from 5 ones, first subtract 1 ten from the 3 tens shown on the Tens-man. That is 2 more than needed; put 2 ones with the 5 ones on the Ones-man to show this. Subtracting 10 and adding 2 is the same as subtracting 8.



The Countingmen show 735 − 8, or 727.

$$\begin{array}{r} 2 \\ 7\cancel{3}5 \\ -\ 268 \\ \hline 7 \end{array}$$

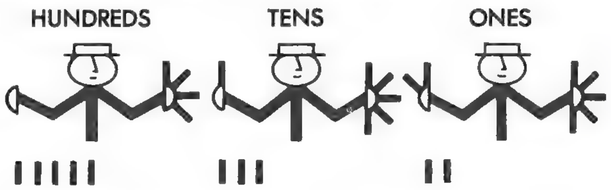
Next, 6 tens (60) are to be subtracted. Subtract 1 hundred from the 7 hundreds shown on the Hundreds-man. Subtracting 10 tens (1 hundred) was subtracting 4 tens more than needed, so add 4 tens to the 2 tens shown on the Tens-man. Subtracting 10 tens and adding 4 tens is the same as subtracting 6 tens.



The Countingmen show 727 − 60, or 667.

$$\begin{array}{r} 62 \\ \cancel{7}\cancel{3}5 \\ -\ 268 \\ \hline 67 \end{array}$$

2 hundreds (200) are to be subtracted. This presents no problem. Just subtract 2 hundreds from the 6 hundreds shown on the Hundreds-man.



The Countingmen show 667 − 200, or 467. They also show 735 − 268, or 467.

$$\begin{array}{r} 62 \\ \cancel{7}\cancel{3}5 \\ -\ 268 \\ \hline 467 \end{array}$$

Adapt this procedure to differences such as 860 − 537, 927 − 464, 5652 − 2386, and 6293 − 3725. In each instance, let one pupil show the computation on the Countingmen and another pupil write the computation on the chalkboard.

► Have a pupil compute the difference 825 − 467 on the chalkboard in the algorism form while he explains it aloud. First he must decide whether or not a ten has to be used to compute the difference of ones. He may say that he will subtract 7 from one of the 2 tens and then add the remaining 3 ones (the difference 10 − 7) to 5 ones with the result 8 ones. Subtracting 7 ones is the same as subtracting 1 ten and adding (10 − 7) ones.

$$\begin{array}{r} 1 \\ 8\cancel{2}5 \\ -\,467 \\ \hline 8 \end{array}$$

Now, let the class decide whether or not 10 tens (1 hundred) has to be used in computing the difference of tens. The pupil at the board should describe this computation of the difference of tens. He may say that he will subtract 6 tens from 10 tens (one of the 8 hundreds) and then add this difference, 4 tens (40), to 1 ten (10) with the result 5 tens (50).

$$\begin{array}{r} 71 \\ \cancel{8}\cancel{2}5 \\ -\,467 \\ \hline 58 \end{array}$$

Finally, tell the pupil at the board to explain the computation of the difference of hundreds. He may say that, since one of the 8 hundreds was used in computing the difference of tens, he should now compute the difference between 7 hundreds and 4 hundreds. The difference is 700 − 400, or 300.

$$\begin{array}{r} 71 \\ \cancel{8}\cancel{2}5 \\ -\,467 \\ \hline 358 \end{array}$$

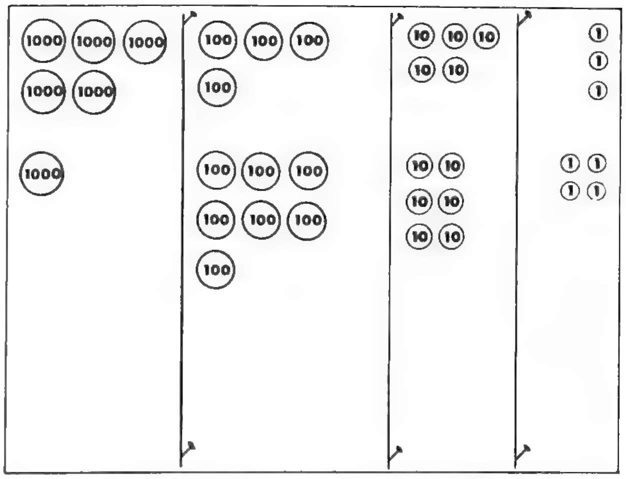
Continue in this way to have the pupils explain the computation of other differences such as these:

709 − 362 843 − 516 7156 − 5374 6312 − 3171

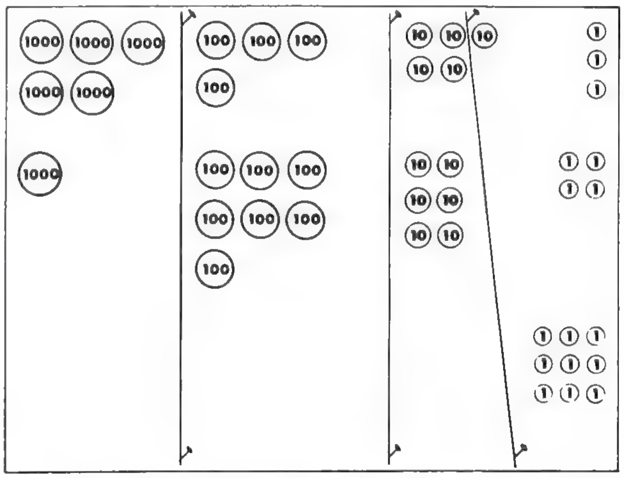
► On a table in the front of the room, place 18 each of tagboard disks labeled 1000, 100, 10, and 1. On the chalkboard write the difference 5453 − 1764 in vertical form.

$$\begin{array}{r} 5453 \\ -\,1764 \\ \hline \end{array}$$

Ask a child to place enough disks on the bulletin board to represent 5453. Have another pupil represent 1764 in the same manner. Pin three lengths of yarn on the board to distinctly separate the ones, tens, hundreds, and thousands.



After the class has decided whether a ten has to be used in the computation of the difference of ones, let a child describe how to compute this difference. He may say that he will subtract 4 from one of the 5 tens and then add this difference (10 − 4, or 6) to 3 with the result 9. To show this procedure on the bulletin board, first place 3 ones at the bottom of the board directly below the 3 ones in the top row. Then move the yarn to include one of the tens with the ones, and put 6 ones in the ones place at the bottom. Point out that what has been done illustrates subtracting 10 and adding 10 − 4.



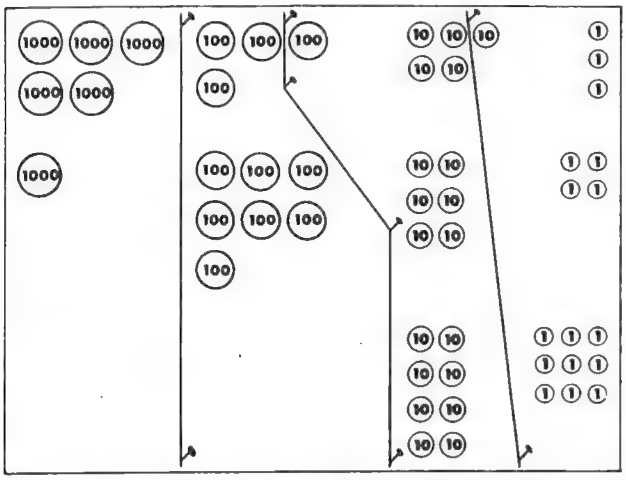
In the algorism on the chalkboard, have a pupil write the result of this computation. He also should cross out 5 tens in 5453 and write 4 above the five.

$$\begin{array}{r} 4 \\ 54\cancel{5}3 \\ -\,1764 \\ \hline 9 \end{array}$$

This will serve as a reminder that since one of the 5 tens has been subtracted, only 4 tens remain to be considered when computing the difference of tens.

Next, ask the class whether a hundred has to be used to compute the difference of tens. A pupil may say that he will subtract 6 tens from 10 tens (one of the 4 hundreds) and then add this difference (10 tens − 6 tens, or 4 tens) to the 4 tens, with the result 8 tens. Help him show this on the bulletin board. Have

him place 4 tens at the bottom of the board aligned with the 4 tens in the top row. Then have him move the piece of yarn between the tens and hundreds to include one of the hundreds with the tens and add 4 tens in the tens place at the bottom. Point out that what has been done illustrates subtracting 10 tens and adding the difference 10 tens − 6 tens.

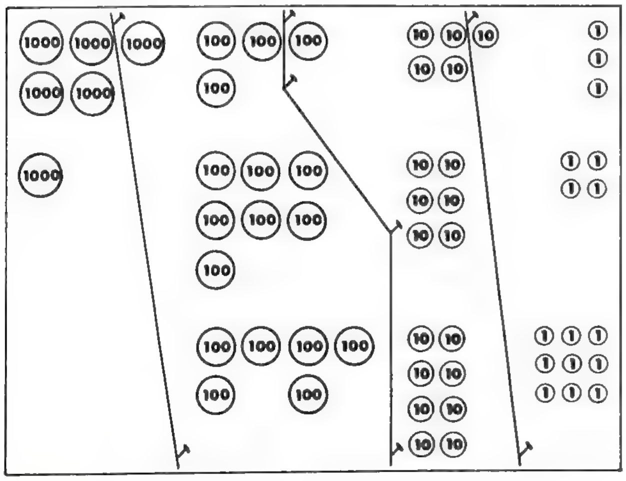


In the algorism, have another pupil write the result of this computation. He also should cross out 4 hundreds in 5453 and write 3 above the 4.

$$\begin{array}{r} 34 \\ 5\cancel{4}\cancel{5}3 \\ -\,1764 \\ \hline 89 \end{array}$$

This will help him to remember that, one of the 4 hundreds having been subtracted, only 3 hundreds remain to be considered when computing the difference of hundreds.

Next, ask the class whether a thousand has to be used in computing the difference of hundreds. Let a pupil describe how to compute this difference. He may say that he will subtract 7 hundreds from 10 hundreds (one of the 5 thousands) and then add this difference (10 hundreds − 7 hundreds, or 3 hundreds) to the 3 hundreds, with the result 6 hundreds. Help him show this procedure on the bulletin board. Point out that what has been done illustrates subtracting 10 hundreds and adding the difference 10 hundreds − 7 hundreds.

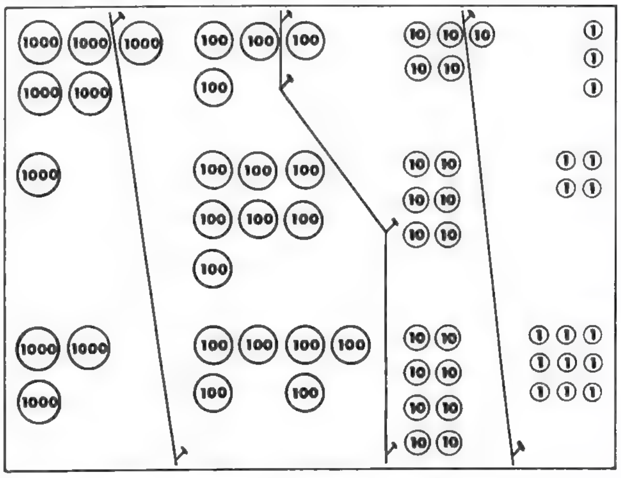


In the algorism, have a pupil write the difference of hundreds. Also have him cross out 5 thousands in 5453 and write 4 thousands above the 5 thousands.

$$\begin{array}{r} 434 \\ \cancel{5}\cancel{4}\cancel{5}3 \\ -\,1764 \\ \hline 689 \end{array}$$

This will help him to remember that one of the 5 thousands has been subtracted and that only 4 thousands remain to be considered.

Then have another pupil describe the computation of the difference of thousands. He may say that since one of the thousands was used in the computation of the difference of hundreds, he must compute the difference between 4 thousands and 1 thousand. This difference is 4000 − 1000, or 3000. Help him show this on the bulletin board.



Have a pupil complete the algorism.

$$\begin{array}{r} 434 \\ \cancel{5}\cancel{4}\cancel{5}3 \\ -\,1764 \\ \hline 3689 \end{array}$$

Continue in this way to let the pupils investigate the computation of other differences such as the following: 8136 − 3854, 7485 − 5517, 509 − 326, and 972 − 625.

► Let a child demonstrate on the chalkboard how to compute the difference 7030 − 4614. He may do this in the following way:

There are not enough ones in the ones place so 4 ones will be subtracted from 10 ones in the tens place. This leaves 2 tens in the tens place. Add the difference (10 ones − 4 ones, or 6 ones) to the zero ones.

$$\begin{array}{r} 2 \\ 70\cancel{3}0 \\ -\,4614 \\ \hline 6 \end{array}$$

There are enough tens in the tens place so 1 ten is subtracted from 2 tens.

$$\begin{array}{r} 2 \\ 70\cancel{3}0 \\ -\,4614 \\ \hline 16 \end{array}$$

There are not enough hundreds in the hundreds place so 6 hundreds will be subtracted from 10 hundreds in the thousands place. This leaves 6 thousands in the thousands place. Add the difference (10 hundreds − 6 hundreds, or 4 hundreds) to the zero hundreds.

$$\begin{array}{r} 6\ \ 2\ \ \\ \cancel{7}0\cancel{3}0 \\ -\ 4614 \\ \hline 416 \end{array}$$

Finally, the thousands are subtracted to complete the algorism.

$$\begin{array}{r} 6\ \ 2\ \ \\ \cancel{7}0\cancel{3}0 \\ -\ 4614 \\ \hline 2416 \end{array}$$

Next, direct a second pupil to compute on the chalkboard the difference 9403 − 5976 explaining each of his computational steps. He might say:

First 6 ones are to be subtracted. Neither the ones place nor the tens place can supply the 6 ones. However, the hundreds place and the tens place together have 40 tens. So subtract 6 ones from one of these 40 tens. This leaves 39 tens (3 hundreds + 9 tens). Then add the difference (10 − 6, or 4) to the 3 ones.

$$\begin{array}{r} 39\ \\ 9\cancel{4}\cancel{0}3 \\ -\ 5976 \\ \hline 7 \end{array}$$

Subtracting 7 tens presents no problem; 7 tens are subtracted from 9 tens.

$$\begin{array}{r} 39\ \\ 9\cancel{4}\cancel{0}3 \\ -\ 5976 \\ \hline 27 \end{array}$$

Next, 9 hundreds are to be subtracted. There are not enough hundreds in the hundreds place, so subtract 9 hundreds from 10 hundreds. This leaves 8 thousands in the thousands place. Then add the difference (10 hundreds − 9 hundreds, or 1 hundred) to the 3 hundreds.

$$\begin{array}{r} 839\ \\ \cancel{9}\cancel{4}\cancel{0}3 \\ -\ 5976 \\ \hline 427 \end{array}$$

Finally, the thousands are easily subtracted to complete the algorism.

$$\begin{array}{r} 839\ \\ \cancel{9}\cancel{4}\cancel{0}3 \\ -\ 5976 \\ \hline 3427 \end{array}$$

Continue to let the pupils explain the computation of other differences, such as the following: 3350 − 1293, 5805 − 2524, 7065 − 2766, and 8000 − 3672. In this last instance, the 1 ten needed will be subtracted from the 800 tens, leaving 799 tens.

$$\begin{array}{r} 799\ \\ \cancel{8}\cancel{0}\cancel{0}0 \\ -\ 3672 \\ \hline 4328 \end{array}$$

► Provide practice in computing products and sums without pencil and paper. Orally present exercises such as the following:

3 tens × 4 (pause) + 7 tens
6 × 4 tens (pause) + 8 tens
2 hundreds × 7 (pause) + 6 hundreds
8 tens × 6 tens (pause) + 4 ten tens
9 tens × 5 tens (pause) + 6 hundreds

If any child responds incorrectly, repeat the exercise so that he can correct his error. Read enough of this type of exercise to give each member of the class a chance to answer at least one of them.

► On the chalkboard, write part of a multiplication check for a division algorism.

$$\begin{array}{r} 4\ 3 \\ \times\ \square\,\square \\ \hline 6 \\ 8\ 0 \\ 1\ 2\ 0 \\ \underline{1\ 6\ 0\ 0} \\ 1\ 8\ 0\ 6 \\ +\ \square\,\square \\ \hline 1\ 8\ 2\ 1 \end{array}$$

Using the information given in the check, have a child set it up in division algorism form.

$$43\overline{)\ 1821}$$

Allow several pupils to take turns at computing the greatest quotient. For example:

$$\begin{array}{r|r} 43\overline{)\ 1821} & \\ \underline{860} & 20 \\ 961 & \\ \underline{860} & 20 \\ 101 & \\ \underline{86} & \underline{2} \\ 15 & 42 \end{array}$$

Then, tell them to place the missing information in the check and to verify it.

Continue in the same way with several other exercises of this kind.

Pages 100 through 106

● The exercises on page 100 involve subtraction. For this page the following procedure is suggested:

Work and discuss the example at the top of the page with the class.

Assign the computational exercises as independent work.

Upon completion of the assignment, let the pupils tell how they computed the differences.

● Page 101 provides more practice in using multiples of 10, 100, or 1000 for the first partial quotient. Discuss the examples at the top of the page with the class. Then work exercises 1 and 2 to be sure the pupils understand the procedure. The pupils need only write a multiple of 10, 100, or 1000 that can be used for the first partial quotient. It does not have to be the greatest multiple that could be used. Afer the pupils have completed exercises 3 through 12 independently and have written equations for three of the exercises, let them work a few of the exercises on the chalkboard and discuss some of their equations.

● The pupils must complete division algorisms on page 102. Work exercises 1 and 2 with the class. Have the pupils complete exercises 3 through 8 independently. It is not necessary that all the pupils complete this page on their own. The less able pupils may be more successful if they work in pairs. After the page has been completed, let the pupils show how they completed the exercises.

Let's take another look at subtraction.

```
  5
  658          5 = 8 − 3
− 273         80 = 100 − 70 + 50
  385        300 = 500 − 200
```

Compute.

1. 521 − 47 = 474	2. 336 − 47 = 289	3. 552 − 87 = 465	4. 771 − 86 = 685
5. 415 − 46 = 369	6. 700 − 89 = 611	7. 283 − 85 = 198	8. 362 − 85 = 277
9. 234 − 83 = 151	10. 418 − 62 = 356	11. 826 − 75 = 751	12. 629 − 351 = 278
13. 206 − 54 = 152	14. 833 − 281 = 552	15. 671 − 508 = 163	16. 937 − 780 = 157
17. 846 − 357 = 589	18. 901 − 322 = 579	19. 4218 − 2009 = 2209	20. 6439 − 5247 = 1192
21. 9186 − 7750 = 1436	22. 1265 − 311 = 954	23. 8799 − 1145 = 7654	24. 5910 − 1431 = 4479

reference page

F-100

Name

Which multiples of 100 can be used for the first partial quotient?

9)3685

3685 = 9 × 400 + 85

300 or 400 can be used.

9 × 300 = 2700

9 × 400 = 3600

9 × 500 = 4500

Which multiples of 10 can be used for the first partial quotient?

7)534

534 = 7 × 70 + 44

50, 60 or 70 can be used.

7 × 50 = 350

7 × 60 = 420

7 × 70 = 490

7 × 80 = 560

7 × 90 = 630

Can any multiple of 100 be used? No

Use a multiple of 10, 100, or 1000 as the first partial quotient. Multiply and subtract. *

1. 3)682 — 10 through 220	2. 9)547 — 10 through 60	3. 6)3695 — 10 through 610	4. 2)938 — 10 through 460
5. 4)2543 — 10 through 630	6. 7)62814 — 10 through 8970	7. 40)1689 — 10 through 40	8. 80)57423 — 10 through 710
9. 500)7842 — 10	10. 11)4852 — 10 through 440	11. 32)1098 — 10 through 30	12. 96)97832 — 10 through 1010

Write equations for any three exercises on this page. *

*Answers will vary.

F-101

Complete each exercise.

```
1.  47)9550
       9400   200
        150
        141     3
          9   203

2.  62)1948
       1860    30
         88
         62     1
         26    31

3.  34)2010
       1700    50
        310
        306     9
          4    59

4.  15)1576
       1500   100
         76
         75     5
          1   105

5.  8)936
      800   100
      136
       80    10
       56
       56     7
        0   117

6.  22)5124
       4400   200
        724
        660    30
         64
         44     2
         20   232

7.  14)3320
       2800   200
        520
        420    30
        100
         98     7
          2   237

8.  37)4586
       3700   100
        886
        740    20
        146
        111     3
         35   123
```

F-102

Pages 103 and 104 provide more practice with the division algorism. Pages 105 and 106 contain quotient-remainder equations to be solved and story problems which require either subtraction or division for their solution. Ask the pupils to write a subtraction equation or a quotient-remainder equation for each story before computing the answer. Upon completion of each assignment, instruct the pupils to tell how they found their solutions.

Name

Compute. Use as few steps as you can. Write an equation to show your results. *

1. 27)213 — $213 = (27 \times 7) + 24$

2. 28)971 — $971 = (28 \times 34) + 19$

3. 54)3921 — $3921 = (54 \times 72) + 33$

4. 67)3912 — $3912 = (67 \times 58) + 26$

5. 29)2761 — $2761 = (29 \times 95) + 6$

6. 11)659 — $659 = (11 \times 59) + 10$

7. 43)3972 — $3972 = (43 \times 92) + 16$

8. 76)3957 — $3957 = (76 \times 52) + 5$

9. 63)7929 — $7929 = (63 \times 125) + 54$

10. 49)7051 — $7051 = (49 \times 143) + 44$

11. 17)8059 — $8059 = (17 \times 474) + 1$

12. 37)752 — $752 = (37 \times 20) + 12$

F-103

* Steps and computation may vary.

Compute. Use as few steps as you can. Write an equation to show your results. *

1. 6)361 — $361 = (6 \times 60) + 1$

2. 7)492 — $492 = (7 \times 70) + 2$

3. 13)1690 — $1690 = (13 \times 130) + 0$

4. 15)2251 — $2251 = (15 \times 150) + 1$

5. 16)2560 — $2560 = (16 \times 160) + 0$

6. 25)6251 — $6251 = (25 \times 250) + 1$

7. 30)9002 — $9002 = (30 \times 300) + 2$

8. 18)1965 — $1965 = (18 \times 109) + 3$

9. 9)8128 — $8128 = (9 \times 903) + 1$

10. 33)1111 — $1111 = (33 \times 33) + 22$

11. 36)21601 — $21601 = (36 \times 600) + 1$

12. 40)79263 — 12. $79263 = (40 \times 1981) + 23$

F-104

* Steps and computation may vary.

Name

Compute the quotient and the remainder for each equation. *

1. $354 = 17q + r$ — $354 = (17 \times 20) + 14$

2. $1496 = 25q + r$ — $1496 = (25 \times 59) + 21$

3. $5997 = 37q + r$ — $5997 = (37 \times 162) + 3$

4. $3891 = 32q + r$ — $3891 = (32 \times 121) + 19$

5. $7926 = 80q + r$ — $7926 = (80 \times 99) + 6$

6. $4993 = 72q + r$ — $4993 = (72 \times 69) + 25$

Write an equation for each story. Answer each question.

7. The science class studied yeast cells. On the second day they had 439 yeast cells. They had 347 more cells on the second day than on the first day. How many cells did they have on the first day? $439 - 347 = 92$

They counted 92 cells on the first day

8. Mrs. Warner baked 2544 cookies for gifts. If she gave each of 35 friends the same number of cookies, how many cookies did she give each friend? $2544 = (35 \times 72) + 24$

Each friend received 72 cookies and there were 24 left over.

reference page

F-105

* Steps and computation may vary.

Compute. *

1. $8\overline{)7643}$

$8\overline{)7643}$
7200 | 900
443
400 | 50
43
40 | 5
3 | 955

2. $13\overline{)2695}$

$13\overline{)2695}$
2600 | 200
95
91 | 7
4 | 207

3. $14\overline{)4321}$

$14\overline{)4321}$
4200 | 300
121
112 | 8
9 | 308

4. $47\overline{)96415}$

$47\overline{)96415}$
94000 | 2000
2415
2350 | 50
65
47 | 1
18 | 2051

5. $84\overline{)7205}$

$84\overline{)7205}$
6720 | 80
485
420 | 5
65 | 85

6. $69\overline{)45303}$

$69\overline{)45303}$
41400 | 600
3903
3450 | 50
453
414 | 6
39 | 656

Answer each question.

7. A baker is to pack 1250 cupcakes for a school picnic. He will put 8 in each box. How many boxes will he fill? How many cupcakes will be left over? $1250 = (8 \times 156) + 2$

He will fill 156 boxes. Two cupcakes will be left over.

8. Golden Gate Bridge in San Francisco was built in 1937. It is 4200 feet long. Rainbow Bridge at Niagara Falls, built in 1941, is 950 feet long. How much shorter is Rainbow Bridge than Golden Gate Bridge? $4200 - 950 = 3250$

Rainbow bridge is 3250 feet shorter.

F-106

* Steps and computation may vary.

Supplemental Experiences

■ On each of several cards write a subtraction equation that contains four-digit numbers. Write correct computed differences in some of the equations and incorrect differences in others.

$\boxed{9341 - 4718 = 4533}$ $\boxed{6573 - 3298 = 3275}$

$\boxed{7038 - 3476 = 4562}$

Separate the class into two teams. Let a member from each team choose a card. Each of these pupils should place his card on the chalktray, compute the difference shown on his card, and then say whether the computed difference on the card is correct. If he finds the computed difference on his card to be incorrect, he must tell the class in which place in this number (ones, tens, hundreds, or thousands) the error occurs. Each pupil who carries out his assignment correctly earns a point for his team. Continue the activity until the entire class has participated. Then have the teams total their points to determine the winning team.

■ Review the names of the first three periods: ones, thousands, millions. Then write the following equation on the chalkboard.

682,591,734
= ___ millions + ___ thousands + ___ ones
= ______ + ______ + ______

Point to the numeral to the left of the equal sign and have the class read this numeral: six hundred eighty-two million, five hundred ninety-one thousand, seven hundred thirty-four. Tell the class that you want them to help write the number named on the board in *expanded period form.* Have a pupil come to the board and write the numeral needed to show the number of millions in the first line of the equation; have a second pupil write the number of thousands and a third pupil write the number of ones. Then, ask the class how many zeros are needed to write the standard numeral for a number of millions (6). In the second line of the equation, let a pupil write the standard numeral for 682 millions.

682,591,734
= <u>682</u> millions + <u>591</u> thousands + <u>734</u> ones
= <u>682,000,000</u> + ______ + ______

Next, ask how many zeros are needed to write the standard numeral for a number of thousands (3). Direct a second pupil to write the standard numeral for 591 thousands in the appropriate place in the equation.

682,591,734
  = 682 millions + 591 thousands + 734 ones
  = 682,000,000 + 591,000 ______ + ___

Finally, have the standard numeral for 734 ones written in the equation.

682,591,734
  = 682 millions + 591 thousands + 734 ones
  = 682,000,000 + 591,000 + 734 ______

Continue in this way to let the children write numbers such as 9021, 46,508, 370,002, and 642,019,101 in expanded period form.

■ Following is a suggested quiz that teachers may use to help uncover any difficulties that the pupils may still have.

SUGGESTED QUIZ

1. Complete the equations for each step in the algorism.

13) 869		$869 = 13q + r$
130	10	$869 = 13 \times 10 + 739$
739		
390	30	$739 = 13 \times \underline{30} + \underline{349}$
349		
260	20	$\underline{349} = 13 \times \underline{20} + \underline{89}$
89		
65	5	$\underline{89} = 13 \times \underline{5} + \underline{24}$
24		
13	1	$\underline{24} = \underline{13} \times \underline{1} + \underline{11}$
		$869 = 13 \times \underline{66} + \underline{11}$

2. Complete.

$7 \times \underline{6} = 42$
$7 \times \underline{60} = 420$
$\underline{6} \times 70 = 420$
$70 \times \underline{60} = 4200$
1400 is 14 hundreds, or 140 tens.

3. Write an equation to show the solution in each exercise.

86) 6205
13 | 72

$6205 = 86 \times 72 + 13$

113) 5655
5 | 50

$5655 = 113 \times 50 + 5$

4. Compute. Write an equation that shows your results.

18) 296
8 | 16

$296 = 18 \times 16 + 8$

25) 761
11 | 30

$761 = 25 \times 30 + 11$

5. Write an equation for this story. Find the greatest partial quotient and the least remainder. Then rewrite the equation using these numbers. Write a sentence that answers each question in the story.

Mary baked 394 cookies. She wanted to pack an equal number of cookies in each of 16 boxes. What is the greatest number of cookies she could pack in each box? How many cookies would be left over?

$394 = 16q + r$ $\quad q = 24, \; r = 10$
$394 = 16 \times 24 + 10$
Mary could pack 24 cookies in each box. There would be 10 cookies left over.

UNIT 9
GEOMETRY: POINTS
Pages 107 Through 120

OBJECTIVE
To explore the geometric concept of point.

The pupil learns to describe the precise location of a point in space by using three references. Grids and maps provide opportunities for him to practice designating locations. He also investigates how many distances are determined by various numbers of points.

See Key Topics in Mathematics for the Intermediate Teacher: Geometry.

KEY IDEAS
A point is a location.

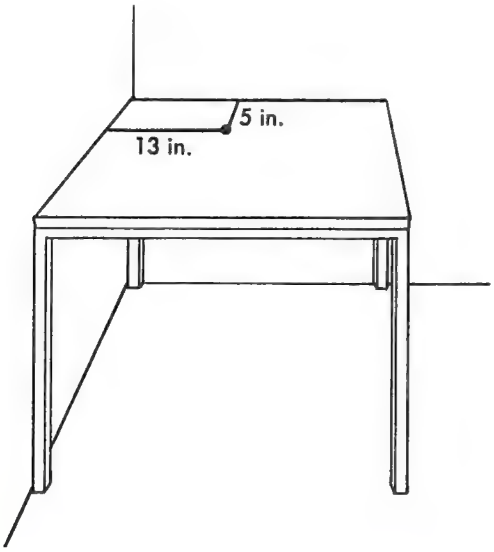
5 points determine $\frac{5 \times 4}{2}$ distances.

CONCEPTS
point

KEY IDEA
A point is a location.

Scope
To develop a concept of point by exploring locations.

Fundamentals
The geometric concept of point is developed from the idea of a precise location. The point is one kind of mathematical answer to the question, "Where?" For example, on the illustrated table top, there is a point 5 inches from the back and 13 inches from the left edge. Three references are used to establish this location. The table top is one reference; the back and left edges are the other two.



The location of a point is more precise than the location of an object having length, breadth, and width. Suppose a paper clip is said to be on the desk 5 inches from the back and 13 inches from the left edge. A difficulty arises because the paper clip would occupy many points. Some particular part of the paper clip could occupy the given location.

The concept of point as a precise location has many applications. In this unit, however, the main focus is simply to develop the concept.

Readiness for Understanding
Ability to measure.

Developmental Experiences
► Tell the class, "I am thinking of a place that is 2 feet above the floor. Where is it?" Encourage the pupils to suggest where the place is. They should not respond with the name of an object. The idea is rather to observe *where* the location "2 feet above the floor" must be.

The pupils will probably point to places in making their suggestions. For each suggested location, ask a question to test whether that location meets the description you have given: "Is this place 2 feet above the floor?"

After a few suggestions have been considered, ask, "Was my description a good way to tell where an object is?" The children will probably agree that it helped but that it was not specific enough. There are many places 2 feet above the floor.

Next tell the class, "This place that is 2 feet above the floor is also 3 feet from the front of the room." Ask the pupils to find the place.

Then ask, "Can we locate the place exactly?" After this question is discussed, suggest a third reference: "It is 5 feet from the right side of the room." After a pupil locates the place that fits all three references, say, "Now there is exactly one place that meets all of our descriptions."

Help the pupils review the three descriptions to be sure that the place they have found fits all three of them. Tell the children that the exact location you have described is a point. Point out that one or two descriptions are not enough to specify a particular point. Three descriptions are necessary.

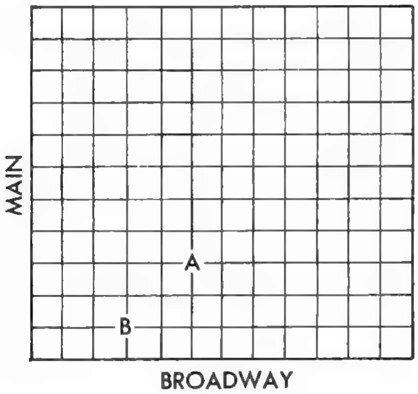
► Have the pupils think of descriptions for particular locations in the classroom. Ask each of three pupils to give a description so that the three together locate a point. For example, one pupil may give a distance above the floor, another may give a distance from a wall, and the third may give a distance from an adjacent wall. Then ask a fourth pupil to find the exact location, or point, specified by the three descriptions. When this is done, the first three pupils should be asked whether their descriptions have been met. If so, the exact location has been found, and the point is specified.

Pages 107 through 113

● Work page 107 as a class activity. Direct the pupils to use the questions on the page as a basis for class discussion. Point out that three references are used to describe the location of each insect. The pupils may need help to recognize the three-dimensional nature of the picture. It may be helpful to use sugar cubes or other kinds of blocks to make a three-dimensional model of the picture.

Ask the class to describe points in the classroom, such as the corner of a desk or the end of a pencil. In each case remind them that three references are needed to establish the precise location.

● Use the following activity as an introduction to pages 108 and 109.



On the chalkboard draw a grid similar to the one shown here. Have the children describe points that are located on the grid. Ask them to tell you the directions needed to direct someone from the intersection of Broadway and Main to point B. ("Go east three blocks and then north one block," or, "Go north one block and then east three blocks").

Continue the discussion of location. The pupils may enjoy thinking up directions for getting to places located on the grid, and trying to stump their classmates with their directions. You may want to label more points on the grid to provide exercises for everyone.

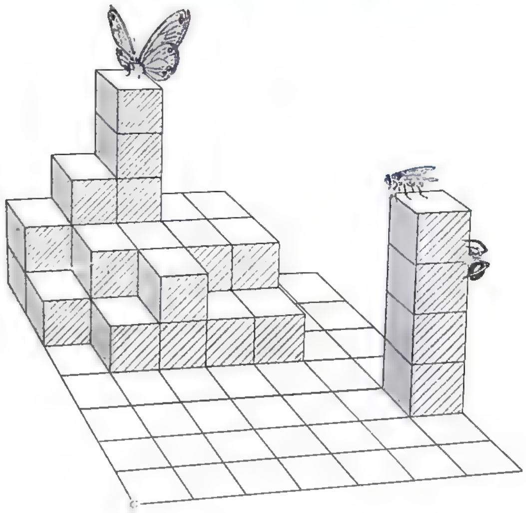
If a starting point on the grid is established, the intersection of Broadway and Main, for example, the directions for locating point B can be reduced to two simple directions: 3 and 1, or (3, 1). The first number tells how far to go east (horizontally, to the right), and the second number tells how far to go north (vertically, upward). The numbers are always given in this order in locating a point. Because of this, any point can be located on the grid merely by giving an ordered pair of numbers. After discussion of this activity, assign these pages as independent work. When they have been completed, discuss the locations of the points named with ordered pairs at the bottom of page 109.

● Use page 110 to give pupils an opportunity to use given ordered pairs of numbers to locate points on a grid. Once procedure for the page has been established, have the page completed independently.

Name

UNIT 9 GEOMETRY: POINTS

For Class Discussion *See pupil page suggestions.*



The spider is over 7, back 2, up _3_.

Where is the fly? Over 6, back _3_, up _4_.

Where is the butterfly? _Over 3, back 7, up 5_

reference page

F-107

How can we describe the location of *M*?

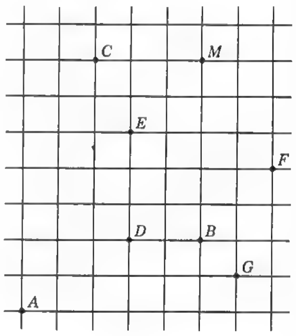
Starting at *D*, go over 2, up 5.

Starting at *E*, go over _2_, up _2_.

Starting at *C*, go over _3_, up _0_.

Starting at *B*, go over _0_, up _5_.

Starting at *A*, go over _5_, up _7_.



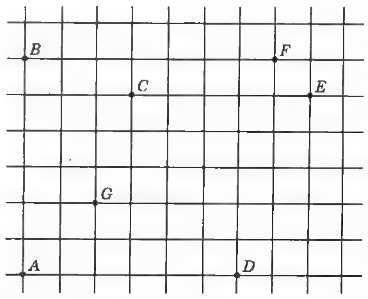
Start at *A*.

Which point is over 7, up 4? _F_

Which point is over 6, up 1? _G_

Name the point at each location. Start at *A*.

1. Over 3, up 5. _C_
2. Over 6, up 0. _D_
3. Over 7, up 6. _F_
4. Over 8, up 5. _E_
5. Over 0, up 6. _B_
6. Over 0, up 0. _A_



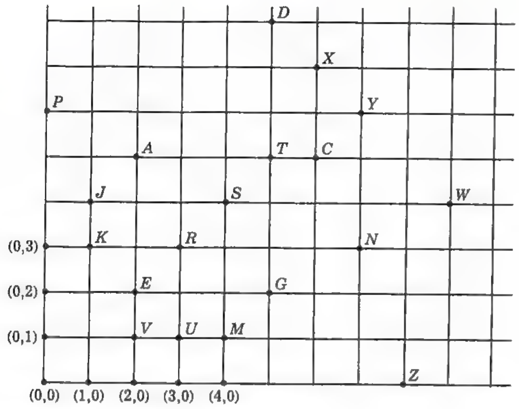
F-108

Use page 111 as a class activity. Since some of the children may not have had map study, be sure to explain that latitude is a distance. The measure of this distance is expressed in degrees. The distance from the Equator to the South Pole, for example, is 90 degrees. The distance from the Equator to the North Pole is also 90 degrees. One degree of latitude is therefore $\frac{1}{90}$ of the distance from the Equator to either pole. (A latitude degree is approximately 69 miles.) Also point out that the word "degree," or "degrees," can be symbolized by a small raised circle (°).

Ask the children to describe the approximate location of your town or city on the map. Discuss its distance in degrees from the Equator.

Discuss with the children the questions at the bottom of the page.

Name



Point K is over 1, up 3. Over 1, up 3 is (1, 3).

Which point is located at (2, 5)? A　　Which point is located at (5, 2)? G

Write the location for each point.

1. R (3, 3)　2. S (4, 4)　3. T (5, 5)　4. U (3, 1)　5. W (9, 4)
6. X (6, 7)　7. Y (7, 6)　8. Z (8, 0)　9. P (0, 6)　10. G (5, 2)

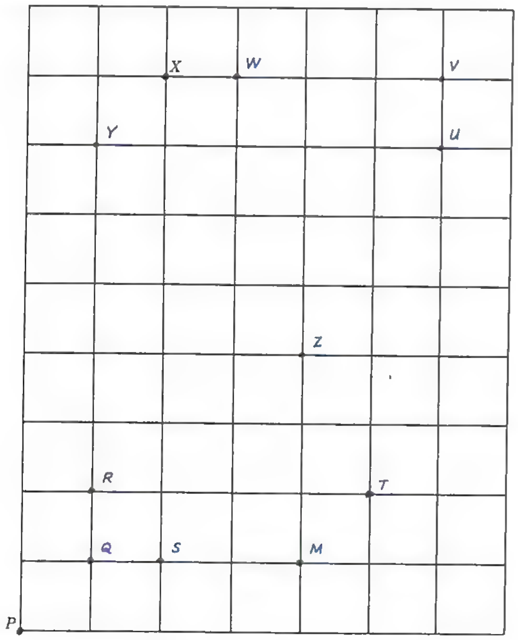
Write the letter for each point.

11. (6, 5) C　12. (5, 8) D　13. (4, 4) S
14. (7, 3) N　15. (3, 3) R　16. (4, 1) M
17. (1, 4) J　18. (2, 1) V　19. (2, 2) E

reference page

F-109

Make a dot for the location of each point. Write the letter.

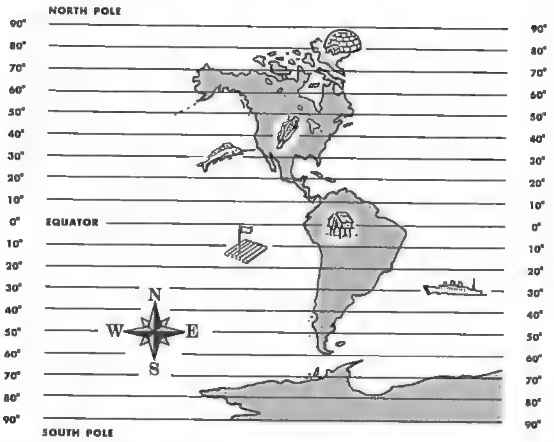


P (0, 0)	S (2, 1)	V (6, 8)	Y (1, 7)
Q (1, 1)	T (5, 2)	W (3, 8)	Z (4, 4)
R (1, 2)	U (6, 7)	X (2, 8)	M (4, 1)

F-110

Name

For Class Discussion



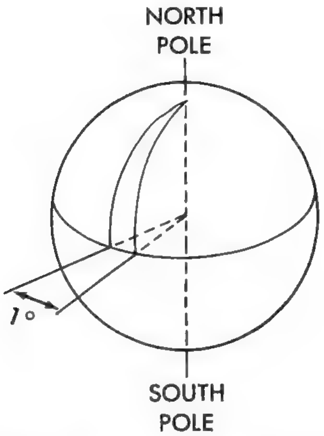
On maps, lines of latitude are used to measure distances north and south of the equator. Find the 0° line on the map. This line is called the *equator*.

1. How many degrees are there from—
   a. the equator to the North Pole? 90°
   b. the equator to the South Pole? 90°
   c. the North Pole to the South Pole? 180°

2. On the map above, what object is shown at each of these latitudes?
   a. 80° north latitude Igloo　b. 0° latitude Hut
   c. 30° south latitude Ship　d. 30° north latitude Swordfish
   e. 40° north latitude Ear of corn　f. 10° south latitude Raft

F-111

● Use page 112 as a class activity. Ask the pupils whether they notice how the lines are drawn on this map (only the vertical lines—lines of longitude—are marked). Explain that the longitude degree is not the same unit of measurement as the latitude degree. At the Equator 1 degree of longitude is the same distance as 1 degree of latitude, 69 miles. However, as this sketch shows, the distance decreases as you approach a pole.



In the United States 1 degree of longitude is about 57 miles in the southern states and about 50 miles in the northern states.

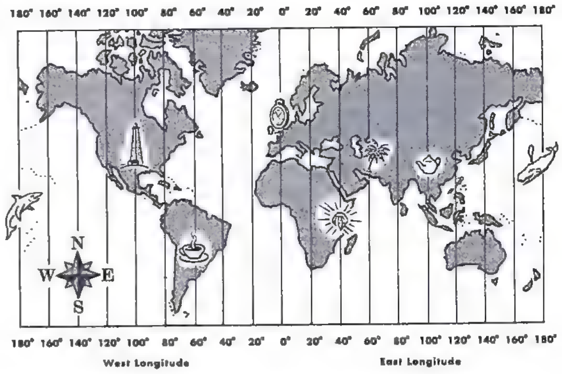
Have the pupils compare their maps to a classroom globe. On the globe they will see that the line of 180° longitude joins the line of 0° longitude to form a circle around the earth. On the pupils' flat maps the fish and the whale appear to be 360° apart. But by looking on the globe, the pupils will see that the fish and the whale are actually at the same degree of longitude.

Discuss with the children the questions at the bottom of the page.

● Have page 113 completed as a class activity. The map on the page contains both lines of latitude and lines of longitude; the compass points are identified, N, S, E, and W. Call attention to the words that show latitude lines as East-West lines and longitude lines as North-South lines. Explain that the reference line for longitude—the Greenwich meridian (0°)—was named for Greenwich, England, the location of the Royal Observatory.

You may want to have the pupils use other classroom maps, if they are available, to locate points described at the bottom of the page. It is customary to name the degree of latitude first, and then the degree of longitude.

For Class Discussion



On maps, lines of longitude are used to measure distances east and west from a line running through Greenwich, England, a town near London. Find the 0° line on the map. This line is called the *prime meridian.*

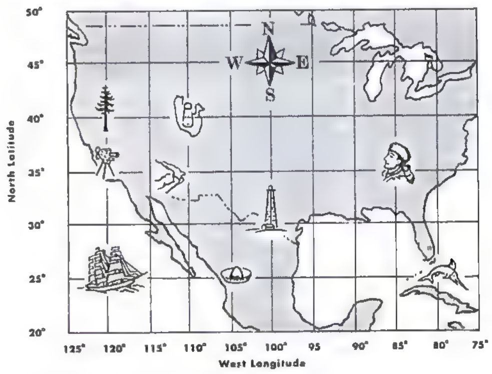
1. How many degrees of longitude are there from—
   a. the whale to Greenwich? 180°
   b. the fish to Greenwich? 180°
   c. the oil well to the diamond? 140°
   d. the whale to the fish? 360° or 0°

2. How many degrees of longitude are there from Greenwich to a point exactly halfway around the world? 180°
   (Count the degrees on the map.)

3. What object is shown on the map at each of these longitudes?
   a. 100° west longitude Oil well
   b. 180° west longitude Fish
   c. 40° east longitude Diamond
   d. 180° east longitude Whale
   e. 100° east longitude Teapot
   f. 60° west longitude Coffee Cup
   g. 60° east longitude Palm Tree
   h. 0° longitude Watch

F-112

Name



This map of the United States shows lines of latitude and lines of longitude. These two sets of lines form a system with which we can locate any point on the earth. The reference lines are the Greenwich meridian (0° longitude) and the equator (0° latitude).

1. Tell what object is shown at each point listed below.
   a. 35° north latitude, 120° west longitude Camera
   b. 35° north latitude, 85° west longitude Man
   c. 25° north latitude, 105° west longitude Sombrero

2. Name the latitude and longitude where the map shows—
   a. a pine tree. 40°N, 120°W
   b. a sailing ship. 25°N, 120°W
   c. the Great Salt Lake. 40°N, 110°W
   d. a swordfish. 25°N, 80°W
   e. an oil well. 30°N, 100°W
   f. the Grand Canyon. 35°N, 110° or 115°W

3. How many degrees of longitude are there between the swordfish and the sailing ship? 40°

   How many degrees of latitude are there between the sombrero and the Grand Canyon? 10°

F-113

KEY IDEA

5 points determine $\frac{5 \times 4}{2}$ distances.

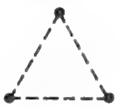
Scope

To count distances determined by points.

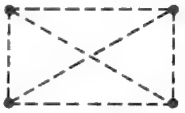
Fundamentals

In this section, the word "distance" refers to each path indicated in the following diagrams. We observe that two distinct points determine one distance. In the diagram below we see that three distinct points determine three distances. There is a pattern revealed by examining the numbers of distances determined by various numbers of points.

2 points
1 distance

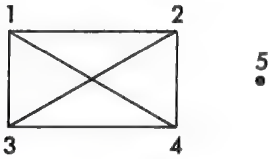


3 points
3 distances

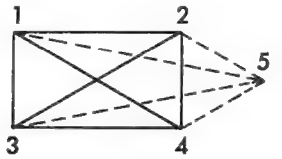


4 points
6 distances

When an additional point is introduced, additional distances can be identified in relationship to each of the already existing points. For example, when a fifth point is added to a 4-point diagram, there are 4 new distances, one from the new point to each of the 4 other points.



4 points
6 distances



5 points
(6 + 4) distances

If a 6th point is added, there are 5 more distances, making a total of 10 + 5. The overall pattern can be seen more clearly if we begin with 2 points and add points one at a time.

2 points	1 distance
3 points	1 + 2 distances
4 points	1 + 2 + 3 distances
5 points	1 + 2 + 3 + 4 distances
6 points	1 + 2 + 3 + 4 + 5 distances

The numbers that make up the above pattern are known historically as triangular numbers because they can be represented by a triangular arrangement of objects. Triangular numbers appear in many different situations.

1   1 + 2   1 + 2 + 3

1 + 2 + 3 + 4   1 + 2 + 3 + 4 + 5

Readiness for Understanding

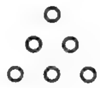
Ability to count.

Developmental Experiences

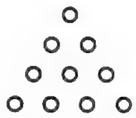
► Arrange some objects in a triangular pattern where all of the class can see the display. For example, draw circles on the chalkboard, place disks on a flannel board, put markers or disks on a table surface, or make a diagram for an overhead projector. Start forming the pattern by placing one object in position.

Then add 2 more.

Add 3 more.

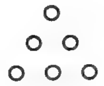


Add 4 more.



Have the children notice the triangular pattern. Explain that the number of objects in each pattern is a triangular number. The children should understand that the pattern can continue as long as you wish. Point out that there are 2 rows in the pattern for the second triangular number, 3.

There are 3 rows in the pattern for the third triangular number, 6.



Ask the children, "What is the fourth triangular number?"

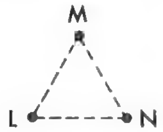
2nd triangular number	3rd triangular number	4th triangular number

The children should be able to answer this question by continuing to build the pattern. They will supply a fourth row, adding 4 objects, and find that the 4th triangular number is 10. Have the children find the 5th, 6th, and 7th triangular numbers.

The pattern of triangular numbers can be expressed as a sequence.

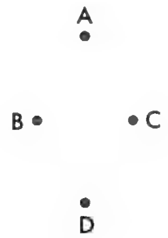
$$1, 3, 6, 10, \ldots$$

▶ Mark three points on the chalkboard, L, M, and N, and connect them, as shown.

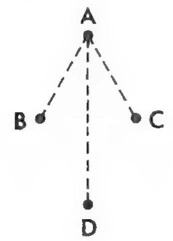


Tell the class that the 3 points determine 3 distances: one from L to M, another from M to N, and a third from L to N.

Then draw 4 dots on the chalkboard, representing 4 points, and label the points as shown.



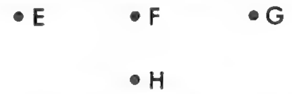
Ask the class how many distances there are from A to each of the other points. Let several children come in turn to the chalkboard, and have each child describe one of the distances. There are three such distances in all.



Mark points P, Q, R, and S on the chalkboard in an arrangement like this:

Ask a child to describe all the distances determined by the points. Tell him to be sure he matches each point with every other point. It may be helpful for the teacher to draw a faint line between each pair of points. Explain that the distances can be named by naming the pairs of points that determine them: PQ, PS, PR, RS, QS, and QR. Ask a child to list on the chalkboard the distances shown in the illustration. Then discuss whether all of the distances have been named and how many there are in all.

Examine other arrangements of 4 points, like this one, where 3 points are on a line.



Ask a child to describe all the distances determined by these points. Be sure that distances EF, FG, and EG are all named. Discuss whether the arrangement of the points makes a difference in the number of distances.

Now arrange 4 points on the chalkboard in a straight line:

Ask a child to describe all the distances determined by these points. It will be easier for the children to be sure they have identified all the distances in this arrangement if each distance is shown in a different color. The distances they should identify are: JK, JL, JM, KL, KM, and LM. Discuss whether this arrangement shows the same number of distances as the previous arrangements of 4 points.

Pages 114 through 120

Before the children begin page 114, place the following diagram on the chalkboard.

Ask the children to find the 3 distances determined by these three points. Direct a child to use dotted lines to show these distances in the illustration. Then use page 114 as a class activity. Ask a pupil to come to the chalkboard to draw a picture and show the distances for the arrangement of three points shown in exercise 1. Work exercises 2 through 4 the same way. Then discuss exercise 5 with the class and tell each child to use pencil and paper to help him answer the question.

For Class Discussion *See pupil page suggestions.*

Two points determine 1 distance.

2 points:

1 distance:

Three points determine 3 distances.

3 points:

3 distances:

Connect the points. Count the distances.

1. 3 points — How many distances? 3
2. 4 points — How many distances? 6
3. 4 points — How many distances? 6
4. 4 points — How many distances? 6
5. 5 points — How many distances? 10

reference page

F-114

In introducing page 115, draw another diagram on the chalkboard.

A B C D E F

Ask how many distances are determined by these 6 points. Have a child write the 15 distances on the chalkboard as different pupils identify them: AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, and EF.

Work the exercises on the page as a class activity. If necessary, draw the diagrams on the chalkboard and label the points for easy identification.

Name

For Class Discussion

Connect the points. Count the distances.

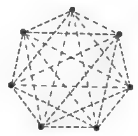
How many distances are determined by six points?

6 points — How many distances? 15

6 points — How many distances? 15

6 points — How many distances? 15

How many distances are determined by seven points? 21



How many distances are determined by eight points? 28

reference page

F-115

● Pages 116 and 117 provide an opportunity for the pupils to explore how many distances are determined by various numbers of points. As the number of points increases, the increase in the number of distances shows a pattern. Place this chart on the chalkboard and instruct the pupils to complete it.

Points	Distances
2	1
3	
4	
5	
6	
7	

Then assign the exercises on page 116 and on page 117.

● Page 118 may be completed as a class activity. After the children have had a chance to examine the patterns of dots, ask them whether they remember how triangular numbers are formed. They should remember that triangular numbers can be obtained from triangular arrangements of dots where each row contains one more member than the row immediately above it. The total number of dots is always a triangular number.

Now look at the series of numbers and encourage the children to discuss their observations and ideas.

First number	1	
Second number	1 + 2	is 3
Third number	1 + 2 + 3	is 6

This pattern may be developed at length on the chalkboard, if helpful.

● Page 119 may be used for a class discussion. Draw two diagrams on the chalkboard similar to the two diagrams shown on the page. One of the diagrams should have all distances shown. Ask a pupil to show on the second diagram the distances between point A and all the other points, while another pupil points out each corresponding distance on the completed diagram. Distances for point A:

AF AE AD AC AB

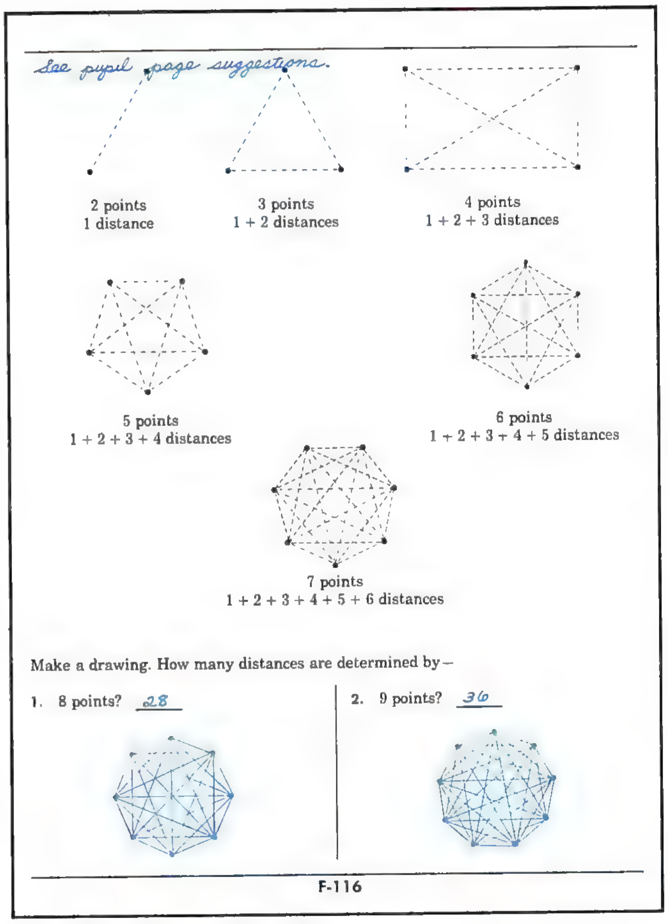
Ask another pupil to make a third diagram and show the distances between point B and all other points. Each distance should be pointed out on the first diagram. Distances for point B:

BF BE BD BC BA

Continue this procedure for each of the other points. Then ask the questions given on the page. As you discuss the distance determined by A and B, guide the pupils to see that the distance from B to A is the same as the distance from A to B. There is only one distance for these two points. After this they will see that each distance has been counted twice.

This shows that we do not have the correct number of distances when we have multiplied 6 and 5, the number of points and the number of distances per point. We must divide the product by 2 to make a correction for having counted each distance twice. The number of distances among 6 points is therefore $\frac{6 \times 5}{2}$, or 15.

● Page 120 may be completed by assigning, for each question, a small team of two or three pupils to work together in finding the answer. Discuss the results with the entire class. Do not expect more than a few pupils to be able to solve the last exercise; it is difficult. If a pupil finds the correct sum (5050), let him explain his procedure to the class.


See pupil page suggestions.

2 points
1 distance

3 points
1 + 2 distances

4 points
1 + 2 + 3 distances

5 points
1 + 2 + 3 + 4 distances

6 points
1 + 2 + 3 + 4 + 5 distances

7 points
1 + 2 + 3 + 4 + 5 + 6 distances

Make a drawing. How many distances are determined by–

1. 8 points? 28

2. 9 points? 36

F-116

Name

Make a drawing if you need to. How many distances are determined by—

1. 10 points? 45

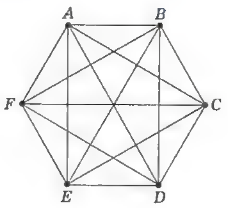
2. 12 points? 66

3. 15 points? 105

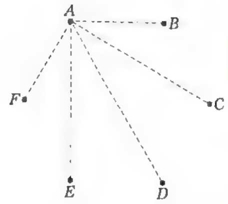
4. 20 points? 190

Name

For Class Discussion See pupil page suggestions.



How many distances are determined by 6 points?



How many distances are determined by—

point A and each of the other 5 points? 5

point B and each of the other 5 points? 5

point C and each of the other 5 points? 5

point D and each of the other 5 points? 5

point E and each of the other 5 points? 5

point F and each of the other 5 points? 5

There are 6 points. How many distances have we counted all together? 30

Was the distance determined by A and B counted twice? Yes

Which distances were counted twice? all

When we count each distance twice we count 6×5 distances.

How many distances are there when we count each distance once? 15

reference page

For Class Discussion See pupil page suggestions.

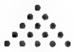
Points, Distances, and Triangular Numbers

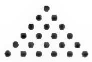
 1 = 1

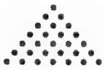
 1 + 2 = 3

 1 + 2 + 3 = 6

 1 + 2 + 3 + 4 = 10

 1 + 2 + 3 + 4 + 5 = 15

 1 + 2 + 3 + 4 + 5 + 6 = 21

 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55

Complete this list of triangular numbers.

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91.

reference page

How many distances are determined by—

4 points? 1 + 2 + 3 = 6; $\frac{4 \times 3}{2} =$ 6

5 points? 1 + 2 + 3 + 4 = 10; $\frac{5 \times 4}{2} =$ 10

6 points? 1 + 2 + 3 + 4 + 5 = 15; $\frac{6 \times 5}{2} =$ 15

7 points? 1 + 2 + 3 + 4 + 5 + 6 = 21; $\frac{7 \times 6}{2} =$ 21

8 points? 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28; $\frac{8 \times 7}{2} =$ 28

9 points? 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36; $\frac{9 \times 8}{2} =$ 36

10 points? 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45; $\frac{10 \times 9}{2} =$ 45

See pupil page suggestions.

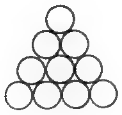
Can you find the sum of the first 100 counting numbers? How?

1 + 2 + 3 + 4 + . . . + 98 + 99 + 100 = 5050

reference page

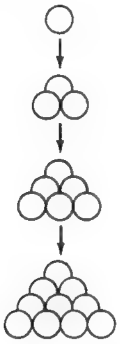
Supplemental Experiences

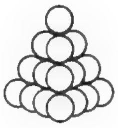
■ Some children will enjoy making a tetrahedron model using marbles and glue. A tetrahedron is a triangular (four-faced) pyramid. Each face of the model will look like this:



(flat view)

It is constructed in layers, each glued to the next as shown here:





After the model is made, the pupils can discuss some questions provoked by it. For example:

How many marbles are in the model?

How many marbles cannot be seen from the outside?

How rigid is this model? Would it be useful for constructing buildings? Can you think of any examples where it has been used?

If another layer were added, how many marbles would be used in that layer?

Could we go on adding layers as long as we wished, if it were physically possible?

Is there a similarity between triangular numbers and the number of marbles in each layer of the model?

$1, 1 + 3, 1 + 3 + 6, 1 + 3 + 6 + 10, \ldots$

If the model is turned in various directions, would the number of marbles in the bottom layer stay the same? How about the number of marbles in each of the other layers?

■ Write these sums on the chalkboard, and ask the pupils to compute them.

$1 + 2 + 3 + \ldots + 50 =$ ____
$1 + 2 + 3 + \ldots + 60 =$ ____
$1 + 2 + 3 + \ldots + 100 =$ ____
$1 + 2 + 3 + \ldots + 151 =$ ____
$1 + 2 + 3 + \ldots + 203 =$ ____
$51 + 52 + 53 + \ldots + 100 =$ ____

Point out that the three dots mean the pattern continues in the same way. If some pupils add each addend to find these sums, allow them to do so. After they have worked two or three exercises, suggest that there might be a shortcut. When the exercises are completed, have the pupils show on the chalkboard how they computed. There may be many shortcuts suggested. If it seems appropriate, point out that a fast way to compute each sum is to use the formula, $n \times \frac{n+1}{2}$. For the sums in the exercises, this formula is applied as shown here:

$$1 + 2 + 3 + \ldots + 50 = \frac{51 \times 50}{2} = 1275$$

$$1 + 2 + 3 + \ldots + 60 = \frac{61 \times 60}{2} = 1830$$

$$1 + 2 + 3 + \ldots + 100 = \frac{101 \times 100}{2} = 5050$$

$$1 + 2 + 3 + \ldots + 151 = \frac{152 \times 151}{2} = 11{,}476$$

$$1 + 2 + 3 + \ldots + 203 = \frac{204 \times 203}{2} = 20{,}706$$

$$\begin{aligned} &51 + 52 + 53 + \ldots + 100 \\ &= (1 + 2 + \ldots + 100) - (1 + 2 + \ldots + 50) \\ &= 5050 - 1275 \\ &= 3775 \end{aligned}$$

Write the following exercise on the chalkboard, and see how many pupils can find the correct computed sum (500,500).

Find the sum of the first 1000 counting numbers.

$1 + 2 + 3 + 4 + \ldots + 998 + 999 + 1000 =$ __

UNIT 10
ROUNDING NUMBERS
Pages 121 Through 136

OBJECTIVE

To introduce the concept of rounding numbers to multiples of 10 in order to estimate sums and products.

The child learns to round numbers to the nearest multiple of 10 and to round numbers up or down to the next multiple of 10. He then uses round numbers to estimate computed sums and products.

KEY IDEAS

Multiples of 10 are round numbers.

Round numbers are useful in estimating sums and products.

CONCEPTS

round number

KEY IDEA

Multiples of 10 are round numbers.

Scope

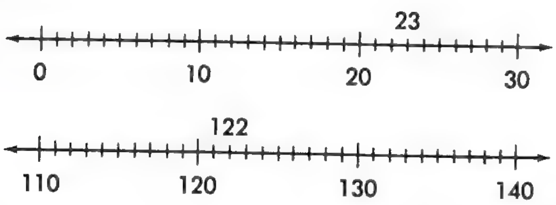
To strengthen and extend the pupil's understanding of place value.

To provide practice in rounding numbers to the nearest ten.

Fundamentals

Rounding numbers is a basic technique used in making estimations. A multiple of 10 is a product of 10 and any whole number. 0, 10, 20, 30, 40, and so forth are multiples of 10. To round a number to the nearest 10 means to replace the number by the multiple of 10 that is nearest to it. Thus the result of rounding 23 to the nearest 10 is 20.

The concept of "the nearest 10" is clearly shown on the number line. The number line shows that 23 is nearer to 20 than to 30, and that 122 is nearer to 120 than to 130.



A special case arises when the number to be rounded is midway between two multiples of 10—numbers such as 25, 35, and 45. There is no fixed rule for treating such cases. In many applications, rounding to the nearest even multiple has certain advantages. A product of 10 and an even number is an even multiple of 10 (20, 40, 60, 80, 100, and so on). For example, 25 would be rounded to 20; 35 would be rounded to 40.

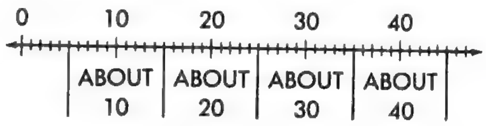
Readiness for Understanding

Ability to recognize multiples of 10.
Knowledge of place value.

Developmental Experiences

► Write on the chalkboard the question, "How many tens?" Below this question write a few multiples of 10, such as 20, 40, 800, 900, and 7000. Let volunteers each select one of the numbers and express that number in terms of tens (20 = 2 tens, 40 = 4 tens, 800 = 80 tens, 900 = 90 tens, 7000 = 700 tens).

Then draw a number line for the numbers from 0 through 100. The entire line should be similar to the segment shown here.



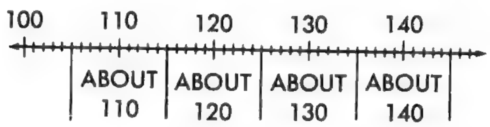
Ask the children, "About how many tens is 32?" (3) Have someone indicate where the 3-tens point is on the number line, and then locate the 32 point. Explain to the children that one way to describe 32 is to say that 32 is about 3 tens, or 30; 30 is a round number for 32. Now repeat this procedure using other numbers, such as 47, 64, 58, and 79. The pupils should notice that all the numbers between two consecutive vertical lines on the number line are rounded to the multiple of 10 that appears between these vertical lines. If someone asks what to do about a number midway between two multiples of 10, tell him that there is no one correct way to round such a number. Otherwise, do not bring up the point at this time.

Ask the children whether they have ever made statements such as these:

"Joe lives about 10 blocks from school."
"There are about 20 houses on Mike's street."
"Bob spent about 60 cents for his lunch."

Tell the children that actually Joe lives 8 blocks from school, there are 17 houses on Mike's street, and Bob spent 63 cents for his lunch. Explain that, in these statements, multiples of 10 were used to tell approximately how many things were involved. Let the children describe occasions when they have used the word *about* in this way.

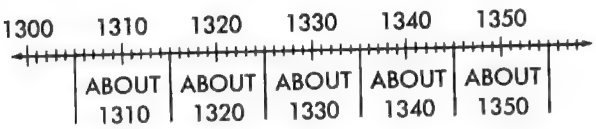
► Write on the chalkboard several three-digit multiples of 10, such as 110, 130, 160, and 190, and ask the children to state the whole number of tens in each number. Then draw a number line for the numbers from 100 through 200, as shown in part here.



Ask a child to point out 110, 130, 160, and 190 on the number line. Then tell him to point to the nearest

whole number of tens for 113 (11 tens, or 110). Use several more examples with this number line. Then extend the activity by using a 200-to-300 line, and then a 300-to-400 line. Continue to use number lines that represent intervals of 100, finally using a 900-to-1000 line. Ask what whole number of tens is nearest to each of the following: 147, 352, 512, 650, and 999. Then let the children locate these numbers on the number line. Continue this activity until every child has had a chance to round a number between 100 and 1000 to the nearest multiple of 10.

Next adapt this procedure to numbers greater than 1000, and have the children locate the whole number of tens for several four-digit numbers. Label the number line on the chalkboard from 1300 through 1400. A segment of the line is illustrated here.



Ask a child to locate 1341 on the number line; then tell him to locate the nearest multiple of 10 (134 tens, or 1340). Let the children locate several numbers between 1300 and 1400 and identify the nearest multiple of ten for each one. Continue the activity by using an 1100-to-1200 line and then a 1400-to-1500 line. Proceed to use line segments that represent intervals of 100, finally using a 1900-to-2000 line.

Have the children identify the nearest multiple of ten for each of these numbers: 1147, 3523, 5122, 6550, and 9999. Continue this activity until every child has had a chance to round a number between 1000 and 10,000. If the children need further practice, draw other parts of the number line and mark off units and intervals of ten. Follow the same procedure as described above and have the pupils round numbers to the nearest multiple of 10.

Pages 121 through 123

● Page 121 provides a short review of the idea of interpreting numerals such as 70, 900, and 8000 as a whole number of tens. Most of the children will have no difficulty interpreting numerals such as 70 and 90, but some may be unsure about numerals such as 900 and 8000. The pupils should see that the digit in the tens place is not the only digit needed to tell the number of tens in a given multiple of ten. The digits to the left of the tens place are equally important. It may also be helpful to reverse the procedure described above and have the pupils name the number that is 6 tens, for example, or the number that is 30 tens, or 700 tens. After the pupils have completed exercises 1 through 12, discuss the examples in the middle of the page. Then assign exercises 13 through 30 to be completed independently. The pupils may use the number line to help them round numbers to the nearest ten. None of the exercises on this page require the children to round a number that is midway between two multiples of 10.

● On page 122, the pupils are asked to round three-digit numbers to the nearest ten. With the class, discuss the examples at the top of the page. Encourage the pupils to think of a three-digit number as a whole number of tens and a number of ones. For example, the number 549 is 54 tens and 9 ones. Since 9 is closer to 10 than to 0, 549 must be closer to 55 tens than to 54 tens. Therefore, a round number for 549 is 550.

Next complete the eight sentences in the middle of the page with the class. Then have the pupils complete exercises 9 through 28 independently. After they have completed the assignment, discuss only those exercises that have caused difficulty.

● Page 123 is concerned with rounding numbers greater than 999. Since rounding to the nearest ten is the main idea, guide the pupils to see the importance of the last two digits of each numeral.

Discuss the example at the top of the page and exercises 1 through 6 with the class. Then assign exercises 7 through 18 for independent work. At this time some pupils may not be ready to work with numerals having more than four digits. The teacher should feel free to assign to these pupils only those exercises involving four-digit numerals. All the pupils should feel free to draw a number line if this will help them find the correct multiple of ten.

Name

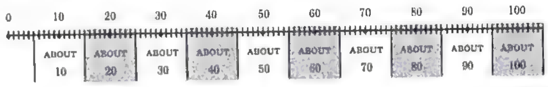
UNIT 10 ROUNDING NUMBERS

See pupil page suggestions

Each of these numbers is how many tens?

90 = 9 tens   900 = 90 tens

1. 20 *2*	2. 70 *7*	3. 700 *70*
4. 200 *20*	5. 400 *40*	6. 1000 *100*
7. 60 *6*	8. 30 *3*	9. 5000 *500*
10. 300 *30*	11. 800 *80*	12. 40 *4*



23 is about 2 tens.   20 (or 2 tens) is a round number for 23.
36 is about 4 tens.   40 is a round number for 36.

Complete each sentence.

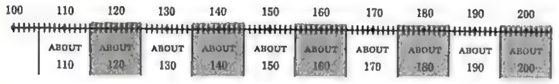
13. 44 is about *4* tens. *40* (or 4 tens) is a round number for 44.
14. 53 is about *5* tens. *50* is a round number for 53.
15. 71 is about *7* tens. *70* is a round number for 71.
16. 48 is about *5* tens. *50* is a round number for 48.
17. 57 is about *6* tens. *60* is a round number for 57.
18. 89 is about *9* tens. *90* is a round number for 89.

Round each number to the nearest ten.

19. 31 *30*	20. 52 *50*	21. 61 *60*	22. 22 *20*
23. 28 *30*	24. 36 *40*	25. 49 *50*	26. 88 *90*
27. 97 *100*	28. 78 *80*	29. 66 *70*	30. 92 *90*

reference page

F-121



132 is about 13 tens. 130 (or 13 tens) is a round number for 132.

341 is about 34 tens. 340 (or 34 tens) is a round number for 341.

549 is about 55 tens. 550 (or 55 tens) is a round number for 549.

382 is about 38 tens. 380 (or 38 tens) is a round number for 382.

Complete each sentence.

1. 121 is about 12 tens. 120 (or 12 tens) is a round number for 121.
2. 132 is about 13 tens. 130 (or 13 tens) is a round number for 132.
3. 521 is about 52 tens. 520 (or 52 tens) is a round number for 521.
4. 938 is about 94 tens. 940 (or 94 tens) is a round number for 938.
5. 257 is about 26 tens. 260 (or 26 tens) is a round number for 257.
6. 626 is about 63 tens. 630 (or 63 tens) is a round number for 626.
7. 933 is about 93 tens. 930 (or 93 tens) is a round number for 933.
8. 468 is about 47 tens. 470 (or 47 tens) is a round number for 468.

Round each number to the nearest ten.

9. 431 430	10. 526 530	11. 964 960	12. 434 430
13. 689 690	14. 197 200	15. 293 290	16. 876 880
17. 323 320	18. 842 840	19. 477 480	20. 212 210
21. 788 790	22. 236 240	23. 633 630	24. 738 740
25. 534 530	26. 128 130	27. 296 300	28. 112 110

F-122

Name

6228 is about 623 tens.

6230 (or 623 tens) is a round number for 6228.

Complete each sentence.

1. 1348 is about 135 tens. 1350 (or 135 tens) is a round number for 1348.
2. 3431 is about 343 tens. 3430 (or 343 tens) is a round number for 3431.
3. 2349 is about 235 tens. 2350 (or 235 tens) is a round number for 2349.
4. 10,436 is about 1044 tens. 10,440 (or 1044 tens) is a round number for 10,436.
5. 15,327 is about 1533 tens. 15,330 (or 1533 tens) is a round number for 15,327.
6. 30,464 is about 3046 tens. 30,460 (or 3046 tens) is a round number for 30,464.

Round each number to the nearest ten.

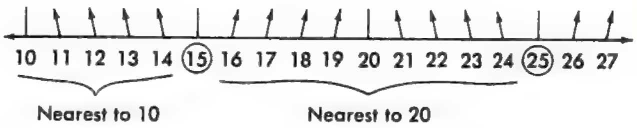
7. 2321 2320	8. 4619 4620	9. 1693 1690
10. 5728 5730	11. 9324 9320	12. 10,131 10,130
13. 56,119 56,120	14. 67,523 67,520	15. 98,986 98,990
16. 472,683 472,680	17. 936,742 936,740	18. 1,432,003 1,432,000

F-123

Supplemental Experiences

■ For the pupils who still do not understand how to find the nearest multiple of 10, draw a number line and label it to show the numbers 10 through 27.

Write the multiples of 10 in red chalk. Have the pupils consider each number in turn, starting at 10, and decide which multiple it is nearest. Draw arrows as shown in the illustration. Circle the numbers that are midway between two multiples. Use braces to show the groupings that result.



Have the pupils extend the number line in both directions and continue the labeling and marking process until they see the resulting pattern.

■ Give the pupils practice in rounding five-and six-digit numbers to the nearest multiple of 10. On the chalkboard write 37 and ask a pupil to round this number to the nearest ten (40). Then have the pupils round the following numbers to the nearest ten.

Number	Round Number
537	540
2537	2540
62,537	62,540
462,537	462,540

Help the pupils observe that, in each instance, they need to consider only the last two digits of the numeral.

Continue to have the pupils round numbers such as these:

24	11	97
724	411	597
6724	3411	7597
46,724	23,411	87,597
346,724	623,411	987,597

KEY IDEA

Round numbers are useful in estimating sums and products.

Scope

To provide practice in estimating sums and products by rounding the given numbers to the nearest multiple of 10.

To help the pupils use their skill in rounding numbers as a check on computation.

Fundamentals

Rounding numbers is a basic technique in estimation. Numbers may be rounded down, rounded up, or rounded to the nearest multiple of 10. To round a number down, round it to the nearest multiple of 10 that is less than the number. For example, 23 is rounded down to 20, 27 is rounded down to 20, 117 is rounded down to 110. Similarly, to round a number up, round it to the nearest multiple of 10 that is greater than the number. For example, 23 is rounded up to 30, 27 is rounded up to 30, and 117 is rounded up to 120. The most common technique involves rounding to the nearest multiple of 10.

When estimating the result of a computation, it is helpful to determine a range of numbers within which the result will be found. For example, in computing a sum, a minimum estimate can be made by rounding all of the addends down and then adding. A maximum estimate can be made by rounding all of the addends up and adding. An estimate that is closer to the actual sum can be made by rounding each addend to the nearest multiple of 10.

Rounding numbers can also be useful in checking computations.

Readiness for Understanding

Ability to compute.

Knowledge of place value and multiples of 10.

Developmental Experiences

► On the chalkboard, write the following numbers.

$12 \longrightarrow 20$
$25 \longrightarrow 30$
$38 \longrightarrow 40$
$49 \longrightarrow 50$
$99 \longrightarrow 100$
$126 \longrightarrow 130$
$488 \longrightarrow 490$
$934 \longrightarrow 940$

Have the pupils read each pair of the numbers aloud. Tell them that each number in the first column has been rounded to a multiple of 10. Ask whether the given multiple of 10 for each number is the nearest multiple of 10. (In some instances it is; in others it is not.) Ask the class to compare the number of tens in the round number with the number of tens in the given number. Some pupil may discover that each round number has one more ten than the given number. For example, there is one ten in 12 but two tens in 20; there are two tens in 25 but three tens in 30. Another pupil may say that each number has been rounded to the next largest multiple of 10. Explain to the class that when numbers are rounded in this way, we say they are rounded up to the next greater 10. Read the following numbers to the class, and ask the pupils to round up each number to the next greater 10: 27, 63, 234, 6761, and 13,456.

Now erase the second column of numbers and, to the left of the remaining column, write the round numbers indicated on the left below.

$10 \longleftarrow 12$
$20 \longleftarrow 25$
$30 \longleftarrow 38$
$40 \longleftarrow 49$
$90 \longleftarrow 99$
$120 \longleftarrow 126$
$480 \longleftarrow 488$
$930 \longleftarrow 934$

Ask the class to observe the ones-digit in the given number and the ones-digit in the round number and to tell what they note. A pupil may say that the ones have been eliminated from each number and only the given number of tens remains. Another pupil may say that each number has been rounded to the next smaller multiple of 10. Accept all explanations that describe the type of rounding that is shown. Then explain that when numbers are rounded in this way, we say they are rounded down to the next lower 10. Read the following numbers and ask several pupils to round down each number: 32, 76, 543, 8469, and 17,255.

► Write the following addition exercise on the chalkboard. Ask a pupil to compute the sum of the four addends.

$$\begin{array}{r} 20 \\ 30 \\ 50 \\ +\ 60 \\ \hline 160 \end{array}$$

Point to the 20 and tell the class to imagine a new addition exercise in which the first addend is 1 less than 20. Ask what number the first addend in this exercise will be. Write 19 on the chalkboard as the first number in a new column. In turn, point to each of the other addends in the first exercise. Tell the class that the second addend in the new exercise is 2 more than 30; the third addend is 1 less than 50; the last addend is the same. Write the new numbers in the second column, and ask a pupil to compute the sum (160).

$$\begin{array}{r} 20 \\ 30 \\ 50 \\ +\,60 \\ \hline 160 \end{array} \qquad \begin{array}{r} 19 \\ 32 \\ 49 \\ +\,60 \\ \hline 160 \end{array}$$

The pupils should see that the sum is the same in both exercises.

Now ask the class to imagine a third exercise. Each of the first three addends will be 1 greater than the corresponding addend in the first exercise (21, 31, and 51). The fourth addend will be 4 less than the 60 of the first exercise. Write the new numbers in a third column on the chalkboard, and ask a pupil to compute the sum (159).

$$\begin{array}{r} 20 \\ 30 \\ 50 \\ +\,60 \\ \hline 160 \end{array} \qquad \begin{array}{r} 19 \\ 32 \\ 49 \\ +\,60 \\ \hline 160 \end{array} \qquad \begin{array}{r} 21 \\ 31 \\ 51 \\ +\,56 \\ \hline 159 \end{array}$$

The pupils will see that in this exercise the sum is not the same as in the first two exercises. Ask whether this sum might be described as about 160. A pupil may say, "Yes" and explain that when 159 is rounded to the nearest 10, the result is 160. Help the pupils see that each addend in the first column is the result of rounding each corresponding addend in the third column to the nearest 10. Point out that the sum of the round numbers in the first column is about the sum of the numbers in the third column.

On the chalkboard write several more addition exercises in columns. For each exercise let one pupil compute the sum of the addends, and have another pupil round the addends to the nearest 10 and compute the sum of the round numbers. The pupils should see that the sum of the round numbers is a good estimate of the sum of the given numbers.

► Write this exercise on the chalkboard.

$$\begin{array}{r} 243 \\ 681 \\ 416 \\ +\,792 \\ \hline \end{array}$$

Tell the class that several children computed the sum of these four addends and that you would like them to decide which child computed correctly. Explain that you would like them to do this without using pencil and paper. On the chalkboard list the result of the children's computations.

Bill	232
Joan	47,232
Mary	2132
Vincent	8232
Paul	13,232

Then ask the pupils to explain their reasons for deciding whether each result is correct. A pupil may say that he knows Bill's result is incorrect because his computed sum, 232, is less than any of the addends. Another pupil may decide that Joan, Vincent, and Paul computed incorrectly since each addend is less than 1000, which means that the computed sum should be less than 4000. Some pupils may round each addend to the nearest 10 and decide that Mary's result is correct since it is the only computed sum that is about 213 tens. The pupils should conclude that Mary computed correctly. Have a pupil compute the sum of the four addends to check Mary's computation.

Follow this procedure for several more addition exercises. In each instance list several results. Direct the pupils to explain their reasons for deciding which result is correct.

► Tell the class to imagine that they have an array of 29 rows with 31 squares in each row. Have a pupil write the product for this array on the chalkboard.

$$\begin{array}{r} 29 \\ \times\,31 \\ \hline \end{array}$$

Tell the class that they can estimate the computed product without actually computing it. Ask them to round both 29 and 31 to the nearest 10 (30 in both cases). To the right of each of the given factors, write the symbol for "is about" and 30. Explain that the symbol $\approx$ means "is about." Have a pupil compute 30×30 and write the result on the chalkboard (3 tens $\times$ 3 tens $=$ 9 ten tens, or 9 hundreds).

$$\begin{array}{r} 29 \\ \times\,31 \\ \hline \end{array} \qquad \begin{array}{r} 30 \\ \times\,30 \\ \hline \end{array}$$

Let the pupils explain why this computation tells us that the product 29×31 is about 900.

Have a pupil compute the product 29×31 to see whether the product of the round numbers is a good estimation.

$$\begin{array}{r} 29 \\ \times\,31 \\ \hline 29 \\ 870 \\ \hline 899 \end{array} \qquad \begin{array}{r} 30 \\ \times\,30 \\ \hline 900 \end{array}$$

The pupils will see that the rounded product is nearly the same as the exact product.

Follow this procedure using other examples. In each instance, ask one pupil to write the product for the given array. Have a second pupil round the factors to the nearest 10 and compute the product of the round factors. Let a third pupil compute the product of the given numbers. The pupils will see that the product of the round numbers provides an estimate of the product of the given numbers.

Pages 124 through 136

● Page 124 provides practice in rounding up numbers to the next higher 10. Discuss the examples at the top of the page with the class, and have the children answer the questions that follow. Assign exercises 1 through 26. Not all the exercises need to be completed by all of the pupils. Be selective in making assignments. The exercises involving five- and six-digit numerals may be omitted for those pupils who have difficulty with them.

● Page 125 provides practice in rounding down numbers to the next lower 10. Discuss the examples at the top of the page with the class, and have the children answer the questions that follow. Assign exercises 1 through 24 for independent work.

● Discuss with the class the example at the top of page 126. After the pupils have answered the questions at the bottom of page 126, let them work and discuss the exercises on page 127. Also discuss how rounding can be used as a rough check for an addition exercise. Estimation will not tell whether the computed sum is correct, but it will indicate whether the computed sum is reasonably close to the actual sum.

● Pages 128 and 129 provide practice in computing rounded sums. Discuss the example at the top of page 128 and work exercises 1 and 2 with the class. Be sure the pupils understand that they need to compute only the rounded sum. Then assign exercises 3 through 9 on page 128 and the exercises on page 129 for independent work. After the assignment has been completed, the teacher may wish to have the pupils compute the exact sum for a few of the exercises and compare it to the rounded sum.

Numbers may also be rounded in this way.

12 rounds up to 20.	91 rounds up to 100.
17 rounds up to 20.	123 rounds up to 130.
49 rounds up to 50.	434 rounds up to 440.

Do you see a pattern? What is it? Yes. Each number is rounded to the next higher 10.

When we round numbers in this way, we are rounding to the next higher 10.

Round each number to the next higher 10.

14 rounds up to 20.

1. 23 rounds up to 30.
2. 336 rounds up to 340.
3. 77 rounds up to 80.
4. 432 rounds up to 440.
5. 85 rounds up to 90.
6. 986 rounds up to 990.
7. 89 rounds up to 90.
8. 15,955 rounds up to 15,960.
9. 46 rounds up to 50.
10. 175,633 rounds up to 175,640.

11. 13 20	12. 39 40	13. 118 120	14. 62,967 62,970
15. 15 20	16. 46 50	17. 521 530	18. 98,435 98,440
19. 22 30	20. 75 80	21. 964 970	22. 764,423 764,430
23. 26 30	24. 87 90	25. 1284 1290	26. 974,386 974,390

reference page

F-124

Name

Numbers may be rounded in another way.

12 rounds down to 10.	32 rounds down to 30.
15 rounds down to 10.	75 rounds down to 70.
19 rounds down to 10.	99 rounds down to 90.

Do you see a pattern? What is it? Yes. Each number is rounded to the next lower 10

When we round numbers in this way, we are rounding to the next lower 10.

Round each number to the next lower 10.

17 rounds down to 10.

1. 14 rounds down to 10.
2. 34 rounds down to 30.
3. 19 rounds down to 10.
4. 745 rounds down to 740.
5. 23 rounds down to 20.
6. 187 rounds down to 180.
7. 28 rounds down to 20.
8. 1164 rounds down to 1160.

9. 12 10	10. 39 30	11. 114 110	12. 1864 1860
13. 14 10	14. 45 40	15. 343 340	16. 9981 9980
17. 21 20	18. 86 80	19. 581 580	20. 15,638 15,630
21. 27 20	22. 99 90	23. 966 960	24. 64,775 64,770

reference page

F-125

Mark went to the pet shop and decided to buy these items.

		Exact sum
Three goldfish	73¢	73¢
Piece of coral	56¢	56¢
Can of fishfood	39¢	39¢
Colored stones	14¢	14¢
		182¢

Mark wanted to be sure that he had enough money. He wrote each rounded price on a piece of paper and added them all together. Which method of rounding do you think Mark used?

Rounding each addend up	Rounding each addend down	Rounding each addend to the nearest 10
80¢	70¢	70¢
60¢	50¢	60¢
40¢	30¢	40¢
20¢	10¢	10¢
200¢	160¢	180¢

Which rounded sum is closest to the exact sum?

1. Which method of rounding gave a rounded sum that is less than the exact sum? Rounding each addend down

2. Which method of rounding gave a rounded sum that is greater than the exact sum? Rounding each addend up

3. Which method of rounding gave a rounded sum that is nearest to the exact sum? Rounding each addend to the nearest 10

reference page

Name

Work each exercise in four ways.

	Exact sum	Rounding each addend up	Rounding each addend down	Rounding each addend to the nearest 10
1.	56	60	50	60
	14	20	10	10
	73	80	70	70
	+ 39	40	30	40
	182	200	160	180
2.	21	30	20	20
	77	80	70	80
	26	30	20	30
	+ 28	30	20	30
	152	170	130	160
3.	47	50	40	50
	82	90	80	80
	29	30	20	30
	+ 93	100	90	90
	251	270	230	250

Which method of rounding in these exercises gave a rounded sum that is nearest to the exact sum? Rounding each addend to the nearest 10

Ted liked to keep track of basketball scores. One season Flash Finnegan scored the most points of any player in the tournament. The coach asked Ted how many points Flash scored.

Flash's scores:

34
62
58
+ 43

First Ted computed the sum of 30 and 60 and got 90. Then he added this sum to 60 and got 150. Finally he added 150 and 40 and got an answer of 190. Was Ted close to Flash's total points? Yes
Which method of rounding numbers did Ted use?
Ted rounded to the nearest 10.

Round each addend to the nearest ten and compute. Try to use Ted's shortcut.

	Addends	Rounded
1.	24	20
	16	20
	33	30
	+ 10	10
		80
2.	18	20
	19	20
	20	20
	+ 13	10
		70
3.	32	30
	11	10
	83	80
	+ 69	70
		190
4.	38	40
	39	40
	98	100
	+ 11	10
		190
5.	59	60
	76	80
	13	10
	+ 12	10
		160
6.	47	50
	49	50
	41	40
	+ 44	40
		180
7.	86	90
	96	100
	76	80
	+ 66	70
		340
8.	71	70
	82	80
	63	60
	+ 94	90
		300
9.	56	60
	37	40
	41	40
	+ 64	60
		200

Name

Round each addend to the nearest 10 and compute. Try to use Ted's shortcut.

	Addends	Rounded
1.	67	70
	80	80
	9	10
	+ 32	30
		190
2.	23	20
	98	100
	37	40
	+ 84	80
		240
3.	30	30
	67	70
	90	90
	+ 78	80
		270
4.	42	40
	52	50
	88	90
	44	40
	+ 74	70
		290
5.	73	70
	6	10
	38	40
	59	60
	+ 2	0
		180
6.	16	20
	61	60
	14	10
	19	20
	+ 68	70
		180
7.	83	80
	82	80
	81	80
	80	80
	+ 83	80
		400
8.	63	60
	74	70
	17	20
	29	30
	+ 11	10
		190
9.	16	20
	16	20
	16	20
	16	20
	+ 16	20
		100
10.	21	20
	21	20
	21	20
	21	20
	+ 21	20
		100
11.	101	100
	101	100
	101	100
	101	100
	+ 101	100
		500
12.	32	30
	32	30
	46	50
	48	50
	+ 53	50
		210

● On pages 130 through 133 the pupils will investigate the uses of rounding in multiplication.

Use page 130 as a discussion page. Direct the pupils to study the example and answer the questions. Then have the children complete the exercises on page 131 independently. Discuss and answer the question at the bottom of the page.

Pages 132 and 133 provide practice in computing rounded products. Discuss the example at the top of page 132 and work exercises 1 and 2 with the class. Be sure the pupils understand that they need compute only the round product. Then instruct the pupils to complete exercises 3 through 13 on page 132 and the exercises on page 133 independently. After the assignment has been completed, the teacher may have the pupils compute the exact product for a few of the exercises and compare it to the rounded product.

● Pages 134 through 136 provide additional practice in rounding.

On pages 134 and 135, the pupils will use rounding to approximate sums and products in story exercises. Discuss the example at the top of page 134 with the class. Make sure the pupils understand that they may round in any way they wish and that the story questions are to be answered with a round number. When you are sure the pupils understand the procedure to be followed, assign the exercises for independent work.

Tell the pupils to read the directions at the top of page 135. They should understand that they are to round the numbers in each exercise to the nearest 10 before finding the rounded sum or product. Exercises 1 through 6 may be assigned for independent work.

For Class Discussion

Study this example.

Can you see the steps followed?

Exact product

$$\begin{array}{r} 73 \\ \times\ 46 \\ \hline 18 \\ 420 \\ 120 \\ 2800 \\ \hline 3358 \end{array}$$

Rounding each factor down	Rounding each factor up	Rounding each factor to the nearest 10
$\begin{array}{r} 70 \\ \times\ 40 \\ \hline 2800 \end{array}$	$\begin{array}{r} 80 \\ \times\ 50 \\ \hline 4000 \end{array}$	$\begin{array}{r} 70 \\ \times\ 50 \\ \hline 3500 \end{array}$

Which rounded product is closest to the exact product?

1. Which method of rounding gave a rounded product that is less than the exact product?

   Rounding each factor down

2. Which method of rounding gave a rounded product that is greater than the exact product?

   Rounding each factor up

3. Which method of rounding gave a rounded product that is nearest to the exact product?

   Rounding each factor to the nearest 10

reference page

F-130

Name

Work each exercise in four ways.

	Exact product	Rounding each factor down	Rounding each factor up	Rounding each factor to the nearest 10
1.	89 × 22 1958	80	2700	90 1800
2.	51 × 77 7 350 70 3500	50 × 70 3500	4800	50 80 4000
3.	92 × 37 14 630 60 2700 3404	90 30 2700		90 40 3600

Which method of rounding the factors in these exercises gave a rounded product that is nearest to the exact product? Rounding each factor to the nearest 10.

reference page

F-131

Exact product	Rounded product
64 × 58 = 3712	60 × 60 = 3600

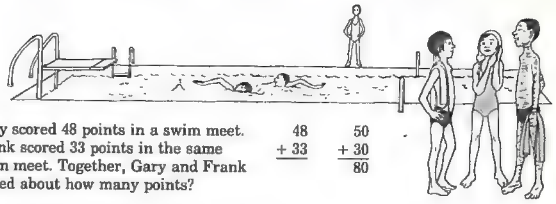
Round each factor to the nearest 10. Compute the rounded product.

1. 57 × 88 — 60 × 90 = 5400
2. 64 × 39 — 60 × 40 = 2400
3. 92 × 81 — 90 × 80 = 7200
4. 76 × 59 — 80 × 60 = 4800
5. 18 × 81 — 20 × 80 = 1600
6. 72 × 67 — 70 × 70 = 4900
7. 89 × 78 — 90 × 80 = 7200
8. 23 × 61 — 20 × 60 = 1200
9. 19 × 12 — 20 × 10 = 200
10. 91 × 92 — 90 × 90 = 8100
11. 97 × 92 — 100 × 90 = 9000
12. 97 × 98 — 100 × 100 = 10,000

13. Which of these products is closest to 10,000? 98 × 101 is closest.

60 × 240   94 × 103   98 × 101

F-132



Gary scored 48 points in a swim meet. Frank scored 33 points in the same swim meet. Together, Gary and Frank scored about how many points?

48 + 33   50 + 30 = 80

1. Betty has 13 perfect attendance ribbons. Ginny has 11. About how many ribbons do they have? about 20 ribbons
2. Daisy has 23 tomatoes. George has 52. About how many tomatoes do they have? about 70 tomatoes
3. Bill's mother baked 37 cookies a day for 24 days. About how many cookies did she bake? about 800 cookies
4. There are 34 pupils in Jim's class, 39 in Greg's, and 18 in John's. About how many children are in the three classes? about 90 children
5. Paul sold 79 tickets for a school play. He collected 50¢ for each ticket. About how much money did Paul collect? about \$40
6. Jack had 12 pencils, Mary had 33, Beth had 16, and Tom had 24. About how many pencils did they have? about 80 pencils
7. Ruth has 33 buttons. Cindy has 12 times as many. About how many buttons does Cindy have? about 300 buttons

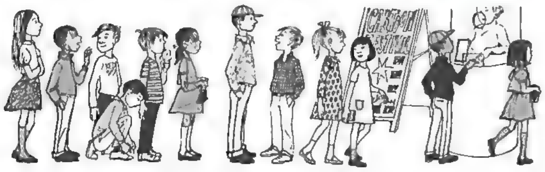
F-134

Name

Round each factor to the nearest 10. Compute the rounded product.

1. 63 × 47 — 60 × 50 = 3000
2. 34 × 51 — 30 × 50 = 1500
3. 40 × 38 — 40 × 40 = 1600
4. 33 × 66 — 30 × 70 = 2100
5. 61 × 92 — 60 × 90 = 5400
6. 58 × 68 — 60 × 70 = 4200
7. 78 × 39 — 80 × 40 = 3200
8. 69 × 82 — 70 × 80 = 5600
9. 102 × 98 — 100 × 100 = 10,000

10. 687 children saw the cartoon show. Tickets cost 48¢ each. About how much is 687 × 48¢? about \$345



F-133

Name

Round the numbers in each exercise to the nearest ten. Then find the rounded sum or product. Answer each story question with a round number.

1. There are 22 classes in Washington School. If there are 27 children in each class, about how many children go to Washington School? about 600 children — 20 × 30 = 600
2. Judy practiced the piano 36 minutes on Monday, 23 minutes on Tuesday, and 47 minutes on Wednesday. About how many minutes did Judy practice during these three days? about 110 minutes — 40 + 20 + 50 = 110
3. Rich earned \$1.50 a week for 22 weeks. About how much money did Rich earn? about \$30 — 1.50 × 20 = 30.00
4. Jack had 12 pencils and Mary had 32. About how many pencils did they have? about 40 pencils — 30 + 10 = 40
5. Jim is gathering sticks to make a model fort. He has gathered 47 sticks. He plans to cut each stick into 8 pieces. About how many pieces can he make from the 47 sticks? about 500 sticks — 50 × 10 = 500
6. Sally bought 36 boxes of candy bars with 24 bars in each box. About how many candy bars did she buy? about 800 bars — 40 × 20 = 800

F-135

● Page 136 gives the pupils an opportunity to test their ability to round numbers to the nearest 10. Discuss the examples at the top of the page with the class. Then assign exercises 1 through 13 for independent work. When the pupils have completed the assignment, discuss those exercises that may have caused difficulty.

Round each addend or factor to the nearest ten. Compute.

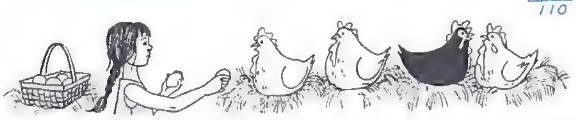
Exact addends	Rounded sum	Exact factors	Rounded product
28 96 63 77 + 54	30 100 60 80 + 50 320	78 × 56	80 × 60 4800

	Exact	Rounded
1.	34 × 52	30 50 1500
2.	39 83 41 + 76	40 80 40 80 240
3.	89 × 48	90 50 4500
4.	36 31 39 + 47	40 30 40 50 160
5.	63 8 57 14 + 91	60 10 60 10 90 230
6.	98 × 98	100 100 10,000
7.	98 97 2 + 91	100 100 0 90 290
8.	41 × 69	40 70 2800
9.	91 4 82 + 9	90 0 80 10 180
10.	42 43 44 46 + 47	40 40 40 50 50 220
11.	28 29 30 31 + 32	30 30 30 30 30 150
12.	79 × 30	80 30 2400

13. During one week Janet gathered 14 eggs, 16 eggs, 13 eggs, 17 eggs, 16 eggs, 14 eggs, and 16 eggs. About how many eggs did Janet gather?

About 110 eggs

10
20
10
20
20
10
+ 20
110



F-136

Supplemental Experiences

■ Write a list of multiples of 10 on the chalkboard. Have the pupils identify all the numbers that result in the first multiple when rounded up. Then have them identify all the numbers that result in the first multiple when rounded down. Repeat this procedure for each multiple of 10 that is listed on the chalkboard.

MULTIPLES OF 10	ROUNDING UP	ROUNDING DOWN
20	11 through 19	21 through 29
80	71 through 79	81 through 89
110	101 through 109	111 through 119
230	221 through 229	231 through 239
460	451 through 459	461 through 469
700	691 through 699	701 through 709
8650	8641 through 8649	8651 through 8659

■ This procedure may help the pupils find the multiple of 10 nearest a given number. Count up from the number until a multiple of 10 is reached. Then count down from the number until another multiple of 10 is reached. The nearest multiple of 10 is the one reached in fewer counting steps. For example, count up from 63: 64, 65, 66, 67, 68, 69, 70 (7 steps). Count down from 63: 62, 61, 60 (3 steps). Thus 60 is the nearest multiple of 10 to 63.

■ Here is a quiz that teachers may wish to use.

SUGGESTED QUIZ

1. Complete each sentence.

40 = 4 tens
400 = 40 tens
4000 = 400 tens

2. Round each number up to the next greater 10.

56	60	71	80
483	490	296	300

3. Round each number down to the next lower 10.

66	60	92	90
449	440	731	730

4. Round each addend to the nearest 10. Compute the sum of the rounded addends.

56 98 47 + 82	60 100 50 + 80 290	38 13 41 + 64	40 10 40 + 60 150

5. Round each factor to the nearest 10. Compute the product of the rounded factors.

32 × 49	30 × 50 1500	88 × 51	90 × 50 4500

UNIT 11
THE DOZENAL SYSTEM
Pages 137 Through 148

OBJECTIVE
To introduce the dozenal system of numeration.

The child learns that the dozenal system of numeration is a place-value system that is based on dozens (twelves), dozen dozens, dozen dozen dozens, and so on.

This examination of a possible alternative helps the child appreciate the decimal numeration system. This unit should be considered for exploration rather than mastery. The child learns to understand better the role of ten in the decimal system of numeration.

KEY IDEA
The dozenal system groups by twelves.

CONCEPTS
dozen
gross
great gross

KEY IDEA
The dozenal system groups by twelves.

Scope
To group by twelves.

Fundamentals
The decimal system of numeration is a place-value system based on powers of 10: tens, ten tens, ten ten tens, and so on. For example:

$$\begin{aligned} 4567 &= 4000 + 500 + 60 + 7 \\ &= 4 \text{ ten ten tens} + 5 \text{ ten tens} + 6 \text{ tens} + 7 \\ &= 4 \times (10 \times 10 \times 10) + 5 \times (10 \times 10) \\ &\quad + (6 \times 10) + 7 \\ &= 4 \text{ thousands} + 5 \text{ hundreds} + 6 \text{ tens} + 7 \end{aligned}$$

In the same way, the dozenal system of numeration is a place-value system based on powers of 12: dozens (twelves), dozen dozens, dozen dozen dozens, and so on. A dozen dozens is called a gross, and a dozen dozen dozens is called a great gross.

$$\begin{aligned} &1 \text{ gross} = 1 \text{ dozen dozens, or } 1 \times (12 \times 12) = 144 \\ &1 \text{ great gross} = 1 \text{ dozen dozen dozens, or} \\ &\qquad 1 \times (12 \times 12 \times 12) = 1728 \end{aligned}$$

Here are some examples that show how to express whole decimal numbers in the dozenal system.

$$\begin{aligned} 27 &= 2 \text{ dozens} + 3, \text{ or } 2 \text{ dozen } 3 \\ 162 &= 1 \text{ dozen dozens} + 1 \text{ dozen} + 6 \\ &= 1 \text{ gross} + 1 \text{ dozen} + 6, \\ &\quad \text{or } 1 \text{ gross } 1 \text{ dozen } 6 \\ 332 &= 2 \text{ dozen dozens} + 3 \text{ dozens} + 8 \\ &= 2 \text{ gross} + 3 \text{ dozens} + 8, \\ &\quad \text{or } 2 \text{ gross } 3 \text{ dozen } 8 \\ 4571 &= 380 \text{ dozens} + 11 \\ &= 31 \text{ dozen dozens} + 8 \text{ dozens} + 11 \\ &= 2 \text{ dozen dozen dozens} + 7 \text{ dozen dozens} \\ &\quad + 8 \text{ dozens} + 11, \\ &= 2 \text{ great gross} + 7 \text{ gross} + 8 \text{ dozens} + 11, \\ &\quad \text{or } 2 \text{ great gross } 7 \text{ gross } 8 \text{ dozen } 11 \end{aligned}$$

Any number written with the standard numerals of the Hindu-Arabic decimal system can also be written in the dozenal system. This is done by using division.

The computations needed to express 332 in the dozenal system, the third example above, are those shown below.

$$\begin{aligned} 332 &= 27 \times 12 + 8 \\ &= 27 \text{ dozens} + 8 \end{aligned}$$

$$\begin{array}{r|l} 12\overline{)\,332} & \\ \underline{120} & 10 \\ 212 & \\ \underline{120} & 10 \\ 92 & \\ \underline{84} & \underline{7} \\ 8 & 27 \end{array}$$

$$\begin{aligned} 27 &= 2 \times 12 + 3 \\ 27 \text{ dozens} &= (2 \times 12 + 3) \text{ dozens} \\ &= 2 \text{ dozen dozens} + \\ &\quad 3 \text{ dozens} \end{aligned}$$

$$\begin{array}{r|l} 12\overline{)\,27} & \\ \underline{24} & \underline{2} \\ 3 & 2 \end{array}$$

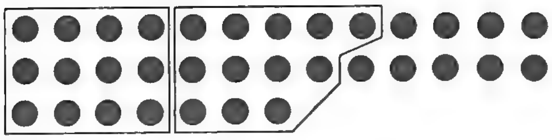
So: $332 = 2$ dozen dozens $+ 3$ dozens $+ 8$, or 2 gross 3 dozen 8

Readiness for Understanding
Ability to group by twelves.

Developmental Experiences
paper strips
rubber bands
paper clips
tagboard cards
felt-tip pen
pocket chart

► Write the word *dozen* on the chalkboard, and ask a child to count a dozen things he can see. Ask another child to give the meaning of the word *dozen*. Then ask the pupils to think of some items that are sometimes packaged or purchased by the dozen (pencils, oranges, eggs).

Draw 33 dots on the chalkboard, as illustrated, and direct a pupil to partition them into sets of a dozen. Express this partitioning in several different ways.



1 dozen + 1 dozen + 9
12 + 12 + 9
2 × 12 + 9
2 dozen 9

Instruct a pupil to write the standard numeral for the number of dots and then to use the division algorism to find out how many dozens of dots there are and how many individual dots are left over.

$$\begin{array}{r|l} 12\overline{)\,33} & \\ 12 & 1 \\ \hline 21 & \\ 12 & 1 \\ \hline 9 & 2 \end{array}$$

$33 = 2 \times 12 + 9$

$33 = 2$ dozen 9

Tell this story to your class:

Sue was counting eggs as she placed them in cartons. Each carton held 1 dozen eggs. As she counted she said, "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11." If Sue was grouping by dozens, what did she say as she placed the next egg in the carton?

Choose a pupil to write the answer to the question on the chalkboard (1 dozen). Continue the story by asking for the count for each of the next three eggs (1 dozen 1, 1 dozen 2, 1 dozen 3). Say that Sue continued counting until she said, "1 dozen 10." Then ask for the count for each of the next four eggs (1 dozen 11, 2 dozen, 2 dozen 1, 2 dozen 2). Continue the story in this manner until the pupils are able to count in the dozenal system.

▶ Have each pupil prepare sets of twelve by placing a rubber band or a paper clip around each of several bundles of twelve strips of paper. Place these bundles in a pile and a few single strips in another pile.

Give a child six bundles of strips, and ask him to count the strips. The easiest method of counting is by dozens: 1 dozen, 2 dozen, 3 dozen, and so on. Then give this pupil nine single strips and tell him to continue counting. He should count thus: 6 dozen 1, 6 dozen 2, and so on. Allow several pupils to count bundles of twelve and single strips in this manner.

▶ Prepare cards on which these words are written.

twelve	great gross
dozen	dozen dozen dozens
dozen dozens	dozen gross
twelve dozens	twelve dozen dozens
gross	twelve gross

Display the cards in random order in a pocket chart or along the chalktray. Tell the class that a gross is one dozen dozens and that a great gross is one dozen dozen dozens. Direct one pupil to collect all the cards that show ways of expressing 144; direct another pupil to collect all cards that show ways of expressing 1728.

Let the pupils take turns holding up their cards and calling on classmates to tell whether the selection is correct. Each classmate called on should explain why the selection is or is not correct. For example:

144 is 12 dozens because 12 dozens is 12 × 12.
144 is a gross because a gross is 12 dozens.
1728 is 12 gross because it is 12 × 144.

In summary, have the class help construct a chart similar to the following:

1 dozen	= 12
1 dozen dozens	= 12 × 12 = 144
1 gross	= 144
1 dozen dozen dozens	= 12 × 12 × 12 = 1728
1 dozen gross	= 12 × 144 = 1728
1 great gross	= 1728

▶ In working with the decimal and dozenal systems, the children should learn to express numbers given in one of the systems in terms of the other system. Read several examples like these to the class.

I wonder how many apples I would have if I had 5 great gross apples.

I wonder how many dozen gumdrops I would have if I had 3000 gumdrops.

After reading each example, let the pupils compute and compare results. The pupils may continue the exercise by making up their own examples for other pupils to solve.

▶ Write on the chalkboard:

144 is 1 dozen dozens, or 1 gross.

Write the exercise "729 = ___ gross ___" on the chalkboard. Ask a pupil to use the division algorism to find the number of gross and the number remaining.

To find the number of dozens

$$\begin{array}{r|l} 12\overline{)\,729} & \\ \hline 9 & 60 \end{array}$$

To find the number of dozen dozens

$$\begin{array}{r|l} 12\overline{)\,60} & \\ \hline 0 & 5 \end{array}$$

or

To find the number of dozen dozens

$$\begin{array}{r|l} 144\overline{)\,729} & \\ \hline 9 & 5 \end{array}$$

If the first method is used, the first part of it gives the number of dozens and the number remaining. Ask a pupil to write an equation showing this.

$$729 = 60 \times 12 + 9$$
$$729 = 60 \text{ dozen } 9$$

Then have another pupil write an equation showing the second step.

$$60 \text{ dozens} = 5 \times 12 \text{ dozens}$$
$$60 \text{ dozens} = 5 \text{ gross}$$

Ask another pupil to write an equation that combines both steps.

$$729 = 5 \text{ gross } 9$$

If the second method is used, ask a pupil to explain why 144 is used as the divisor (because 144 = 1 gross). Ask what the remainder 9 means (there are 9 ones remaining). Have a pupil write an equation illustrating this method.

$$729 = 5 \times 144 + 9$$
$$729 = 5 \text{ gross } 9$$

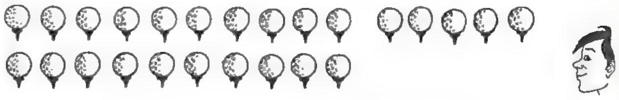
Work several examples in which the pupils must find the standard decimal numeral for a number given in the dozenal system.

$$\begin{aligned} 3 \text{ dozen dozen } 5 \text{ dozen } 6 &= 3 \times 144 + 5 \times 12 + 6 \\ &= 432 + 60 + 6 \\ &= 498 \end{aligned}$$

Name

UNIT 11 THE DOZENAL SYSTEM

Jim had 25 golf balls that he wanted to arrange in dozens, or sets of 12. How many sets of 12 golf balls could he make? How many golf balls would remain?



Can you use division to find the number of sets of 12 golf balls Jim could make and the number of golf balls remaining?

$25 = (2 \times 12) + 1$

Another name for the number 12 is 1 dozen.

Another name for the number 24 is 2 dozen.

Another name for the number 25 is 2 dozen 1.

Write each number in the dozenal system by using dozens and ones.

Number	Dozens and ones	Number	Dozens and ones
9	9	15	1 dozen 3
10	10	16	1 dozen 4
11	11	17	1 dozen 5
12	1 dozen	18	1 dozen 6
13	1 dozen 1	23	1 dozen 11
14	1 dozen 2	24	2 dozen

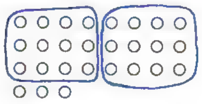
reference page

F-137

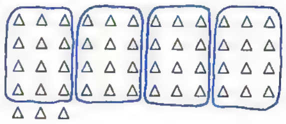
Write each number in the dozenal system by using dozens and ones.

Number	Dozens and ones	Number	Dozens and ones
25	2 dozen 1	61	5 dozen 1
26	2 dozen 2	62	5 dozen 2
27	2 dozen 3	89	7 dozen 5
28	2 dozen 4	96	8 dozen
37	3 dozen 1	100	8 dozen 4
46	3 dozen 10	144	12 dozen
47	3 dozen 11	178	14 dozen 10
48	4 dozen	546	45 dozen 6

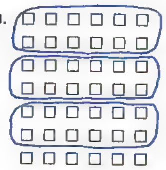
How many? Write your answer in the dozenal system. Circle each dozen.

1. 

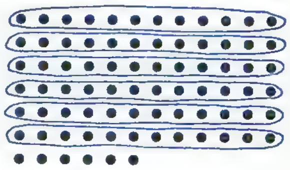
2 dozen 3

2. 

4 dozen 3

3. 

3 dozen 6

4. 

6 dozen 6

F-138

Pages 137 through 148

● On pages 137 and 138 the pupils will work with the idea of expressing numbers as dozens and ones. Use the story at the top of page 137 and the text that follows to develop the idea that another name for 25 is 2 dozen 1. Be sure the children understand that a quotient-remainder equation is useful in expressing numbers in dozens and ones. Discuss the completed portion of the chart at the bottom of the page. Then have the children complete the chart and the exercises on page 138 independently. Since multiples of 12 play such an important part in this unit, you may wish to make a chart of these multiples and post it in the classroom.

On pages 139 and 140 another way of expressing numbers greater than 143 is used. As soon as the pupils reach 144, they can see that this is not only 12 dozens, but also 1 dozen dozens. Since 12 is one dozen, the name 1 dozen can be substituted for 12. Thus 12 dozen 3 is 1 dozen dozens 3. Ask the pupils, "If 144 is 1 dozen dozens, what number is 2 dozen dozens?" Help the children understand that any number that is greater than 143 but less than 288 contains only 1 dozen dozens plus a number of dozens and a number of ones. The array on page 140 contains 331 dots. Ask the class how many groups of dozen dozens they can see, and how many dozens and ones are left. Assign the two pages for independent work.

Another name for 1 dozen dozens is 1 gross. This word is used on page 141. Use the example at the top of the page to help the pupils understand that any number greater than 143 will contain at least 1 dozen dozens, or 1 gross. Given any number greater than 143, first find the number of dozens in it and the number of ones remaining. Then the number of dozens can itself be expressed in terms of dozens. The result will be a number of dozen dozens, or gross, plus a remainder of dozens and ones.

Some children may see that since 144 is 1 gross, they can find the number of gross in one step by dividing by 144. This idea should come from the class, not from the teacher.

Assign the exercises at the bottom of the page for independent work.

Use page 142 to give children an opportunity to test themselves on their ability to express numbers in the dozenal system. Assign the exercises for independent work. Discuss only those exercises that may have caused difficulty.

Pages 143 and 144 introduce another name for 1 dozen gross—1 great gross. First have the children complete exercises 1 through 10 on page 143 independently. Then use the top of page 144 to develop the concept of a great gross. Have the pupils make a chart, such as the one shown below, that explains all the names used in the dozenal system.

1 great gross = 1 dozen gross
= 1 dozen dozen dozens
= 12 dozen dozens
= 144 dozens
= 1728

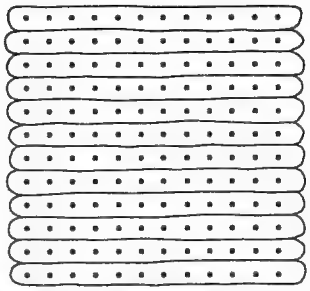
1 gross = 1 dozen dozens
= 12 dozens
= 144

1 dozen = 12

Then assign exercises 1 through 8 on page 144 for independent work. Since an extra division step is needed to find the number of great gross in any given number, some pupils may need additional help in completing the exercises on this page.

Name

For Class Discussion



Each ring shows 1 dozen.
144 is 12 dozen.

There are 1 dozen sets of a dozen.
144 = 1 dozen dozens

Imagine that you are placing more dots in the array and counting as you add each dot. Complete the table below and express the number of dots in two different ways. Continue the table through 159 dots.

Number of dots	Count by dozens	
144	12 dozen	1 dozen dozens
145	12 dozen 1	1 dozen dozens 1
146	12 dozen 2	1 dozen dozens 2
147	12 dozen 3	1 dozen dozens 3
148	12 dozen 4	1 dozen dozens 4
149	12 dozen 5	1 dozen dozens 5
150	12 dozen 6	1 dozen dozens 6
151	12 dozen 7	1 dozen dozens 7
152	12 dozen 8	1 dozen dozens 8
153	12 dozen 9	1 dozen dozens 9
154	12 dozen 10	1 dozen dozens 10
155	12 dozen 11	1 dozen dozens 11
156	13 dozen	1 dozen dozens 1 dozen
157	13 dozen 1	1 dozen dozens 1 dozen 1
158	13 dozen 2	1 dozen dozens 1 dozen 2
159	13 dozen 3	1 dozen dozens 1 dozen 3

F-139

Jim and Sue were asked to express the number 154 in dozens.
Sue worked the exercise this way.

$$\begin{array}{r|l} 12\,\overline{)154} & \\ 120 & 10 \\ \hline 34 & \\ 24 & 2 \\ \hline 10 & 12 \end{array}$$

Sue said that 154 equals 12 dozen 10.
Jim said 154 could be named another way.
154 = 12 dozen 10
= 1 dozen dozens 10

Express each number as dozens and ones, and then as dozen dozens, dozens, and ones.

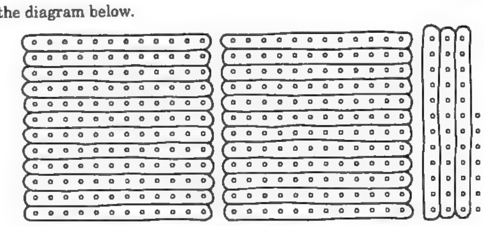
1. 144 = 12 dozen 0
= 1 dozen dozens

2. 178 = 14 dozen 10
= 1 dozen dozens 2 dozen 10

3. 153 = 12 dozen 9
= 1 dozen dozens 9

4. 245 = 20 dozen 5
= 1 dozen dozens 8 dozen 5

Study the diagram below.



331 = 27 dozen 7
= 2 dozen dozens 3 dozen 7

Express each number as dozen dozens, dozens, and ones.

5. 336 = 2 dozen dozens, 4 dozen

6. 782 = 5 dozen dozens, 5 dozen 2

7. 498 = 3 dozen dozens, 5 dozen 6

F-140

Name

Another name for 1 dozen dozens is 1 gross. Jim expressed 315 in this way.

315 = 26 dozen 3
= 2 dozen dozens 2 dozen 3
= 2 gross 2 dozen 3

Jim computed 315 ÷ 12 to find the number of dozens in 315.

```
12|315
   240|20
    75
    72| 6
     3|26
```

315 = 26 dozen 3

Since there are 26 dozens, and since 12 dozens make 1 gross, Jim computed 26 ÷ 12 to find the number of dozen dozens or gross.

```
12|26
   24|2
    2|2
```

26 dozen = 2 gross 2 dozen

So, 315 = 2 gross 2 dozen 3.

Express each of these numbers in the dozenal system, using gross, dozens, and ones.

1. 590 = 4 gross 1 dozen 2
2. 642 = 4 gross 5 dozen 6
3. 581 = 4 gross 5
4. 880 = 6 gross 1 dozen 4
5. 1283 = 8 gross 10 dozen 11
6. 1019 = 7 gross 11

reference page

F-141

Name

Write the number that is 1 more than each of these numbers. Use the dozenal system.

1. 3 dozen 4 3 dozen 5
2. 3 dozen 11 4 dozen
3. 11 dozen 11 1 gross
4. 6 dozen 11 7 dozen
5. 1 gross 3 dozen 5 1 gross 3 dozen 6
6. 5 gross 7 dozen 11 5 gross 8 dozen
7. 8 gross 11 dozen 11 9 gross
8. 11 gross 10 dozen 11 11 gross 11 dozen
9. 11 gross 11 dozen 10 11 gross 11 dozen 11
10. 11 gross 11 dozen 11 12 gross

F-143

Express each of these numbers in the dozenal system, using gross, dozens, and ones.

1. 1235 = 8 gross 6 dozen 11
2. 1248 = 8 gross 8 dozen
3. 1460 = 10 gross 1 dozen 8
4. 1592 = 11 gross 8
5. 1765 = 12 gross 3 dozen 1
6. 1824 = 12 gross 8 dozen
7. 2024 = 14 gross 8
8. 3248 = 22 gross 6 dozen 8
9. 4163 = 28 gross 10 dozen 11
10. 1728 = 12 gross

F-142

12 gross is a dozen gross or a dozen dozen dozen.
A dozen gross is a great gross.

1 dozen gross = 1 great gross
1 dozen dozens = 1 gross
12 = 1 dozen

In the dozenal system, we express numbers in ones, dozens, gross, great gross, and so on.

```
12|2000          12|166          12|13
   1200|100         120|10          12|1
    800              46              1|1
    720| 60          36| 3
     80              10|13
     72|  6
      8|166
```

13 gross = 1 great gross 1 gross

166 dozen = 13 gross 10 dozen

2000 = 166 dozen 8

So, 2000 = 1 great gross 1 gross 10 dozen 8

Express each number in the dozenal system.

1. 1730 = 1 great gross 2
2. 1897 = 1 great gross 1 gross 2 dozen 1
3. 2745 = 1 great gross 7 gross 9
4. 3772 = 2 great gross 2 gross 2 dozen 4
5. 9640 = 5 great gross 6 gross 11 dozen 4
6. 8129 = 4 great gross 8 gross 5 dozen 5
7. 4937 = 2 great gross 10 gross 3 dozen 5
8. 2483 = 1 great gross 5 gross 2 dozen 11

reference page

F-144

● On page 145 the pupils are given numbers expressed in the dozenal system and are asked to write standard decimal numerals for them. Discuss the example at the top of the page and then assign exercises 1 through 13 for independent work.

● Pages 146 through 148 contain exercises that will improve the pupils' skills in computing with the multiplication and division algorisms. These pages should be assigned for independent work.

Name ______________________________

Sue wanted to know the standard numeral for 6 dozen 3.
Jim said that since 1 dozen is 12, 6 dozen 3 must be $(6 \times 12) + 3$.
What is the standard numeral? 75

Sue decided that 4 gross 5 dozen 6 must be:

$$(4 \times 12 \times 12) + (5 \times 12) + 6$$

Is Sue correct? yes What is the standard numeral? 642

Express these numbers with standard numerals.

1. 6 dozen 3 = 75
2. 1 gross 7 = 151
3. 3 dozen dozens = 432
4. 5 gross 7 dozen 9 = 813
5. 11 dozen 8 = 140
6. 3 gross 7 dozen = 516
7. 10 gross = 1440
8. 6 dozen gross = 10,368
9. 6 great gross 5 gross 1 dozen = 11,100
10. 6 great gross 1 dozen dozens = 10,512
11. 1 dozen dozen dozens = 1728
12. 2 great gross 5 dozen 7 = 3523
13. 3 great gross 5 gross 7 dozen 8 = 5996

F-145

Answer each question.

1. Sam has 3672 ants in his ant farm.
   a. How many dozen does he have? 306
   b. How many gross does he have? 25
   c. How many great gross does he have? 2

2. Tess has 6783 kernels of popcorn.
   a. How many dozen does she have? 565
   b. How many gross does she have? 47
   c. How many great gross does she have? 3

3. Seth has 5372 freckles.
   a. How many gross does he have? 37
   b. How many great gross does he have? 3

4. Lisa has 10,000 hairs on her head.
   a. How many great gross does she have? 5
   b. How many gross does she have? 69

5. Fred's front lawn has 12,765 blades of grass.
   a. How many gross does he have? 88
   b. How many great gross does he have? 7

6. Al's house has 15,760 nails in it.
   a. How many gross are there? 109
   b. How many great gross are there? 9

F-146

Name ______________________________

Express each number in the dozenal system.

1. 129 = 10 dozen 9
2. 1042 = 7 gross 2 dozen 10
3. 215 = 1 gross 5 dozen 11
4. 847 = 5 gross 10 dozen 7
5. 1440 = 10 gross
6. 399 = 2 gross 9 dozen 3
7. 11,999 = 6 great gross 11 gross 3 dozen 11
8. 98 = 8 dozen 2

Express each number in standard numerals.

9. 3 dozen 9 = 45
10. 7 gross 2 dozen 3 = 1035
11. 2 gross 1 dozen = 300
12. 4 great gross 6 dozen = 6984
13. 1 great gross 8 dozen 5 = 1829
14. 9 dozen 9 = 117
15. 3 great gross 4 = 5188
16. 3 great gross 7 = 5191

F-147

Express each number in the dozenal system.

1. 580 = 4 gross 4
2. 924 = 6 gross 5 dozen
3. 7859 = 4 great gross 6 gross 6 dozen 11
4. 8547 = 4 great gross 11 gross 4 dozen 3

Express each number in standard numerals.

5. 2 great gross 5 gross 2 dozen 11 = 4211
6. 6 gross 11 = 875
7. 5 gross 3 dozen = 756
8. 7 great gross 7 gross 7 = 13,111

Answer each question. Express your answer in the dozenal system.

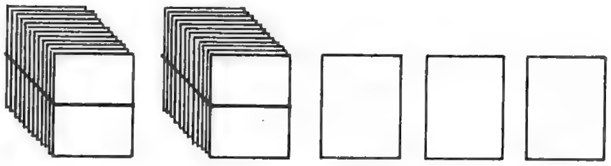
9. An orange grove produced 21,849 oranges in one harvest.
How many oranges were produced?
12 great gross 7 gross 8 dozen 9 oranges were produced.

10. Fred estimated that the oak tree in his yard had 31,285 leaves.
How many leaves did he estimate were on the oak tree? He estimated that 18 great gross 1 gross 3 dozen 1 leaves were on the tree.

F-148

Supplemental Experience

■ Give the pupils some paper strips, or index cards, and ask them to arrange the strips in dozens—sets of twelve—putting a rubber band or paper clip around each set. These sets may be displayed in a pocket chart or placed on the chalktray. The strips that are not part of an even dozen should be displayed so that each individual strip can be seen.



Ask a child to write on the chalkboard the number of dozens he counts and the number of individual strips left over (2 dozen 3 in the previous illustration).

Remove the rubber bands, put the cards in one pile, and ask a pupil to count them. Ask him how many he counts (27). Then show how he can find the standard numeral for 2 dozen 3 without counting.

2 dozen 3

$$\begin{array}{r} 12 \\ \times\ 2 \\ \hline 24 \\ +\ 3 \\ \hline 27 \end{array}$$

UNIT 12
MEASUREMENT
Pages 149 Through 164

OBJECTIVE
To understand measurement.

The child compares various quantities of length and weight. He learns that measurement is comparison and that measurement in standard units is comparison with a quantity that has been assigned the number 1. Standard units are those which have been assigned the number 1 by law or custom.

See Key Topics in Mathematics for the Intermediate Teacher: Measurement.

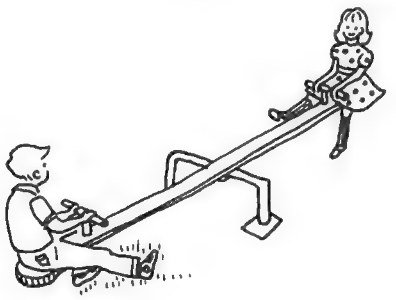
KEY IDEA
Measurement is comparison.

CONCEPTS
comparison
standard unit

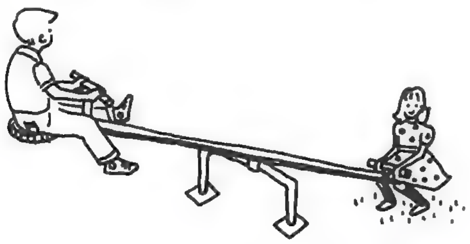
KEY IDEA
Measurement is comparison.

Scope
To compare lengths, areas, and weights.

Fundamentals
The balance scale uses a principle of comparison. Consider how a teeter-totter can be used to compare the weight of two children. Suppose a boy and a girl are seated on a teeter-totter and both are the same distance from the balance point. If the boy weighs more than the girl, the situation would look like this.

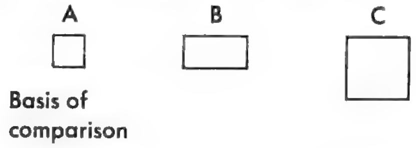


If the boy and girl weigh the same, the teeter-totter would balance. And if the girl weighs more than the boy, the situation would look like this.

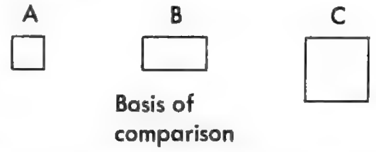


This is the simplest kind of comparison—one object is found to be more, less, or the same as another.

For more refined comparisons, numbers are often used. A particular quantity is usually chosen as a basis of comparison. Other quantities are then compared with it. In the following diagram, for example, if the area of A is chosen as a basis of comparison, the areas of B and C are compared to the area of A.



The area of B is twice the area of A; the area of C is four times the area of A.



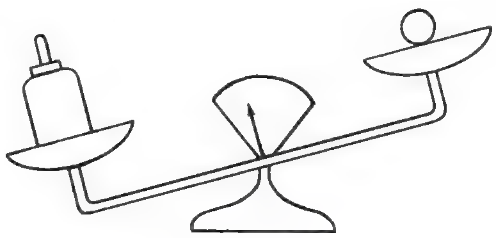
Also the area of A is $\frac{1}{2}$ the area of B and the area of C is twice the area of B. The choice of a basis of comparison may be made at will. Note that the numbers used in comparing are fractional numbers. If the number one is assigned to the basis of comparison, the basis is a unit. However, numbers other than one are frequently assigned to bases of comparison.

A standard unit is most generally used as a basis of comparison. Such standard units as the inch, pound, and so on, are bases of comparison which have been established by law or by custom. Regardless of whether measuring is done in terms of a standard unit or some other basis, the perception of measurement as comparison is essential.

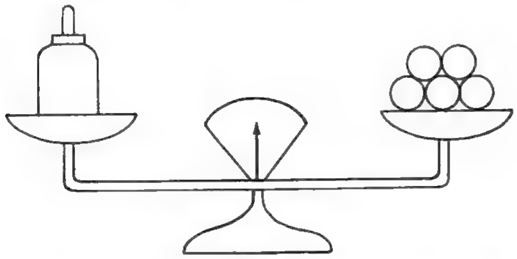
Readiness for Understanding
Familiarity with some fractional numbers.

Developmental Experiences
balance scale
modeling clay
paste

▶ For this activity, you will need a balance scale, 5 balls of modeling clay all the same size, and a full jar of paste. The 5 balls of clay together should weigh the same as the jar of paste. Tell the class that measuring weight involves making comparisons and that a balance scale can be used to compare weights. Place the jar of paste on one pan of the scale and one ball of clay on the other pan. Hold the objects in place, if necessary, but allow the scale to tip naturally.



The class should observe the result and tell which weighs more (the jar of paste). Place another ball of clay on the scale and ask the pupils how the weight of the jar of paste compares with the weight of a ball of clay (the weight of the jar of paste is greater than 2 times the weight of a ball of clay). In turn, place a third ball of clay and then a fourth ball of clay on the scale. Each time another ball of clay is put on the scale, ask how the weight of the jar of paste compares to the weight of a ball of clay (the jar of paste weighs more than 3 times as much as a ball of clay and more than 4 times as much as a ball of clay). Then place the fifth ball of clay on the scale.



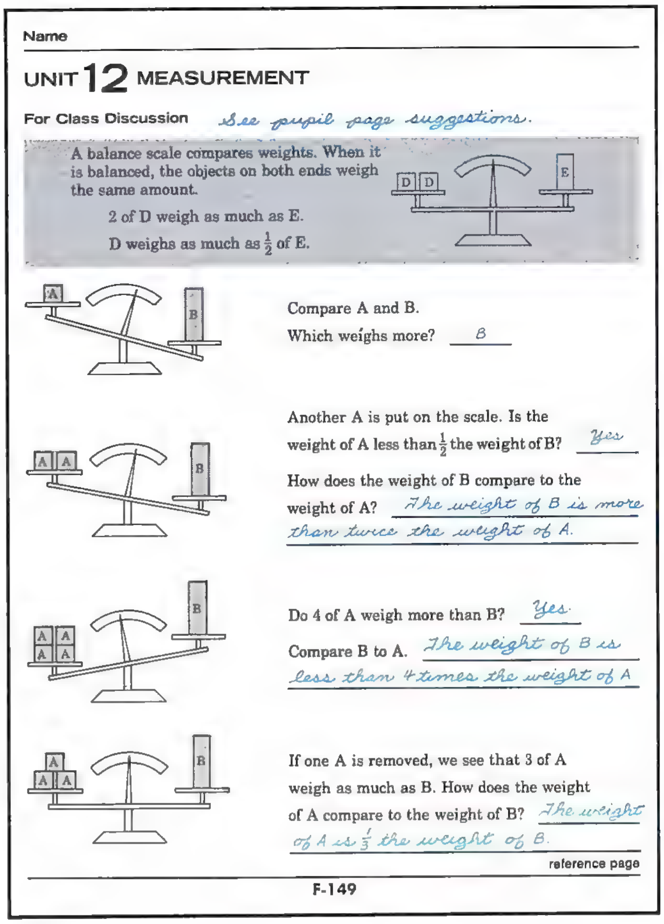
Since 5 balls of clay balance the jar of paste, the pupils can see that the jar of paste weighs 5 times as much as a ball of clay. Ask the pupils how the weight of a ball of clay compares to the weight of the jar of paste (a ball of clay weighs $\frac{1}{5}$ as much as the jar of paste).

The teacher may wish to follow a similar procedure using objects that weigh the same as 3 balls of clay or 4 balls of clay.

Pages 149 and 150

● Discuss the comparisons of weight on page 149. If possible, use a balance scale, four small wooden blocks, and a piece of wood that weighs the same as 3 of the blocks. Let a pupil demonstrate each step that is presented on page 149. After two A's have been placed on the scale, ask the pupils how the weight of one A compares to the weight of B (it is less than $\frac{1}{2}$ the weight of B). Ask the pupils how they know this. One pupil may say that the scale shows that two A's together weigh less than B, but the two A's weigh the same, so each must be less than $\frac{1}{2}$ of B. Continue the presentation of this page, and ask the pupils to explain their answers to the last question.

● Do page 150 as a class activity. Help the pupils make the comparisons of weight asked for in exercises 1 through 4. The problem at the bottom of the page reinforces the idea that a given comparison can be made in many ways. For example, 2J weighs the same as 3K and 1J weighs $\frac{1}{2}$ as much as 3K.



Name

UNIT 12 MEASUREMENT

For Class Discussion See pupil page suggestions.

A balance scale compares weights. When it is balanced, the objects on both ends weigh the same amount.

2 of D weigh as much as E.

D weighs as much as $\frac{1}{2}$ of E.

Compare A and B.
Which weighs more? B

Another A is put on the scale. Is the weight of A less than $\frac{1}{2}$ the weight of B? Yes

How does the weight of B compare to the weight of A? The weight of B is more than twice the weight of A.

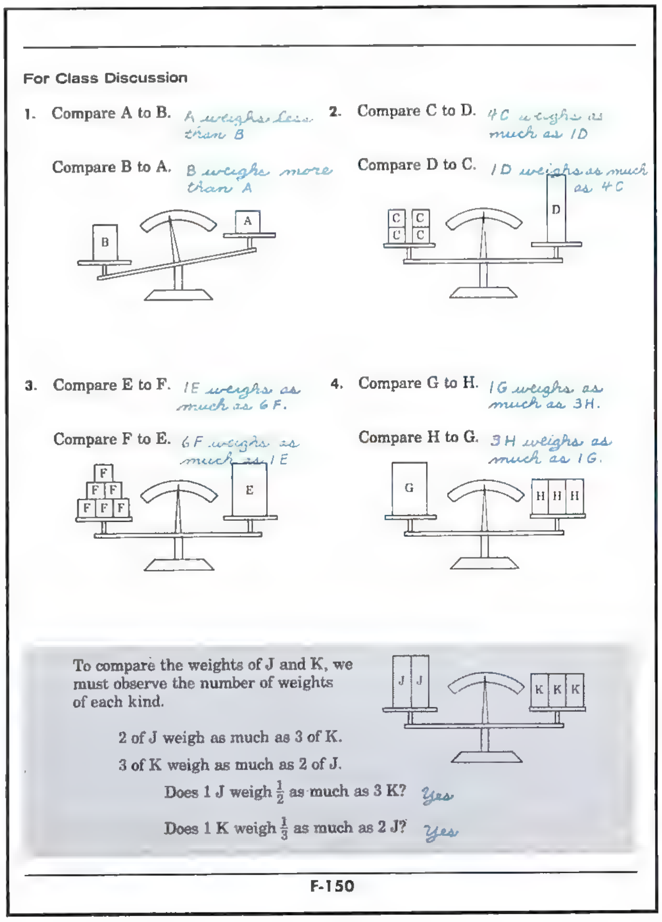
Do 4 of A weigh more than B? Yes.

Compare B to A. The weight of B is less than 4 times the weight of A

If one A is removed, we see that 3 of A weigh as much as B. How does the weight of A compare to the weight of B? The weight of A is $\frac{1}{3}$ the weight of B.

reference page

F-149



For Class Discussion

1. Compare A to B. A weighs less than B

Compare B to A. B weighs more than A

2. Compare C to D. 4C weigh as much as 1D

Compare D to C. 1D weighs as much as 4C

3. Compare E to F. 1E weighs as much as 6F.

Compare F to E. 6F weighs as much as 1E

4. Compare G to H. 1G weighs as much as 3H.

Compare H to G. 3H weighs as much as 1G.

To compare the weights of J and K, we must observe the number of weights of each kind.

2 of J weigh as much as 3 of K.

3 of K weigh as much as 2 of J.

Does 1 J weigh $\frac{1}{2}$ as much as 3 K? Yes

Does 1 K weigh $\frac{1}{3}$ as much as 2 J? Yes

F-150

Developmental Experiences

for each child
pencil
paper clip
notebook

▶ Write these questions on the chalkboard.

How tall is Janet?
How long is this room?
How wide is the desk?

Ask the pupils how to get the information they need to answer these questions. They will probably suggest using a ruler or a yardstick to measure Janet's height, the room's length, and the desk's width.

Point out that measurement is comparison. An inch, a foot, a yard, a book-width, a pencil-length, a line segment, or any other length can be used in measuring.

Have the pupils compare the length of a paper clip and the length of a pencil. Then have them compare the length of an edge of their notebook and the length of the pencil. Write the results of these comparisons on the chalkboard. For example, if the pencil is about 5 times as long as the paper clip, the paper clip is about $\frac{1}{5}$ as long as the pencil. If an edge of the notebook is about 2 times as long as the pencil, the pencil is about $\frac{1}{2}$ as long as the edge of the notebook.

Next, tell the class that you would like them to compare the length of various objects in the room to the length of the pencil. Write the results of each comparison on the chalkboard. For example:

The chalktray is about 11 times as long as the pencil.
The crayon is about $\frac{1}{2}$ as long as the pencil.
The edge of a desk is about 4 times as long as the pencil.
The rubber eraser is about $\frac{1}{3}$ as long as the pencil.

Page 151

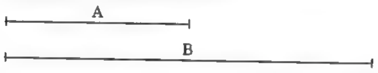
● Use page 151 to discuss comparisons of length. With the class, read the paragraph at the top of the page and discuss the comparison of the lengths of segments A and B. Have the pupils compare the lengths of segments D, E, and F as required in exercise 2. The comparisons may be expressed in two ways. For example, in comparing segments D and E, the pupils may say that E is 2 times as long as D and D is $\frac{1}{2}$ as long as E. Then, have the class make the comparisons asked for in exercises 3 through 5.

Name ______

For Class Discussion

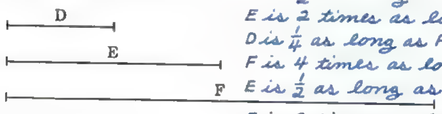
Measurement is comparison. We may compare lengths, weights, amounts of liquid, amounts of time, and many other quantities.

1. Compare the lengths of A and B.



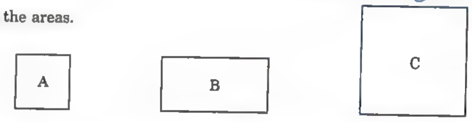
Compare B to A: B is 2 times as long as A.
Compare A to B: A is $\frac{1}{2}$ as long as B.

2. Compare the following lengths.



D is $\frac{1}{2}$ as long as E.
E is 2 times as long as D.
D is $\frac{1}{4}$ as long as F.
F is 4 times as long as D.
E is $\frac{1}{2}$ as long as F.
F is 2 times as long as E.

Compare the areas.



3. Compare B to A. B is 2 times as large as A.
Compare C to A. C is 4 times as large as A.

4. Compare A to B. A is $\frac{1}{2}$ as large as B.
Compare C to B. C is 2 times as large as B.

5. Compare A to C. A is $\frac{1}{4}$ as large as C.
Compare B to C. B is $\frac{1}{2}$ as large as C.

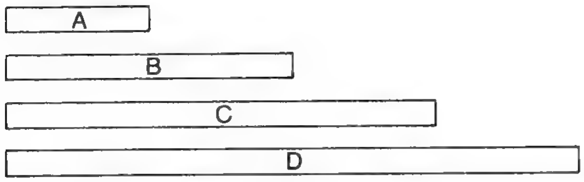
reference page

F-151

Developmental Experiences

for flannel board felt-tip pen
tagboard strips (1″ × 6″, 1″ × 12″, 1″ × 18″, 1″ × 24″)

▶ Cut out four strips of tagboard with a width of 1 inch and the following lengths: 6 inches, 12 inches, 18 inches, and 24 inches. Label these strips A, B, C, and D, respectively. Then arrange them on a flannel board or bulletin board as shown.



Tell the pupils that you want them to compare the length of the A-strip with the length of each of the other strips. As a pupil makes a comparison, have him write it on the chalkboard.

B is 2 times as long as A.
A is $\frac{1}{2}$ as long as B.
C is 3 times as long as A.
A is $\frac{1}{3}$ as long as C.
D is 4 times as long as A.
A is $\frac{1}{4}$ as long as D.

Next, ask the pupils to tell the length of B if A is 3 (the length of B will be 2 × 3). Let the pupils tell the lengths of C and D, if the length of A is 3 (the length of C is 3 × 3 and the length of D is 4 × 3). Explain to the class that if the length of B is 4 the length of A is 4 ÷ 2, and if the length of B is 9 the length of A is 9 ÷ 2. Assign various numbers to the length of A, and ask pupils to give the corresponding numbers for the length of B. Then, assign various numbers to B and ask pupils to name the corresponding numbers for A. Tabulate the results of this activity on the chalkboard.

A	B
3	2 × 3
5	2 × 5
10	2 × 10

B	A
6	6 ÷ 2
11	11 ÷ 2
3	3 ÷ 2

Vary the activity by writing the number assigned to A or B with numerals that are not in standard form. Place the following table on the chalkboard.

A	B
	2 × 5
7 ÷ 2	
	13
8	
75 ÷ 2	

Ask several pupils to come to the chalkboard and complete the table. Be sure they understand that numbers are being assigned in this activity even when they are not expressed by standard numerals. 2 × 3, 12 ÷ 2, and 6 are all names for a number, though they are different in form.

Pages 152 through 155

● Use page 152 as a class activity to investigate comparisons of area. Let pupils discuss and explain their ideas. Some pupils may be sure that B is 2 times as large as A and that C is 3 times as large as A. They may also say that 2A is as much as B or that A is $\frac{1}{2}$ of B. As they compare B and C, some pupils may see that 3B is as much as 2C. Some pupils may wish to trace and cut out the regions to be sure that 3B will just cover 2C. Discuss and investigate exercise 4 in a similar way.

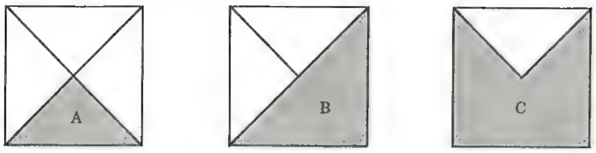
● Use page 153 for class discussion to continue the comparisons of area. Since the shaded areas are shown on graph paper, square units may be used as a basis of comparison.

● Complete page 154 as a class activity. Help the pupils make the comparisons of length asked for in exercises 1 through 6. Allow the pupils to attempt exercises 7 through 10 on their own. For those pupils who have difficulty with exercises 7 through 10, work the example at the bottom of the page. Ask pupils how the picture helps. Then encourage the pupils to try a similar procedure for the other exercises. Note that the length of A is 3 and that each of the other four lengths is a multiple of A's length.

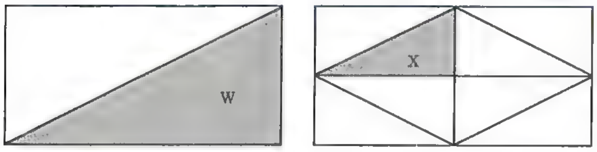
● Page 155 uses the idea of assigning a number to a length. Given two lengths, such as those of segments A and B, assigning a number to one of the lengths determines a number for the other length. If the number 3 is assigned to the length of A, the length of B will be 2 × 3 because B is 2 times as long as A. If 5 is assigned to the length of A, the length of B will be 2 × 5. The numbers for the lengths are always related in the same way as the lengths. Discuss examples 1 through 4 with the pupils. Then ask them to complete the table independently.

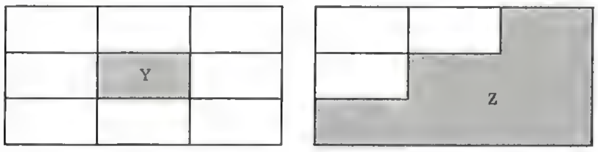
For Class Discussion

Compare the areas.



1. Compare A and B. *A is $\frac{1}{2}$ as large as B. B is 2 times as large as A.*
2. Compare A and C. *A is $\frac{1}{3}$ as large as C. C is 3 times as large as A.*
3. Compare B and C. *B is $\frac{2}{3}$ as large as C. C is $\frac{3}{2}$ as large as B.*
4.





The area of W is 72. What is the area of X? *18*

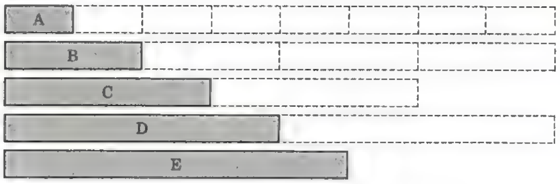
What is the area of Y? *16*

What is the area of Z? *96*

reference page

For Class Discussion

Compare the lengths.



Compare the lengths of B and D. Notice that 2 of B is as long as D.

So B is $\frac{1}{2}$ as long as D, and D is 2 times as long as B. Also, 4B is as long as 2D.

1. Compare A and B. *A is $\frac{1}{2}$ as long as B. B is 2 times as long as A.*
2. Compare A and C. *A is $\frac{1}{3}$ as long as C. C is 3 times as long as A.*
3. Compare A and D. *A is $\frac{1}{4}$ as long as D. D is 4 times as long as A.*
4. Compare A and E. *A is $\frac{1}{5}$ as long as E. E is 5 times as long as A.*
5. Compare B and C. *B is $\frac{2}{3}$ as long as C. C is $\frac{3}{2}$ as long as B.*
6. Compare B and D. *B is $\frac{1}{2}$ as long as D. D is 2 times as long as B.*

If the length of B is 6, then–

7. what is the length of A? *3*
8. what is the length of C? *9*
9. what is the length of D? *12*
10. what is the length of E? *15*

Mike drew a picture for exercise 8.

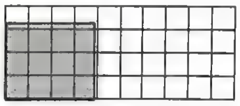
Mike said, "C is the missing factor in $2 \times \square = 18$."

Do you think the picture helped Mike? *See pupil page suggestions.*

Name

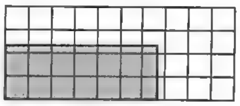
For Class Discussion

What do you know about the area of the shaded figures?



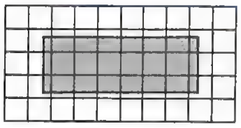
greater than 12 square units
less than 16 square units

How do you know? *More than 12 are covered. Less than 16 are covered.*

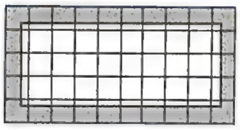


greater than 12 square units
less than 21 square units

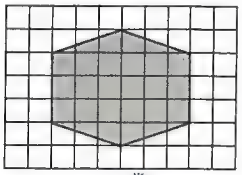
How do you know? *More than 12 are covered. Less than 21 are covered.*



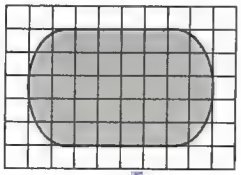
greater than *6* square units
less than *24* square units



greater than *13** square units
less than *26* square units



greater than *18** square units
less than *30* square units

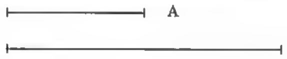


greater than *20** square units
less than *40* square units

** Answers may vary.*

Name

We can assign any number to a length.



1. If the length of A is 3, the length of B is 2×3.
2. If the length of A is 5, the length of B is 2×5.
3. If the length of B is 4, the length of A is $4 \div 2$.
4. If the length of B is 7, the length of A is $7 \div 2$.

Complete the table using the numbers assigned to A or B.

	A	B
1.	3	2 × 3
2.	5	*2 × 5*
3.	4 ÷ 2	4
4.	*7 ÷ 2*	7
5.	8	*2 × 8*
6.	10	*2 × 10*
7.	14	*2 × 14*
8.	28 ÷ 2	*28*
9.	*12 ÷ 2*	12
10.	*13 ÷ 2*	13

	A	B
11.	*50 ÷ 2*	50
12.	*75 ÷ 2*	75
13.	10 ÷ 2	*10*
14.	64 ÷ 2	*64*
15.	*9*	2 × 9
16.	*16*	2 × 16
17.	22 ÷ 2	*22*
18.	*53 ÷ 2*	53
19.	86	*2 × 86*
20.	*345*	2 × 345

reference page

Developmental Experiences

yardstick
items varying in weight from
1 ounce to 1 pound
scale
liquid containers
cup quart
pint gallon

for each child
ruler

► Have pupils measure the width of their desks in thumbwidths. Then have them report their measurements. Write the measurements on the chalkboard. It is likely that there will be a variety of measurements. Ask pupils to suggest reasons for this variety.

Next, have several pupils measure the length of a mark on the chalkboard using handspans as the unit of measure. It is likely that there will be a variety of measurements. Ask the pupils why this is so.

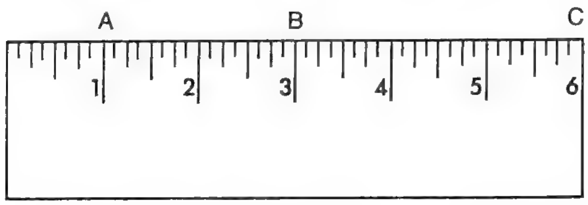
Discuss the desirability of obtaining the same measurement or as close to the same measurement as possible, each time an object is measured. Then tell the pupils that standard units such as the inch are used to help resolve the problem. Ask the pupils to identify some standard units that are used for measuring length (inch, foot, yard, mile, etc.). Let the children examine some rulers and yardsticks. Draw a 1-inch segment, a 1-foot segment, and a 1-yard segment on the chalkboard and compare the lengths of these standard units. Also discuss the use of the 1-mile unit for measuring longer distances. Help the pupils see that standard units do not vary in size as do books, pencils, thumbwidths, handspans, and sheets of paper.

Next, discuss the fact that standard units are also used for measuring weight. Have the pupils identify two of the standard units that are used for measuring weight (the ounce and the pound). Bring to class several standard weights (1 ounce, 8 ounces, 1 pound, etc.). Let the children pick them up and compare them. You might also provide a scale so that the children can verify some of these weights.

Finally, have the pupils identify the standard units used for measuring liquids. Let the children compare containers of the following sizes: a cup, a pint, a quart, and a gallon.

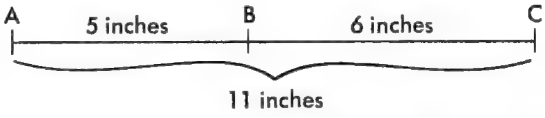
► Have the pupils examine their rulers to locate the 0-inch mark. If a pupil cannot find a 0-inch mark on his ruler, tell him that the end of his ruler represents the 0-mark. Mention to the class that sometimes the end of a ruler is rounded and that using its end as the 0-inch mark may not be accurate. Then explain that any point on the ruler may be used as a starting point when measuring lengths.

Draw a 6-inch ruler on the chalkboard. Place points A, B, and C on the ruler as illustrated.



Ask someone what the distance is from 0 to A. Ask what the distance is from 0 to B (the distance from 0 to B is 3 times the distance from 0 to A, or 3 inches). Similarly, ask for the distances from A to B, from 0 to C, from A to C, and from B to C.

► On the chalkboard, draw a 5-inch line segment and label the endpoints A and B. Ask a pupil to measure the segment and to write its length above it. Then extend the line segment 6 inches from point B and mark C at the new endpoint. Ask a pupil to measure and write the length of line segments BC and AC.

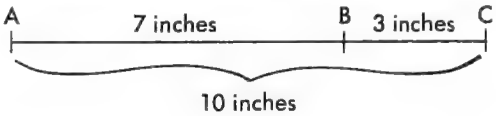


Ask another pupil to compute the combined length of segments AB and BC on the chalkboard.

$$\begin{array}{r} 5 \text{ inches} \\ +\ 6 \text{ inches} \\ \hline 11 \text{ inches} \end{array}$$

Compare the measured length of the combined segments to the computed length. The pupils should see that they can find the combined length of two segments by adding the numbers that describe their respective lengths.

Next, draw a line segment 10 inches long. Ask a pupil to partition it into two segments, one of which is 7 inches long. Ask another pupil to measure and mark the length of the second segment.



Direct a third pupil to compute the difference between the 10-inch segment and the 7-inch segment.

$$\begin{array}{r} 10 \text{ inches} \\ -\ 7 \text{ inches} \\ \hline 3 \text{ inches} \end{array}$$

Compare the measured length of the second segment and the computed length. The pupils should see that they can find the difference between the lengths of two segments by subtracting the numbers that describe their respective lengths.

Continue in this way to have the pupils compute the sum and difference of other lengths described in terms of inches, feet, and yards. Then, using a similar procedure, let the class compute sums and differences of weights in terms of pounds and ounces.

Pages 156 through 164

● Pages 156 and 157 introduce standard units for measuring lengths.

Direct the pupils to read the two paragraphs at the top of page 156. Discuss the meaning of phrases such as "piece of paper 7 inches long" or "10 inches of ribbon." Then assign exercises 1 through 6 and the exercise at the bottom of the page for independent work. The pupils may express their answers in either of two ways. For example, in response to exercise 5, the pupils may say the distance from 2 to 4 is 2 times the distance from 0 to 1, or the distance from 2 to 4 is 2 inches.

Use the top of page 157 to discuss the foot, yard, and mile as standard units for measuring length. Then tell the children to study the examples. With the class, work exercises 1 through 6. Exercises 7 through 14 may be assigned for independent work. After the pupils have completed the assignment, ask them what standard unit should appear in each computed sum, difference, or product.

● Page 158 introduces the idea of assigning a number to a weight. Follow a procedure similar to that of page 155.

There are standard units of measurement.
One standard unit is the inch. To measure the length of an object, we compare its length with the inch.

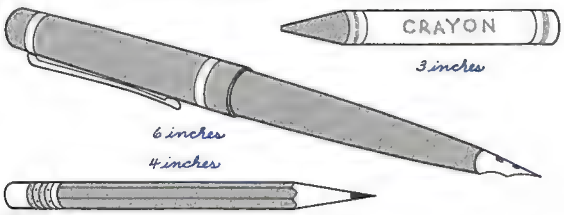
1 inch

When we say that a piece of paper is 5 inches long, we mean that it is 5 times as long as the 1-inch standard unit.

The distance from 0 to 1 on a ruler or yardstick is 1 inch.
The distance from 0 to 2 is 2 times the distance from 0 to 1, or 2 inches.

1. What is the distance from 0 to 3? 3 inches
2. What is the distance from 0 to 5? 5 inches
3. What is the distance from 2 to 3? 1 inch
4. What is the distance from 1 to 5? 4 inches
5. What is the distance from 2 to 4? 2 inches
6. What is the distance from 2 to 5? 3 inches

Use your ruler to measure the length of each object shown below.



reference page

F-156

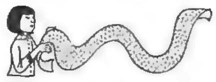
Name

Some other standard units for measuring length are the foot (ft.), the yard (yd.), and the mile (mi.).

Jim is 4 feet tall.

Jim's height is 4 times as much as the 1-foot unit.

Nancy ordered 3 yards of cloth.



The cloth is 3 times as long as the 1-yard unit.

Polly traveled 10 miles.

10 miles is 10 times as much as 1 mile.

Compare each length to the standard unit used.

7 inches — 7 inches is 7 times as much as the 1-inch unit.

$\frac{1}{2}$ inch — $\frac{1}{2}$ inch is $\frac{1}{2}$ as much as 1 inch.

1. 8 yards — 8 yards is 8 times as much as the 1-yard unit.
2. 6 feet — 6 feet is 6 times as much as the 1-foot unit.
3. 2 miles — 2 miles is 2 times as much as the 1 mile unit.
4. $\frac{1}{2}$ yard — $\frac{1}{2}$ yard is $\frac{1}{2}$ as much as the 1-yard unit.
5. $\frac{1}{4}$ foot — $\frac{1}{4}$ foot is $\frac{1}{4}$ as much as the 1-foot unit.
6. $\frac{1}{3}$ mile — $\frac{1}{3}$ mile is $\frac{1}{3}$ as much as the 1-mile unit.

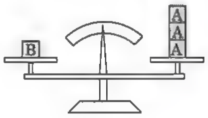
Compute.

7. 8 yards + 9 yards = 17 yards
8. 32 feet − 7 feet = 25 feet
9. 18 yards × 6 = 108 yards
10. 218 feet − 92 feet = 126 feet
11. 11 inches + 23 inches + 8 inches = 42 inches
12. 450 yards − 83 yards = 367 yards
13. 32 miles × 28: 256, 640, 896 miles
14. 37 inches × 41: 37, 1480, 1517 inches

reference page

F-157

We can assign any number to a weight.



1. If the weight of A is 1, the weight of B is 3 × 1.
2. If the weight of A is 6, the weight of B is 3 × 6.
3. If the weight of B is 9, the weight of A is 9 ÷ 3.
4. If the weight of B is 18, the weight of A is 18 ÷ 3.

Complete the table using the numbers assigned to A or B.

	A	B
1.	1	3 × 1
2.	6	3 × 6
3.	9 ÷ 3	9
4.	8 ÷ 3	8
5.	3	3 × 3
6.	5	3 × 5
7.	10	3 × 10
8.	13	3 × 13
9.	5 ÷ 3	5
10.	6 ÷ 3	6

	A	B
11.	15 ÷ 3	15
12.	26 ÷ 3	26
13.	30 ÷ 3	30
14.	99 ÷ 3	99
15.	9	3 × 9
16.	41	3 × 41
17.	78 ÷ 3	78
18.	219 ÷ 3	219
19.	101	3 × 101
20.	896	3 × 896

reference page

F-158

● Pages 159 and 160 introduce standard units for measuring weight.

Complete page 159 as a class activity. Have the pupils study the examples at the top of the page. Then discuss the use of balance scales on which standard weights are used. If possible, obtain a scale that uses ounce-weights or pound-weights. Explain to the class that when using this kind of scale, the object to be weighed is placed on one pan and enough standard weights are placed on the other pan to balance it. The weight of the object is computed by totaling the weights used to balance it.

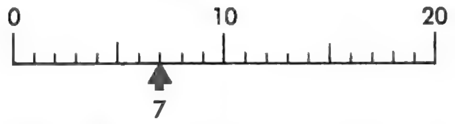
Then have the pupils discuss the comparisons pictured at the bottom of the page. Each picture suggests a different piece of information about the weight of a single bag of marbles. Picture 2, for example, suggests that a single bag weighs less than 1 pound. Picture 4 suggests that a single bag weighs less than $\frac{1}{3}$ pound.

The five pictures at the bottom of page 159 show that:

1. 1 bag weighs more than 0 pounds.
2. 1 bag weighs less than 1 pound.
3. 2 bags weigh less than 1 pound.
4. 3 bags weigh less than 1 pound.
5. 4 bags weigh 1 pound.

   1 bag of marbles weighs $\frac{1}{4}$ pound.

● Discuss how to read each type of scale pictured on page 160. Remind the pupils to look for the unit used in each instance. Emphasize the fact that usually not every mark on a scale is labeled. Draw a number line and label only a few of the marks that represent whole numbers. Then draw an arrow pointing to one of the unlabeled marks and ask a pupil to tell the number represented by it.



Have several pupils identify the unlabeled marks that represent whole units on the scales shown in exercises 1, 2, 5, 6, and 8.

Then ask the pupils to describe situations for which each scale might be used. Exercise 8 shows only the scale arm, fixed weights, and sliding weight of a platform scale—the kind found in feed mills, coal yards, and doctors' offices.

When you are sure the pupils understand the procedure to be followed, assign exercises 1 through 8 for independent work. After the pupils have completed the page, discuss only those exercises that may have caused difficulty.

● Page 161 introduces standard units for measuring liquids. Ask the class to study the illustrations at the top of the page. Then assign exercises 1 through 18 for independent work. After the pupils have completed the assignment, ask them what standard unit should be used in each computed sum, difference, or product.

● Page 162 consists of story exercises that require computation. Assign the page for independent work. When the assignment has been finished, have some pupils show at the chalkboard how they worked specific exercises.

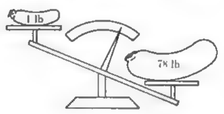
● The exercises on pages 163 and 164 deal with comparison of one standard unit to another. Form teams of two or three pupils each and have each team work the exercises separately.

Name

For Class Discussion *

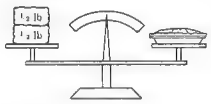
A standard unit for measuring weight is the pound (lb.).

The large dog weighs 78 pounds.
The small dog weighs 1 pound.

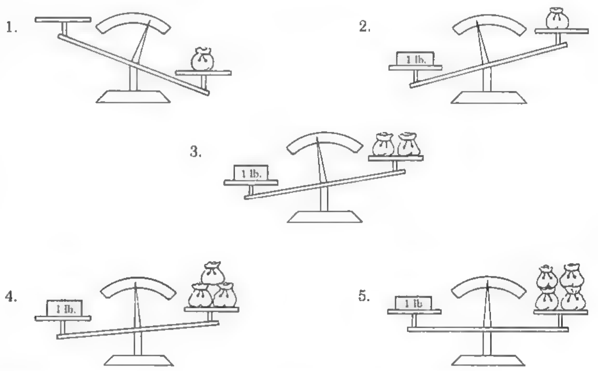


The large dog weighs 78 times as much as the 1-pound dog.

One cake weighs $\frac{1}{2}$ lb.
The pie weighs 1 pound.



One cake weighs $\frac{1}{2}$ as much as the pie. Two of the cakes weigh the same as the 1-lb. pie.



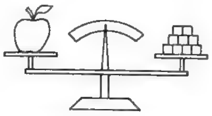
What do these pictures tell about the weight of a bag of marbles?
What does each picture tell about the weight of one bag of marbles?
How much does one bag of marbles weigh? * ________

reference page

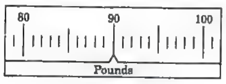
* See pupil page suggestions. F-159

Another standard unit for measuring weight is the ounce (oz.).

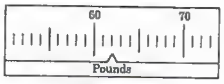
An apple weighs 9 ounces. It weighs 9 times as much as the 1-ounce unit.



The pictures below represent dials found on scales that are used to measure weight. What weight is shown on each dial?

1. 

90 pounds

2. 

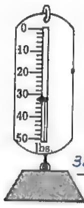
62 pounds

3. 

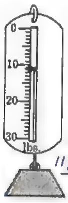
3 ounces

4. 

6 ounces

5. 

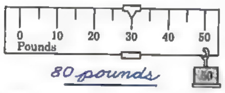
32 pounds

6. 

11 pounds

7. 

22 pounds

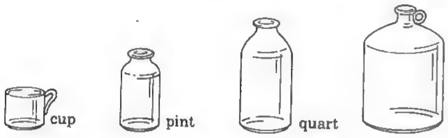
8. 

80 pounds

reference page

Name

Some standard units for measuring liquids are the cup (c.), the pint (pt.), the quart (qt.), and the gallon (gal.).



Compute.

1. 3 pints + 2 pints = 5 pints
2. 19 pints + 2 pints = 21 pints
3. 39 quarts + 11 quarts = 50 quarts
4. 6 cups × 5 = 30 cups
5. 26 gallons + 73 gallons = 99 gallons
6. 93 quarts − 13 quarts = 80 quarts
7. 45 pints − 29 pints = 16 pints
8. 30 quarts × 9 = 270 quarts
9. 25 gallons × 12 = 300 gallons
10. 4 cups, 11 cups, 29 cups + 45 cups = 89 cups
11. 8 gallons, 17 gallons, 26 gallons + 73 gallons = 124 gallons
12. 8 quarts, 13 quarts, 39 quarts + 93 quarts = 153 quarts
13. 24 pints × 36 = 864 pints
14. 300 gallons − 119 gallons = 181 gallons
15. 15 cups × 87 = 1305 cups
16. 428 gallons − 282 gallons = 146 gallons
17. 84 pints + 99 pints = 183 pints
18. 36 cups × 38 = 1368 cups

reference page

Write a sentence to answer each question.

1. Before it rained, the water in Mr. Drew's rain barrel was 17 inches deep. After it rained, the water was twice as deep. How deep was the water after it rained?
   The water was 34 inches deep.
2. On the way to the zoo Marian and Joan bought two bags of peanuts. They paid 10 cents for both bags. How much did one bag of peanuts cost?
   Each bag cost 5¢.
3. Sara weighs 68 pounds. Her sister, Diana, weighs $\frac{1}{2}$ as much as she does. How much does Diana weigh?
   Diana weighs 34 pounds.
4. Joe threw a baseball 60 feet. His older brother Roger threw it 3 times as far as Joe did. How far did Roger throw the baseball?
   Roger threw the ball 180 feet.
5. The man at the store told Bill that a grapefruit weighed 8 ounces, an orange 6 ounces, and a lemon 4 ounces. How much did the three pieces of fruit weigh?
   The 3 pieces of fruit weighed 18 ounces.
6. Terry cut off a 27-inch length of tape from a roll of tape. The roll now has 26 inches on it. How much tape was on the roll before Terry cut it?
   There were 53 inches on the roll before Terry cut it.
7. Tom had 8 stones. They weighed 1 ounce, 2 ounces, 3 ounces, 5 ounces, 6 ounces, 7 ounces, 8 ounces, and 9 ounces. His toy crane can move a load of only 12 ounces at one time without breaking. What is the least number of loads he needs to move all the stones?
   The least number of loads is 4.

Name

For Class Discussion

Length:
12 inches is as long as 1 foot. 3 feet is as long as 1 yard.
5280 feet is as long as 1 mile. 1 inch is as long as $\frac{1}{12}$ foot.

1. Compare 1 foot and 1 yard. 1 ft. is as long as $\frac{1}{3}$ yd.
2. Compare 1 foot and 2 inches. 1 ft. is 6 times as long as 2 in.
3. Compare 1 foot and 6 inches. 1 ft. is 2 times as long as 6 in.
4. Compare 1 inch and 1 yard. 1 in. is as long as $\frac{1}{36}$ yd.
5. Compare 1 yard and 18 inches. 1 yd. is 2 times as long as 18 in.
6. Compare 1 foot and 2 miles. 1 ft. is as long as $\frac{1}{10,560}$ of 2 mi.

Liquid measure:
4 quarts is as much as 1 gallon. 2 pints is as much as 1 quart.

7. Compare 1 pint and 4 quarts. 1 pt. is $\frac{1}{8}$ of 4 qt.
8. Compare 1 pint and 1 gallon. 1 pt. is $\frac{1}{8}$ of a gal.
9. Compare 1 gallon and 1 quart. 1 gal. is 4 times as much as a qt.
10. Compare 1 quart and 1 pint. 1 qt. is 2 times as much as a pt.
11. Compare 1 gallon and 4 pints. 1 gal. is 2 times as much as 4 pt.
12. Compare 1 quart and $\frac{1}{2}$ pint. 1 qt. is 4 times as much as $\frac{1}{2}$ pt.

For Class Discussion

Weight:

1 pound is as much as 16 ounces. 1 ton is as much as 2000 pounds.

1. Compare 1 pound and 1 ounce. 1 lb. is 16 times as much as 1 oz.

2. Compare 8 ounces and 1 pound. 8 oz. is $\frac{1}{2}$ of 1 lb.

3. Compare 1 ton and 1000 pounds. 1 ton is 2 times as much as 1000 lb.

4. Compare 8 ounces and 2 pounds. 8 oz. is $\frac{1}{4}$ of 2 lb.

5. Compare 1 ton and 1 ounce. 1 ton is 32,000 times as much as 1 oz.

6. Compare 1 ton and 16,000 ounces. 1 ton is 2 times as much as 16,000 oz.

7. Compare 4 ounces and 1 pound. 4 oz. is $\frac{1}{4}$ of 1 lb.

8. Compare 1 ton and 4000 pounds. 1 ton is $\frac{1}{2}$ as much as 4000 lb.

Money:

9. Compare 1 nickel and 1 dollar. 1 nickel is $\frac{1}{20}$ of a dollar.

10. Compare 1 dime and 1 dollar. 1 dime is $\frac{1}{10}$ of a dollar.

11. Compare 1 quarter and 1 dollar. 1 quarter is $\frac{1}{4}$ of a dollar.

12. Compare 1 dime and 1 nickel. 1 dime is 2 times as much as 1 nickel.

13. Compare 1 quarter and 1 dime. 1 quarter is $2\frac{1}{2}$ times as much as 1 dime.

F-164

Supplemental Experiences

■ Ask the pupils to look at labels on bottles and packages at home to find which standard units were used to measure the contents. The pupils may bring empty containers or their labels to class, or they may make lists of products and the amount included in a container of each. Suggest that they look at laundry supplies, food, hobby items, and so on. Discuss the things they have found and make a list to display on the bulletin board so that the pupils can make their own comparisons. Use the activity to emphasize the importance of standard units. For example, a quart of milk is a standard unit of measure that has been precisely determined, while a bottle of milk or a jug of milk may vary in size.

■ The class may find it interesting to know that the standard units inch and foot were derived from parts of the body. The inch was originally based on the distance from the end of a man's thumb to the first joint. Have each pupil measure this distance on his own thumb and compare it to an inch. The foot was based on the average length of a man's foot. Let each pupil measure his own foot and compare it to the standard foot. Use the activity to review the advantages of having a standard unit of length.

■ Cut 6 pieces of string measuring from 1 to 6 feet in length and place each piece in an envelope. Write the length of the string on the outside of the envelope. Call on a pupil to choose one of the envelopes and use a ruler to draw a line segment on the chalkboard that has the length that is written on the envelope. Tell the pupil to take the string from the envelope and compare it to the line segment. The class will see that the lengths are the same. They should conclude that the use of a standard unit guaranteed that the string and the line segment would have the same length. Continue in this way with the other envelopes containing pieces of string.

■ Write sentences such as the following on the chalkboard.

John is 3 feet tall.
John's father is twice as tall.
His father is 6___tall.
His father is 72___tall.

Ask the pupils to complete each sentence with a unit of length that would be reasonable in the sentence.

Next, on the chalkboard, write phrases such as the following:

The length of a bookshelf,
The length of a crayon,
The height of a building,
The length of a school hall,
The length of a person's hand.

Ask pupils to name a unit of measure that could be conveniently used to express the length of each object.

■ Ask several pupils to write story exercises that involve computations with weight measure. Ask them to read the stories aloud so that the class can solve them.

UNIT 13
THE SET OF FRACTIONAL NUMBERS
Pages 165 Through 184

OBJECTIVE

To introduce the concept of fractional number.

Through the use of measurement, the pupil learns that the set of quotients (fractional numbers) is required in "How much?" situations. Through this development, he learns that a fractional number is a quotient—a whole number divided by a counting number.

See Key Topics in Mathematics for the Intermediate Teacher: The Set of Fractional Numbers.

KEY IDEAS

$5 \div 6 = \frac{5}{6}$.
$6 \times \frac{5}{6}$ is 5.

CONCEPTS

quotient
fractional number
fraction

KEY IDEA

$5 \div 6 = \frac{5}{6}$.

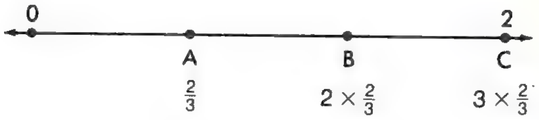
Scope

To see quotients as answers to the question, "How much?"

Fundamentals

Everyday experiences show that in many instances the counting numbers fail in physical application. We may have 1 quart, 2 quarts, 3 quarts, or any whole number of quarts of a liquid. The amount can, however, be divided into any number of equal parts. If 1 quart is divided into 2 equal parts, each part is 1 quart *divided by* 2. If 2 quarts is divided into 2 equal parts, each part is 2 quarts *divided by* 2. If 3 quarts is divided into 2 equal parts, each part is 3 quarts *divided by* 2. And if 4 quarts is divided into 2 equal parts, each part is 4 quarts *divided by* 2. To summarize, if m quarts is divided into n equal parts, each part is m quarts *divided by* n.

A fractional number is a quotient of two whole numbers. The fractional number $\frac{2}{3}$, for example, is 2 *divided by* 3. To locate $\frac{2}{3}$ on the number line, the segment from 0 to 2 is partitioned into three equal parts, as shown below. Each of the parts has a length of $\frac{2}{3}$, so the distance from 0 to A is $\frac{2}{3}$. Accordingly, we name distance A, $\frac{2}{3}$.

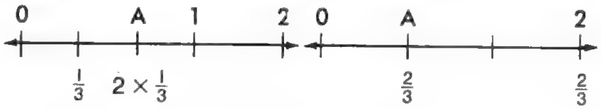


Since the distance from 0 to B is two times the distance from 0 to point A, this distance is $2 \times \frac{2}{3}$. Point B is thus named $2 \times \frac{2}{3}$. The distance from 0 to C is three times the distance from 0 to A, so point C is named $3 \times \frac{2}{3}$. As the distance from 0 to C is also 2, we see that $3 \times \frac{2}{3}$ is another name for 2. This defines $\frac{2}{3}$ as the number that, when multiplied by 3, yields the product 2: $3 \times \frac{2}{3} = 2$. If $3 \times \square = 2$, then $\square = \frac{2}{3}$.

This approach to fractional numbers differs somewhat from the unit-fraction approach but they are compatible. The unit-fraction approach views $\frac{2}{3}$ as two of 3 equal parts of 1, or $2 \times \frac{1}{3}$.

Since 2 is equal to $1 + 1$, $2 \times \frac{1}{3}$ equals $(1 + 1) \times \frac{1}{3}$, or $\frac{1}{3} + \frac{1}{3}$. Therefore $\frac{2}{3}$ is $\frac{1}{3} + \frac{1}{3}$, or two thirds.

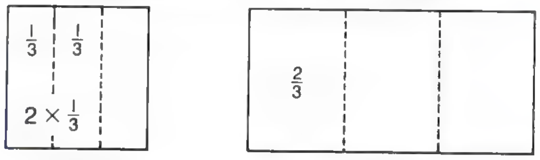
Unit-Fraction Approach   Quotient Approach



Thus, 2 of 3 equal parts of 1 is the same as 1 of 3 equal parts of 2.

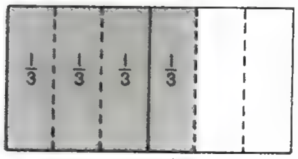
We can also show the above relationship by using regions.

Unit-Fraction Approach   Quotient Approach



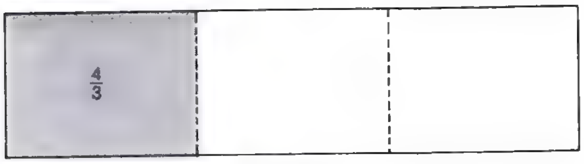
1-region   2-region

Since the unit-fraction approach depends upon the addition of equal parts of 1, a fractional number greater than 1 in this approach is treated as the sum of a whole number and a fractional number less than 1. For example, $\frac{4}{3}$ is understood as $1 + \frac{1}{3}$. The form $\frac{4}{3}$ is considered improper and called an improper fraction.



$1 + \frac{1}{3}$

In the quotient approach, a 4-region is partitioned into 3 equal parts.



4-region

The shaded region in this diagram is the same as the shaded region in the previous diagram, so $1 + \frac{1}{3}$ is equal to $\frac{4}{3}$.

Readiness for Understanding

Ability to compare regions.

Developmental Experiences

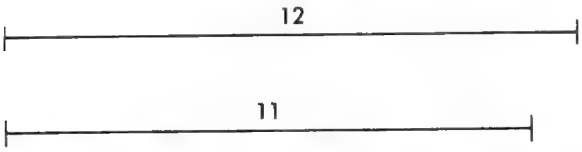
paper rectangular regions	*for each child*
felt-tip pen	12 counters
masking tape	12″ paper strip
	11″ paper strip
	two cards (3″ × 5″)
	pair of scissors

▶ Give each child 12 counters and 2 strips of paper, one strip 12 inches long and the other 11 inches long. Ask the children to arrange the 12 counters in 4 equal piles. Then write these equations on the chalkboard:

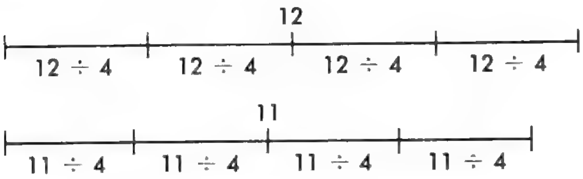
$$4 \times \square = 12$$
$$\square = 12 \div 4$$

Ask a pupil to give the missing factor (3). Remind the children that 3 is the quotient of the whole numbers 12 and 4. Point out that each of their 4 piles of counters has the same number of members (3). Now ask the pupils to try to arrange 11 counters in 4 piles. They will see that the counters cannot be arranged in 4 piles of the same size.

Draw two line segments on the chalkboard, one slightly longer than the other. Label them 12 and 11.



Have a pupil divide each segment into 4 equal parts. Then ask the class to suggest ways of labeling the parts. They might suggest using quotients; if not, suggest this possibility yourself by labeling the parts, as shown.



Emphasize that each part of the 12-segment is $12 \div 4$ and that each part of the 11-segment is $11 \div 4$.

On the chalkboard, below the first pair of missing-factor equations, write the following:

$$4 \times \square = 11$$
$$\square = 11 \div 4$$

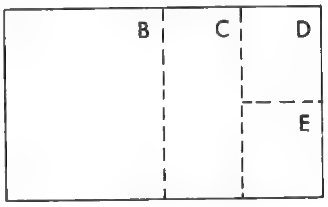
Explain that the missing factor in these equations is a number. Explain that it is the quotient of the whole numbers 11 and 4, but that it is not a whole number itself.

Tell the pupils that the longer strip of paper they have been given is a 12-segment and the shorter strip is an 11-segment. Ask them to fold each strip in half, and then to fold each strip in half again. Ask the pupils to unfold the strips and see whether both have been divided into a whole number of equal parts. Tell the children to label each part of the two strips as a quotient of two whole numbers ($12 \div 4$ in each part of the longer strip; $11 \div 4$ in each part of the shorter strip).

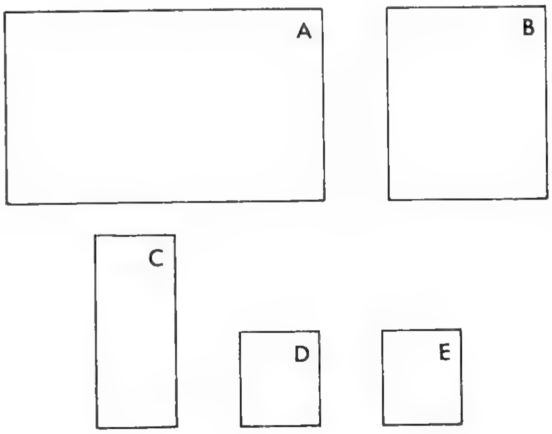
On the diagram on the chalkboard, rewrite each quotient $11 \div 4$ in fraction form, $\frac{11}{4}$. Tell the class that the quotient $12 \div 4$ can also be written in fraction form; call on a pupil to rewrite the quotients on the 12-segment. Explain that since $\frac{12}{4}$ is the quotient of two whole numbers, it is a fractional number. But $\frac{12}{4}$ is also the whole number 3. A whole number, therefore, can be a fractional number as well ($3 = \frac{12}{4}$).

You may want to repeat the activity to illustrate $8 \div 4$ and $9 \div 4$.

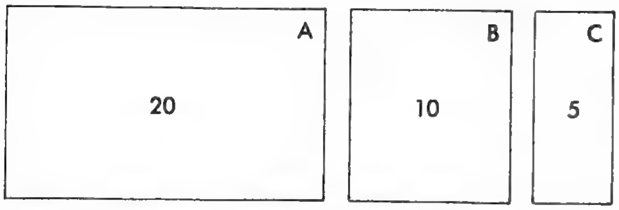
▶ Give each pupil two 3 by 5 cards and a pair of scissors. Show the class how to cut one of their cards as illustrated below. Part B is $\frac{1}{2}$ of the card, part C is $\frac{1}{2}$ of the remainder, and D and E are the same size.



Then have each pupil arrange his second card and the pieces of the first card as shown below, labeling them A, B, C, D, and E (or coloring each piece a different color).



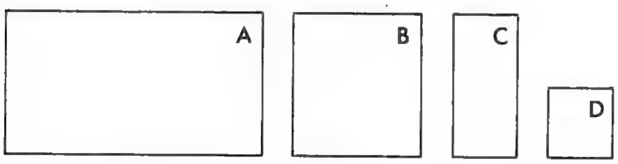
Tell the pupils to label the large card 20. Then ask someone to compare the sizes of cards A and B (card A is twice as large as B, or B is $\frac{1}{2}$ as large as A, or B is as much as A *divided by* 2). Tell the class that B should be labeled 10, since 2×10 is 20. Ask someone to compare B and C. How should C be labeled? (5)



Now ask whether D should be labeled with a whole number; ask for an explanation (no—C is twice as large as D, and 5 is not 2 times a whole number).

Tell the class that since no whole number can be used for D, we need a new kind of number. Write $5 \div 2$ on D. Tell the class this number means that two D pieces are as much as one C piece. Ask the class how much each part represents if C is divided into 2 equal parts ($5 \div 2$). Ask how E should be labeled ($5 \div 2$). Encourage the pupils to explain (their answer may indicate a comparison with either D or C).

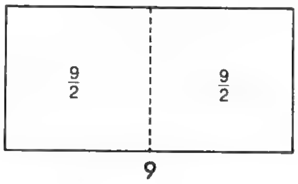
Continue this activity by drawing rectangles on the board, as shown.



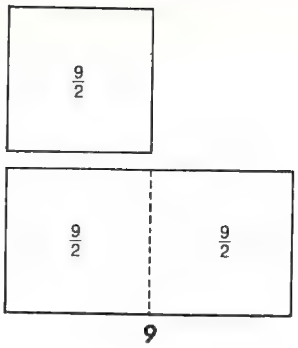
Let card A represent 12, and let the class determine the numbers for B, C, and D (6, 3, $3 \div 2$).

Continue, using such numbers as 28, 16, 36, and 4 for card A. Note that if, say, 9 is used for A, C becomes $9 \div 4$ because 4C is as much as A; D becomes $9 \div 8$ because 8D is as much as A.

▶ Draw on the chalkboard a rectangle measuring 16 inches by 8 inches. Tell the pupils that this is a region and that any number can be assigned to a region. Suggest 9, and write 9 below the figure. Then divide the region in half with a vertical line, labeling each of the two parts $\frac{9}{2}$.



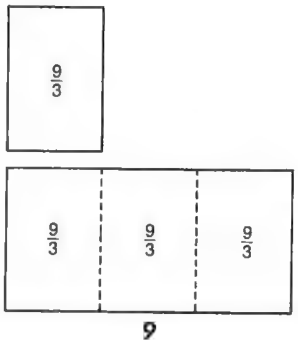
Place a piece of paper 8 inches by 8 inches over one of the two parts so that the pupils can see that the paper is the same size as half the 9-region. Label the piece of paper $\frac{9}{2}$, and tape it to the chalkboard above the 9-region.



Ask the pupils to compare the paper $\frac{9}{2}$-region and the chalk 9-region (the 9-region is 2 times as large as the $\frac{9}{2}$-region, or the $\frac{9}{2}$-region is $\frac{1}{2}$ as large as the 9-region). On the chalkboard write these sentences:

$$9 \div 2 \text{ is } \tfrac{9}{2}$$
$$\tfrac{1}{2} \text{of } 9 = \tfrac{9}{2}$$

Erase the $\frac{9}{2}$'s and the line dividing the 9-region, and then divide the region into 3 equal parts. Have a pupil label each part $\frac{9}{3}$. Replace the paper $\frac{9}{2}$-region with a paper $\frac{9}{3}$-region ($5\frac{1}{3}$ inches by 8 inches).



Ask the children to compare the paper $\frac{9}{3}$-region and the chalk 9-region (the 9-region is 3 times as large as the $\frac{9}{3}$-region; the $\frac{9}{3}$-region is $\frac{1}{3}$ as large as the 9-region). Below the previous sentences write these sentences:

$$9 \div 3 \text{ is } \tfrac{9}{3}$$
$$\tfrac{1}{3} \text{ of } 9 \text{ is } \tfrac{9}{3}$$

Repeat this procedure with the 9-region divided into 4, 5, and 6 equal parts.

You may want to continue the activity by assigning to the region numbers other than 9. The numbers 5, 8, 11, and 12 are suggested.

Pages 165 through 171

● Use page 165 as a class activity to give the children experience in assigning numbers to regions and in recognizing the number for each part of a region. Have the class discuss the examples on the page. Then let the children assign other numbers to the regions shown and tell the number for each part.

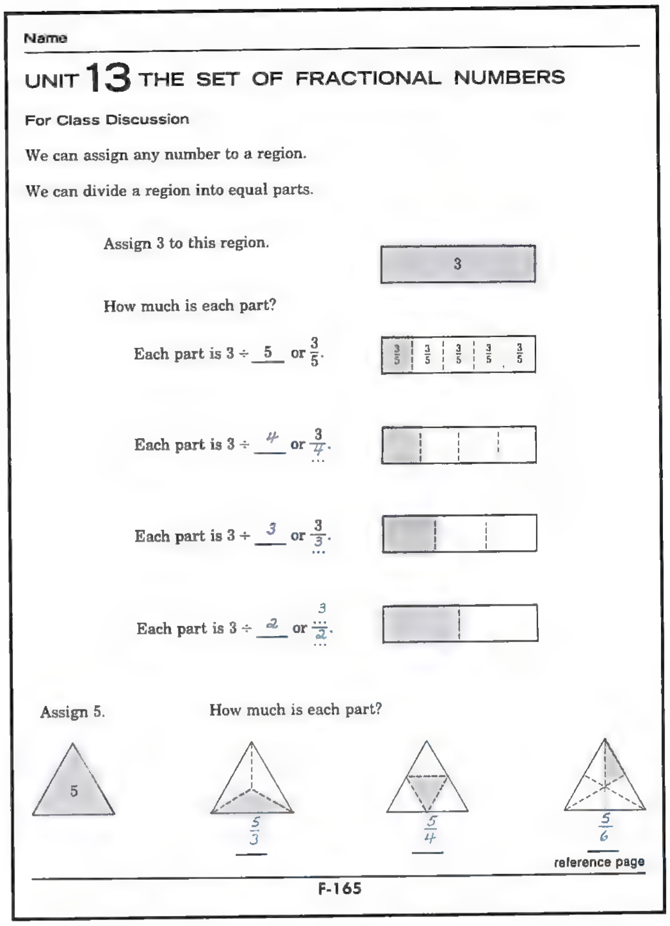
● Page 166 provides an opportunity for discussion of the idea that five divided by four is one-fourth of five. Reading $\frac{5}{4}$ as "five divided by four" is vital to understanding.

Let the children explain why each part in the first illustration is $\frac{5}{4}$ and why each part is also one-fourth of 5. Help the class understand that one of the four equal parts of 5 is $\frac{5}{4}$. Proceed with the other illustrations in a similar manner.

● On page 167 the pupils have an opportunity to identify the number for each shaded part. After the pupils have completed the exercises independently, let them discuss their decisions. In exercise 2, pupils should see that the large triangle is four times as large as $\frac{11}{4}$, or 11.

● Pages 168 and 169 provide experience in showing a region for a given number. After each pupil has completed the exercises independently, let them work in pairs to inspect each other's work. Then let the class raise questions about things they noticed and show interesting ways in which the regions were partitioned.

● Discuss the example on page 170 with the class. The pupils should notice that 13 is divided in 3 equal parts; the number for each part is 13 divided by 3. The number for the shaded part is 2 times 13 ÷ 3. Assign the exercises for independent work. Then let pupils tell how they completed each exercise.



Name

UNIT 13 THE SET OF FRACTIONAL NUMBERS

For Class Discussion

We can assign any number to a region.

We can divide a region into equal parts.

Assign 3 to this region.

How much is each part?

Each part is 3 ÷ 5 or $\frac{3}{5}$.

Each part is 3 ÷ 4 or $\frac{3}{4}$.

Each part is 3 ÷ 3 or $\frac{3}{3}$.

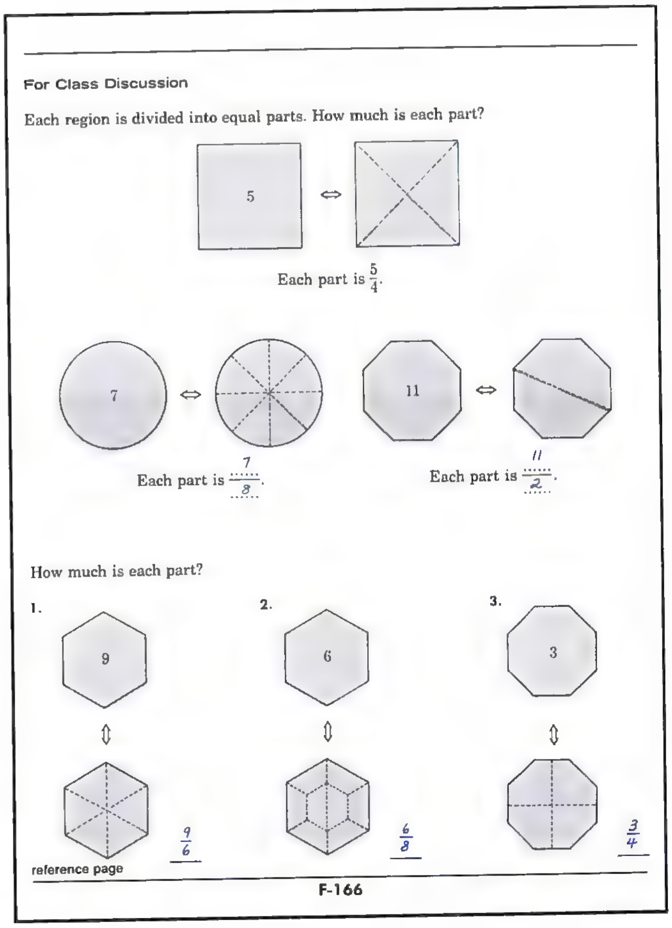
Each part is 3 ÷ 2 or $\frac{3}{2}$.

Assign 5. How much is each part?

$\frac{5}{3}$ $\frac{5}{4}$ $\frac{5}{6}$

reference page

F-165



For Class Discussion

Each region is divided into equal parts. How much is each part?

Each part is $\frac{5}{4}$.

Each part is $\frac{7}{8}$.

Each part is $\frac{11}{2}$.

How much is each part?

1. $\frac{9}{6}$

2. $\frac{6}{8}$

3. $\frac{3}{4}$

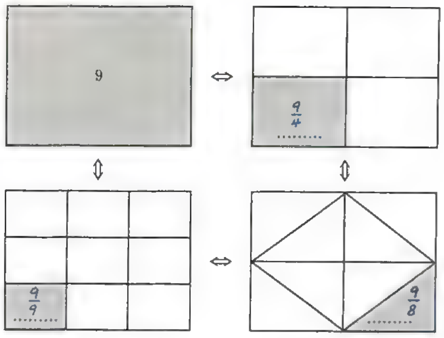
reference page

F-166

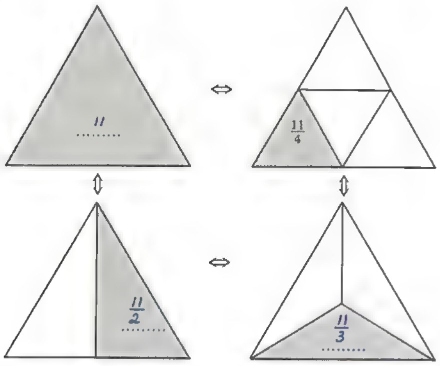
Name

What is the number for the shaded part?

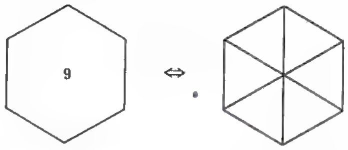
1.



2.

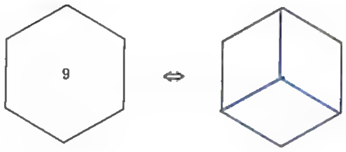


Name

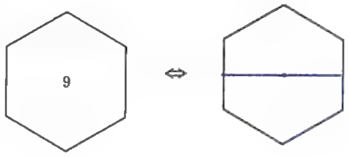


Each part is $9 \div 6$. How do you know?

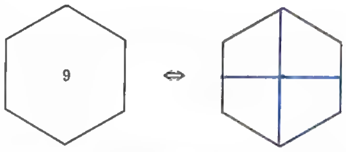
1. Show $9 \div 3$.



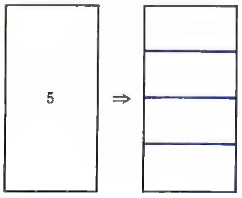
2. Show $9 \div 2$.



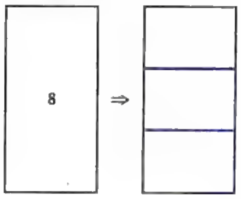
3. Show $9 \div 4$.



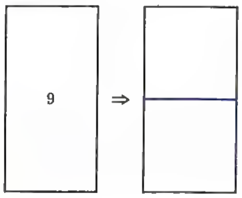
1. Show $5 \div 4$.



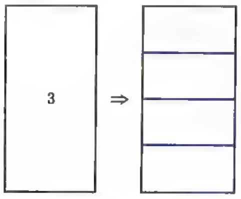
2. Show $8 \div 3$.



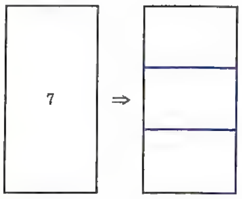
3. Show $9 \div 2$.



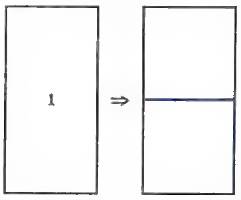
4. Show $\frac{3}{4}$.

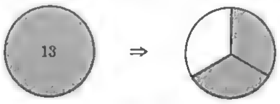


5. Show $\frac{7}{3}$.



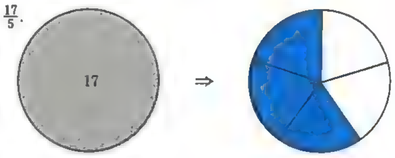
6. Show $\frac{1}{2}$.



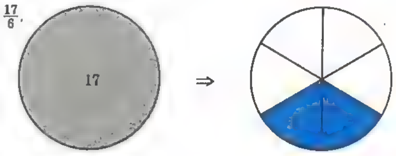


The shaded part is $2 \times \frac{13}{3}$. How do you know?

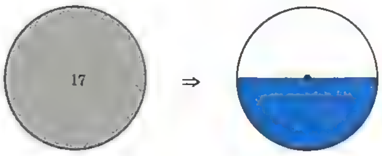
1. Shade $3 \times \frac{17}{5}$.



2. Shade $2 \times \frac{17}{6}$.



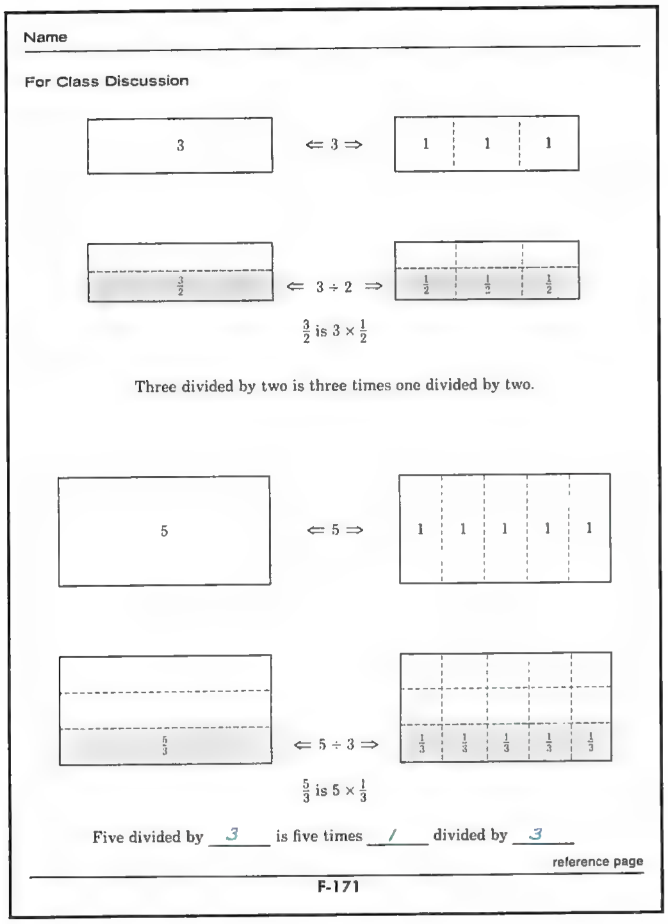
3. Shade $\frac{17}{2}$.



reference page

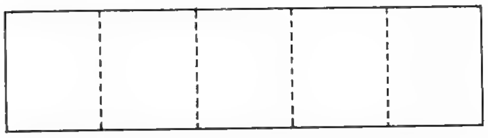
● On page 171 the children's perception of fractional numbers is extended. Discuss the diagrams with the class. In the first four diagrams, the class should see that 3 is 3×1 and that $\frac{3}{2}$ is $3 \times \frac{1}{2}$ (3 *divided by* 2 is 3 times 1 *divided by* 2). In the last four diagrams a similar perception of $\frac{5}{3}$ is developed.

After the discussion of the page is completed, draw diagrams on the chalkboard to illustrate that $\frac{3}{4}$ is $3 \times \frac{1}{4}$ and $\frac{2}{5}$ is $2 \times \frac{1}{5}$. Let the children interpret what they see.



Developmental Experiences

▶ Draw a rectangular region on the chalkboard, and divide it into 5 equal parts.



Use this diagram for the following questions. Ask the pupils to write their answers.

If the whole region is 5, what number is each part? (1, or $\frac{5}{5}$)

If each part is $\frac{7}{5}$, what number is the whole region? (7)

If the whole region is 11, what number is each part? ($\frac{11}{5}$)

If each part is $\frac{3}{5}$, what is the whole region? (3)

If the whole region is 4, what is each part? ($\frac{4}{5}$)

If each part is $\frac{5}{5}$, what is the whole region? (5)

If the whole region is 100, what is each part? (20, or $\frac{100}{5}$)

If each part is $\frac{10}{5}$, what is the whole region? (10)

If each part is 2, what is the whole region? (10)

If each part is $\frac{1}{5}$, what is the whole region? (1)

If the whole region is 15, what is each part? (3, or $\frac{15}{5}$)

If the whole region is 555, what is each part? (111, or $\frac{555}{5}$)

If each part is $\frac{83}{5}$, what is the whole region? (83)

If the whole region is 9, what is each part? ($\frac{9}{5}$)

If each part is $\frac{4}{5}$, what is the whole region? (4)

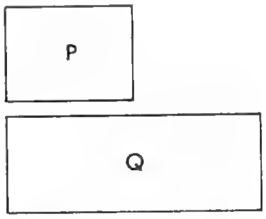
If each part is 1, what is the whole region? (5)

If the whole region is 1, what is each part? ($\frac{1}{5}$)

If each part is $\frac{6}{5}$, what is the whole region? (6)

Encourage the pupils to discuss their answers.

▶ Draw a region on the chalkboard, and label it P. Draw another region, twice as large as P, and label it Q.



Ask the class: "If P is 1, what number is Q?" (2) "If P is 2, what number is Q?" (4) "If P is 7, what number is Q?" (14) "If P is $\frac{2}{3}$, what number is Q?" ($2 \times \frac{2}{3}$) Then have a pupil assign a number to P. Ask another pupil to give the corresponding number for Q. Ask pupils to assign fractional numbers to P, and have other pupils give corresponding numbers for Q. Then assign whole numbers to Q, and ask what number P is. For example, "If Q is 7, what number is P?" ($\frac{7}{2}$)

Next draw another region three times as large as P, and label it R.

Assign various numbers to P, and ask what number R is. For example, "If P is 3, what number is R?" (9) Then assign numbers to R, and ask for numbers for P and Q. For example, "If R is 5, what is P?" ($\frac{5}{3}$) "What is Q?" ($2 \times \frac{5}{3}$)

Continue the activity by assigning numbers to Q and having the pupils find corresponding numbers for P and R. For example, "If Q is 3, what is P?" ($\frac{3}{2}$) "What is R?" ($3 \times \frac{3}{2}$)

Pages 172 through 175

● On page 172 the pupils have an opportunity to compare numbers. Have the class discuss the numbers for regions B, C, D, and E when A is 2. The pupils should realize that B is 2 times as much as A, C is 3 times as much as A, and so on. With this realization there should be no difficulty in stating the number for each region. Then have the class tell the number for each part of each region. Through observation they should discover that each part is 2. With further investigation the pupils may realize that the parts of the various regions are $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, and $\frac{10}{5}$. However, this is an exercise in identifying quotients, rather than equivalent fractions; therefore, refrain from calling attention to this concept.

As regions P, Q, R, S, and T are investigated, the children must compare each region with one of the parts of P. The clues indicate that Q is $5 \times \frac{1}{6}$.

Continue the discussion by using the same illustrations and assigning other numbers to the regions. For example, assign 3 and then 5 to A; assign 2 and then 4 to P.

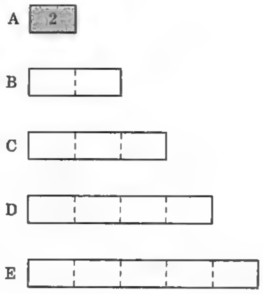
● Page 173 provides further experience in comparing numbers. For exercise 1 call attention to the fact that region F is 1 and E is 2. Ask the class to write the numbers for D, C, B, and A, and then discuss the results.

Continue in this way with exercises 2 through 5. For exercise 3 the pupils must realize that $\frac{2}{2}$ is 2 times as much as $\frac{1}{2}$.

Explain to the class that the table at the bottom of the page shows an easy way to arrange information on their papers. Have the children complete the table. If the class does not understand how to complete the table, tell them to cover everything in the table except the first row across. Have a child tell the number assigned to region A in this row (1). Direct attention to region B. Then have the child tell the number for B when A is 1 ($\frac{5}{6}$). Ask him to give his reasons. Have the pupils write the number for region B in the second square of the first row of the table. Continue this way to find the numbers for C, E, and F when A is 1, thus completing the first row. Then have the children complete the rest of the table.

For Class Discussion *See pupil page suggestions.*

How much?



A is 2.

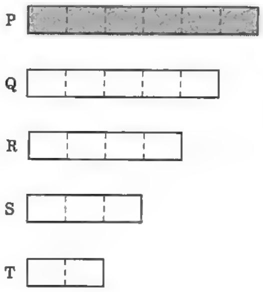
B is $\underline{4}$. Each part is $\frac{4}{2}$.

C is $\underline{6}$. Each part is $\frac{6}{3}$.

D is $\underline{8}$. Each part is $\frac{8}{4}$.

E is $\underline{10}$. Each part is $\frac{10}{5}$.

How much?



P is 1. Each part is $\frac{1}{6}$.

Each part is $\frac{1}{6}$. Q is $5 \times \frac{1}{6}$, or $\frac{5}{6}$.

Each part is $\frac{1}{6}$. R is $4 \times \frac{1}{6}$, or $\frac{4}{6}$.

Each part is $\frac{1}{6}$. S is $\underline{\frac{3}{6}}$.

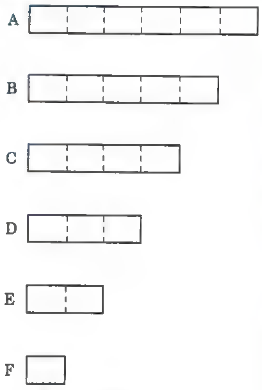
Each part is $\frac{1}{6}$. T is $\underline{\frac{2}{6}}$.

reference page

F-172

Name

How much? *See pupil page suggestions.*



1. If F is 1, then E is 2, D is $\underline{3}$, C is $\underline{4}$, B is $\underline{5}$, A is $\underline{6}$.

2. If F is 2, then E is 4, D is $\underline{6}$, C is $\underline{8}$, B is $\underline{10}$, A is $\underline{12}$.

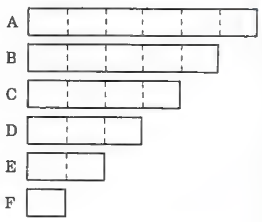
3. If F is $\frac{1}{2}$, then E is $\frac{2}{2}$, D is $\underline{\frac{3}{2}}$, C is $\underline{\frac{4}{2}}$, B is $\underline{\frac{5}{2}}$, A is $\underline{\frac{6}{2}}$.

4. If A is 1, then F is $\frac{1}{6}$, E is $\underline{\frac{2}{6}}$, D is $\underline{\frac{3}{6}}$, C is $\underline{\frac{4}{6}}$, B is $\underline{\frac{5}{6}}$.

5. If A is 2, then F is $\underline{\frac{2}{6}}$, E is $\underline{\frac{4}{6}}$, D is $\underline{\frac{6}{6}}$, C is $\underline{\frac{8}{6}}$, B is $\underline{\frac{10}{6}}$.

6. In each row of the table, one of the regions is 1. Complete each row by writing the numbers for the other regions.

A	B	C	D	E	F
1	$\frac{5}{6}$	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
$\frac{6}{5}$	1	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
$\frac{6}{4}$	$\frac{5}{4}$	1	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{4}$
$\frac{6}{3}$	$\frac{5}{3}$	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$
$\frac{6}{2}$	$\frac{5}{2}$	$\frac{4}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$
6	5	4	3	2	1

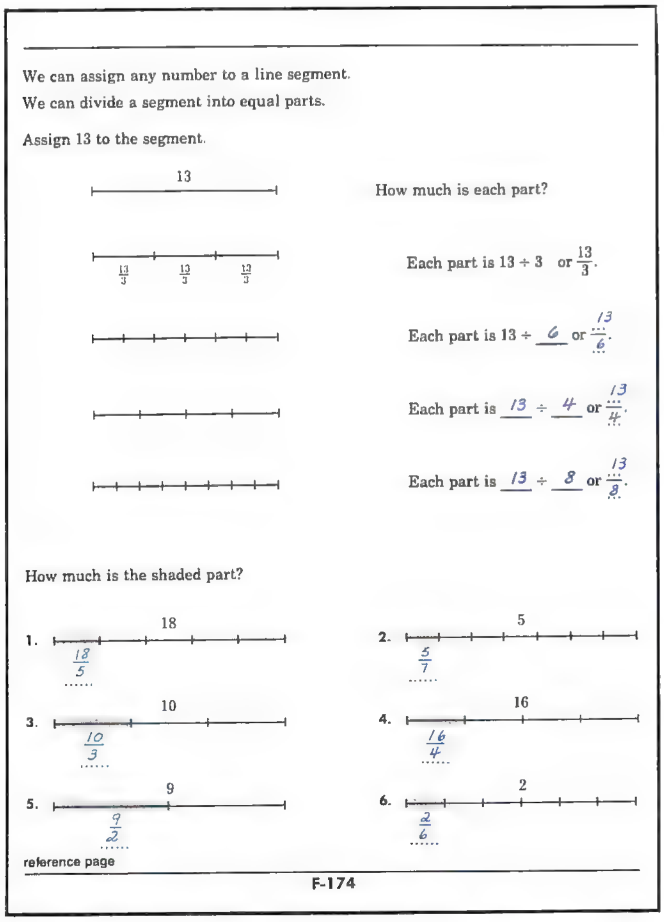


F-173

● Page 174 introduces the idea of assigning a number to a line segment. As the class discusses the top of the page, the children should give reasons for their decisions. Assign exercises 1 through 6 for independent work. Then for each exercise let pupils tell what the number for each part is.

● Use page 175 for class discussion. Have the children discuss the numbers for segments B, C, D, E, and F when A is 3. They should realize that B is 2 times as much as A, C is 3 times as much as A, and so on. With this realization there should be no difficulty in stating the correct number for each segment. Then have the class tell the number for each part of each segment. Through observation the pupils should realize that each part is 3. With further investigation they may realize that the parts of the various segments are $\frac{6}{2}$, $\frac{9}{3}$, $\frac{12}{4}$, $\frac{15}{5}$, and $\frac{18}{6}$. Though the models show equivalent fractions, refrain from calling attention to this concept at this time.

As segments J, K, L, M, and N are investigated, the children must compare each segment with one of the parts of J. The clues indicate that K, for example, is $5 \times \frac{1}{7}$. The pupils should also mention that $5 \div 7$ is 5 times as much as $1 \div 7$. Use the questions at the bottom of the page for discussion. Answers to each question may be expressed as products. For question 1, M is $3 \times \frac{35}{7}$. It is not necessary to compute. Accept any answer that shows an understanding of the model.



We can assign any number to a line segment.
We can divide a segment into equal parts.

Assign 13 to the segment.

How much is each part?

Each part is $13 \div 3$ or $\frac{13}{3}$.

Each part is $13 \div$ 6 or $\frac{13}{6}$.

Each part is 13 ÷ 4 or $\frac{13}{4}$.

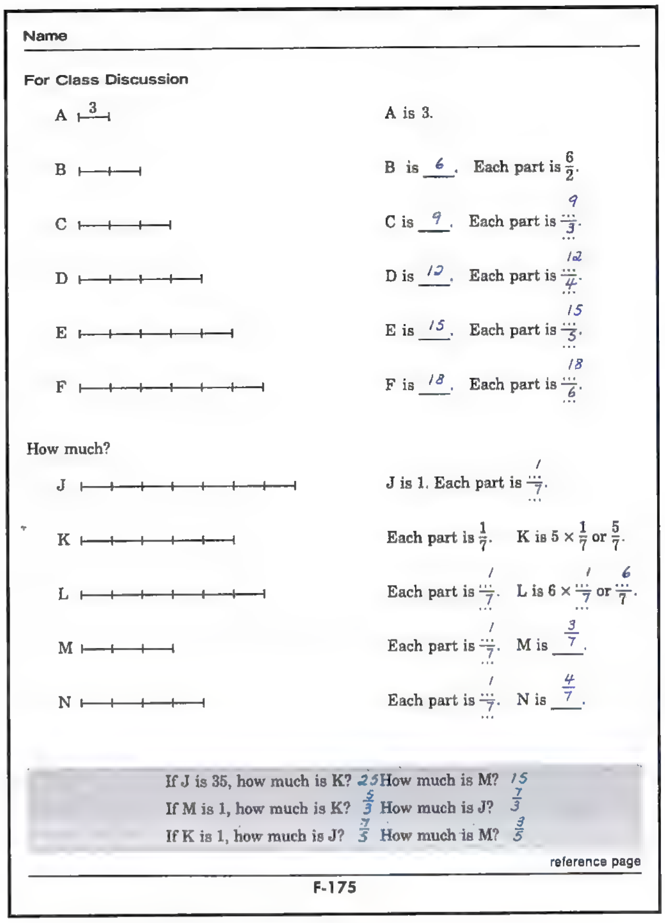
Each part is 13 ÷ 8 or $\frac{13}{8}$.

How much is the shaded part?

1. 18 — $\frac{18}{5}$
2. 5 — $\frac{5}{7}$
3. 10 — $\frac{10}{3}$
4. 16 — $\frac{16}{4}$
5. 9 — $\frac{9}{2}$
6. 2 — $\frac{2}{6}$

reference page

F-174



Name

For Class Discussion

A is 3.

B is 6. Each part is $\frac{6}{2}$.

C is 9. Each part is $\frac{9}{3}$.

D is 12. Each part is $\frac{12}{4}$.

E is 15. Each part is $\frac{15}{5}$.

F is 18. Each part is $\frac{18}{6}$.

How much?

J is 1. Each part is $\frac{1}{7}$.

Each part is $\frac{1}{7}$. K is $5 \times \frac{1}{7}$ or $\frac{5}{7}$.

Each part is $\frac{1}{7}$. L is $6 \times \frac{1}{7}$ or $\frac{6}{7}$.

Each part is $\frac{1}{7}$. M is $\frac{3}{7}$.

Each part is $\frac{1}{7}$. N is $\frac{4}{7}$.

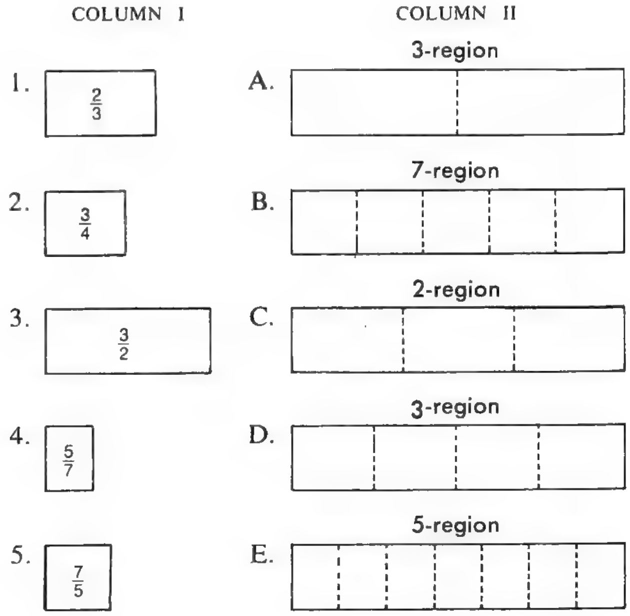
If J is 35, how much is K? 25 How much is M? 15
If M is 1, how much is K? $\frac{5}{3}$ How much is J? $\frac{7}{3}$
If K is 1, how much is J? $\frac{7}{5}$ How much is M? $\frac{3}{5}$

reference page

F-175

Supplemental Experiences

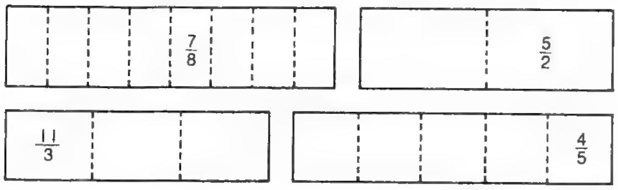
■ The following illustration may be duplicated so that each pupil has a copy, or it may be placed on the chalkboard.



Have the pupils match each region in column I with a region in column II (1C, 2D, 3A, 4E, 5B). Let the

children discuss their choices and their reasons for them.

■ Draw several rectangular regions on the chalkboard, and divide each one into a number of equal parts. Assign a number to each region but do not label the entire region. Instead label one of the parts of each region with the appropriate fraction.



Call attention to the number assigned to the labeled part of the first region. Ask whether this information can be used to tell what number has been assigned to the entire region. For example, the pupils can see that one of the 8 parts of the first region is a $\frac{7}{8}$-region, so the entire region must be a 7-region.

Follow this procedure for each of the other regions.

KEY IDEA

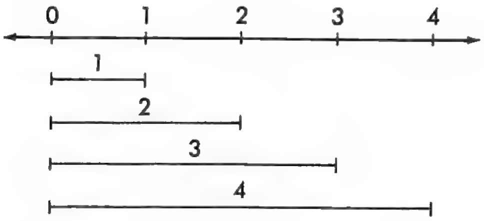
$6 \times \frac{5}{6}$ is 5.

Scope

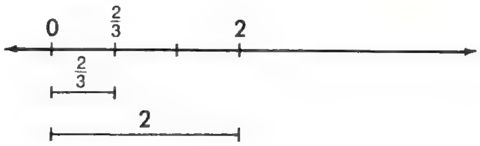
To locate quotients on the number line.

Fundamentals

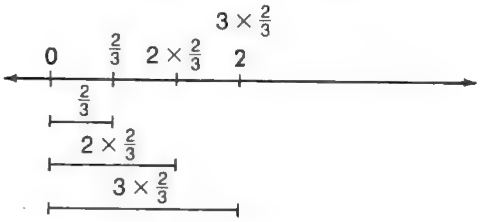
Any fractional number can be located on the number line. You can do this by using the concept of a fractional number as a quotient of two whole numbers. Remember that the length of the line segment between 0 and any number on the number line is that number itself. In other words, the distance from 0 to 1 is 1, the distance from 0 to 2 is 2, and so on. This is shown by the following illustration:



The location of a fractional number on the number line can be found by dividing a whole-number segment into a whole number of equal parts. To locate $\frac{2}{3}$ on the number line, for example, divide the segment between 0 and 2 into 3 equal parts. Each part of the segment has the length $\frac{2}{3}$, so the distance from 0 to the first point dividing the segment is $\frac{2}{3}$. This point, therefore, is the location of the number $\frac{2}{3}$ on the number line.



The second point that divides the segment between 0 and 2 is twice as far from 0 as the first point. This second point, therefore, is the location of the number $2 \times \frac{2}{3}$. The next point on the number line, 2, is 3 times as far from 0 as the first point, so this point is also the location of the number $3 \times \frac{2}{3}$.



The number $3 \times \frac{2}{3}$ therefore is the number 2.

$$3 \times \frac{2}{3} = 2$$

Similarly, it can be shown that $3 \times \frac{1}{3}$ is 1, $3 \times \frac{3}{3}$ is 3, and $3 \times \frac{4}{3}$ is 4, or that $4 \times \frac{3}{4}$ is 3, $4 \times \frac{5}{4}$ is 5, and so on.

Number lines are also used to demonstrate relationships like the following: $2 \times \frac{1}{3} = \frac{2}{3}$, $3 \times \frac{1}{3} = \frac{3}{3}$, $4 \times \frac{1}{3} = \frac{4}{3}$ $2 \times \frac{1}{4} = \frac{2}{4}$, and $3 \times \frac{1}{4} = \frac{3}{4}$.

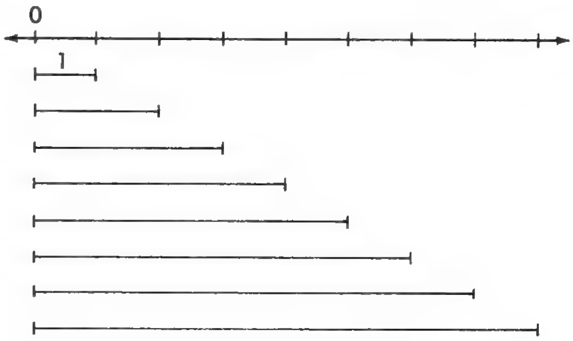
Readiness for Understanding

Knowledge of number line.

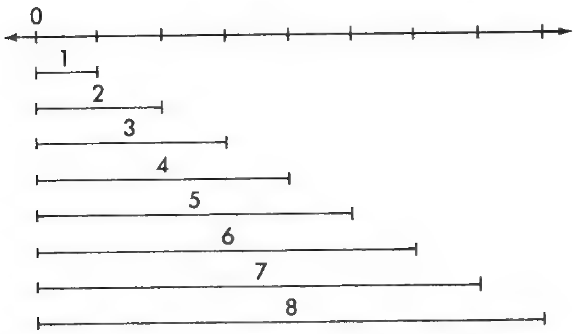
Developmental Experiences

paper rectangular regions
felt-tip pen
box
string

► Draw the following diagram on the chalkboard. Note that the number line could be drawn on a transparency and projected on the chalkboard.

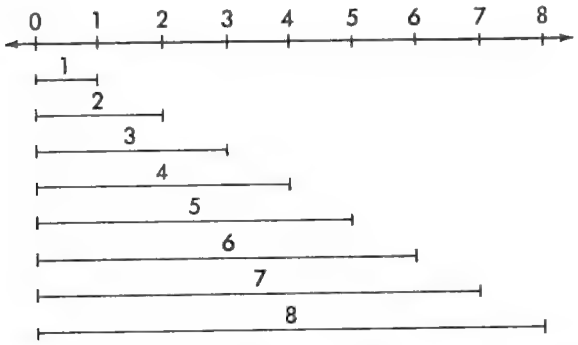


Tell the class that the top figure is called a number line, and that you have assigned the number 1 to the first segment below it. Ask the pupils to label each of the segments below the 1 segment.



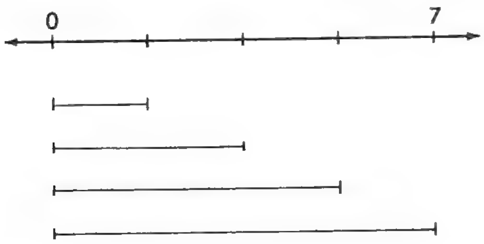
Point to the first mark to the right of 0 on the number line, and ask a pupil to tell the distance from 0 to that mark (1). Have him write this number above the mark.

Point to the next mark to the right, and ask another pupil to tell the distance from 0 to that mark (2). Have him write this number above the mark. Proceed in this way until all the marks shown on the number line are labeled.

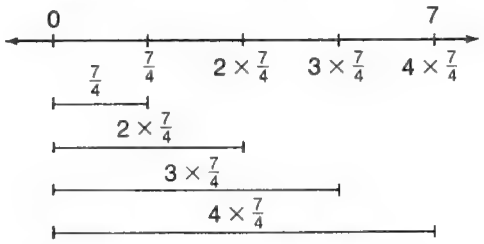


Do not erase this diagram. Keep it on the chalkboard for the remainder of the activity.

Now draw another number line on the chalkboard. Mark and label only the points 0 and 7. Divide the segment between 0 and 7 into 4 equal parts. Then finish the diagram, as shown.



Ask a pupil how many parts the segment between 0 and 7 has been divided into (4). Ask another pupil to give the distance from 0 to the first mark to the right ($\frac{7}{4}$). Have him label both this mark and the first segment below the number line. Then ask another pupil to compare the second segment below the number line with the first. He may say that it is twice as long. Have him label the second segment ($2 \times \frac{7}{4}$). Ask a third pupil to give the distance on the number line from 0 to the second mark to the right and to label this mark ($2 \times \frac{7}{4}$). Have pupils compare the third and fourth segments with the first one. They are 3 and 4 times as long, and should be labeled $3 \times \frac{7}{4}$ and $4 \times \frac{7}{4}$ respectively. Other pupils may then tell the distances on the number line from 0 to the third and fourth marks to the right. They should label these marks accordingly.

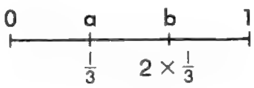


Now ask a pupil to compare the segment labeled $4 \times \frac{7}{4}$ with the segment between 0 and 7 on the number line (they are the same length). Ask another pupil to write an equation showing this ($4 \times \frac{7}{4} = 7$). Point out that the labels 7 and $4 \times \frac{7}{4}$ on the number line name the same distance. This means that 7 is $4 \times \frac{7}{4}$ and $4 \times \frac{7}{4}$ is 7.

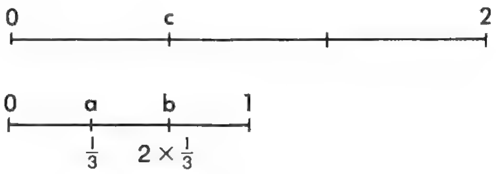
Vary this activity by dividing the segment between 0 and 7 on the number line into 3 or 5 equal parts. Use other segments too. For example, use the segment between 0 and 5, the segment between 0 and 8, and the segment between 0 and 10.

▶ On the chalkboard draw a line segment. Partition the segment into 3 equal parts, and label it as indicated below. Point out that this is a 1-segment, and ask for the distance from 0 to *a*.

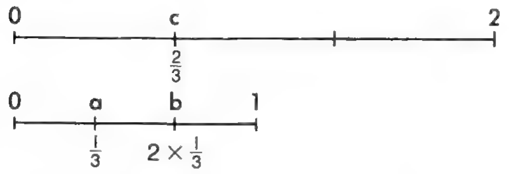
When a pupil responds that the distance from 0 to *a* is 1 *divided by* 3, or one third, have him write the fraction for this number in the appropriate place below the line. Then ask for the distance from 0 to *b*. When a pupil responds that the distance from 0 to *b* is 2 times 1-*divided-by*-3, or $2 \times \frac{1}{3}$, have him write this number in the appropriate place.



Now, directly above the segment, draw a segment that is twice as long. Tell the class that since the length of this segment is 2 times the length of the 1-segment, it will be a 2-segment. Partition the 2-segment into 3 equal parts, and label it as indicated.



Ask for the distance from 0 to c. When a pupil responds that the distance from 0 to c is 2 *divided by* 3, have him write the fraction for this number in the appropriate place.



Next help a pupil cut a piece of string that is the same length as the distance from 0 to c. Have another pupil use this string to compare the distance from 0 to c with the distance from 0 to b. Have the class tell what they notice about the distance from 0 to c when compared with the distance from 0 to b (they are the same distance). Ask for the number telling the distance from 0 to c ($\frac{2}{3}$); ask for the number that gives the distance from 0 to b ($2 \times \frac{1}{3}$). Tell the class this comparison shows that $\frac{2}{3}$ is $2 \times \frac{1}{3}$. On the chalkboard have a pupil write an equation giving this information.

$$\frac{2}{3} = 2 \times \frac{1}{3}$$

Have the equation read: "Two thirds is two times one third," or "2 *divided by* 3 is 2 times 1-*divided-by*-3."

Follow the same procedure in having the pupils use number lines to compare $\frac{5}{6}$ and $5 \times \frac{1}{6}$, $\frac{3}{5}$ and $3 \times \frac{1}{5}$, $\frac{3}{8}$ and $3 \times \frac{1}{8}$, $\frac{4}{7}$ and $4 \times \frac{1}{7}$.

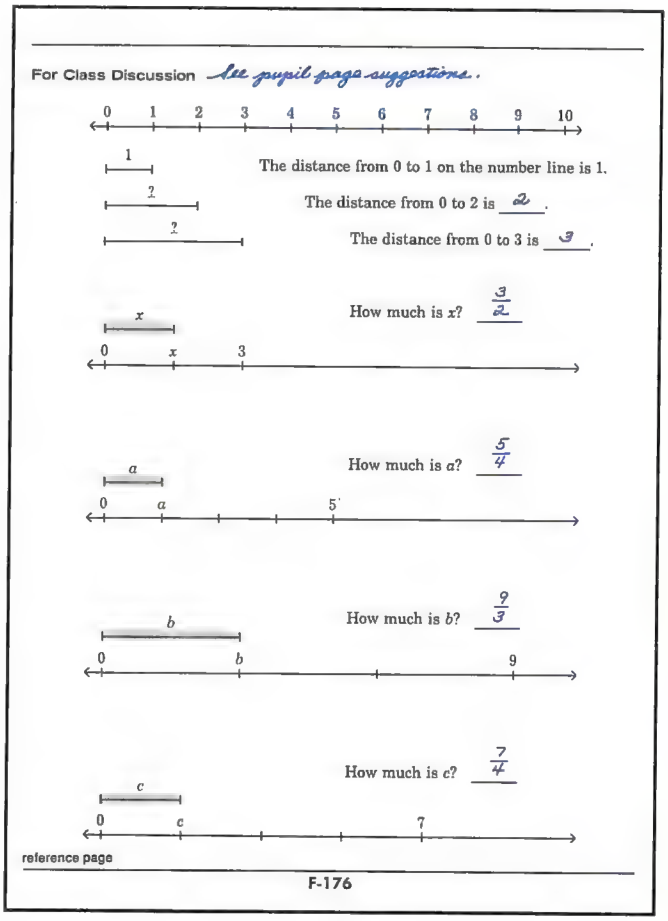
► Cut out several rectangular regions of varying sizes, and use a felt-tip pen to divide each region into some number of equal parts. Place the regions in a box. Have a pupil select one of the regions and place it on the flannel board or bulletin board. Ask him to assign any number he wishes to the region. Then have him call on a classmate to tell how many parts the region has been divided into and to give the fractional number for each part. For example, if a pupil assigns the number 7 to a region that is divided into 11 parts, the pupil he calls on should say that there are 11 equal parts and that each part is a $\frac{7}{11}$-region. Then point out to the class that it takes eleven $\frac{7}{11}$-regions to make a 7-region. Ask a third pupil to write an equation showing this ($11 \times \frac{7}{11} = 7$). Ask this pupil to assign a different number to the same region. Then he should select another pupil to tell what number is now associated with each part of the region. For example, if the third pupil assigns 12 to the region, the pupil he calls on should say that each part is $\frac{12}{11}$. Now choose another pupil to write an equation that shows how many $\frac{12}{11}$-regions make up the 12-region ($11 \times \frac{12}{11} = 12$).

Begin the activity again using a new region from the box. Follow a similar procedure until each pupil participates at least once in the activity.

Pages 176 through 184

● Page 176 provides experience with the number line. Use the page for discussion. On the number line at the top of the page, have some pupils point to the ends of those segments that extend from 0 to 1, 0 to 2, 0 to 3, 0 to 7, and 0 to 10.

Then discuss the distances shown by each segment. Tell the class that each of the other four pictures on the page shows a segment whose length is given on the number line. Ask them to tell the length of the x-segment. They should see that x is the distance from 0 to x on the number line and that x is $\frac{3}{2}$. Discuss the a-segment, the b-segment, and the c-segment in a similar way.



● Page 177 provides practice in identifying distances on the number line. In exercises 2 and 3, the distance to be identified is both a whole number and a fractional number. For example, in the second exercise x is both $\frac{4}{2}$ and 2. Be sure the distance is described in both ways.

Assign exercises 1 through 14 for independent work. As the results are discussed, ask the children to give reasons for their decisions. Exercises 10 through 14 should cause considerable discussion. Some pupils may see that each illustration is the same as one of the illustrations in exercises 5 through 9. More important is the idea that if one distance is 3 *divided by* 2, two such distances are 3. Some pupils may explain this as follows: "Start with 3 and divide it into 2 equal parts. Put the 2 parts together and you have 3 again." The pupils should see that $2 \times \frac{3}{2}$ is 3, $4 \times \frac{5}{4}$ is 5, $3 \times \frac{7}{3}$ is 7, $2 \times \frac{2}{2}$ is 2, and $4 \times \frac{2}{4}$ is 2. Perception of this idea, rather than computation, is desired.

● Page 178 provides experience with recognition of related distances. Discuss the examples at the top of the page with the class. Then assign exercises 1 through 4 for independent work. Discuss the results.

● Page 179 provides practice in the interpretation of fractional numbers. Two views of the number line are shown in each exercise. One distance is shown in two ways.

Discuss the example with the class. They should see that the distance 1 is divided into 3 equal parts. The distance c therefore is $2 \times \frac{1}{3}$. The distance 2 is divided into 3 equal parts, so the distance c is also $\frac{2}{3}$. Since c is the same distance on both number lines, 2 times 1-*divided-by*-3 is 2 *divided by* 3.

Assign the exercises for independent work. Let the class discuss the results.

● Page 180 states that quotients of whole numbers are fractional numbers. The pupils should also realize that fractional numbers are quotients of whole numbers. Consecutive numbering on the number line is introduced.

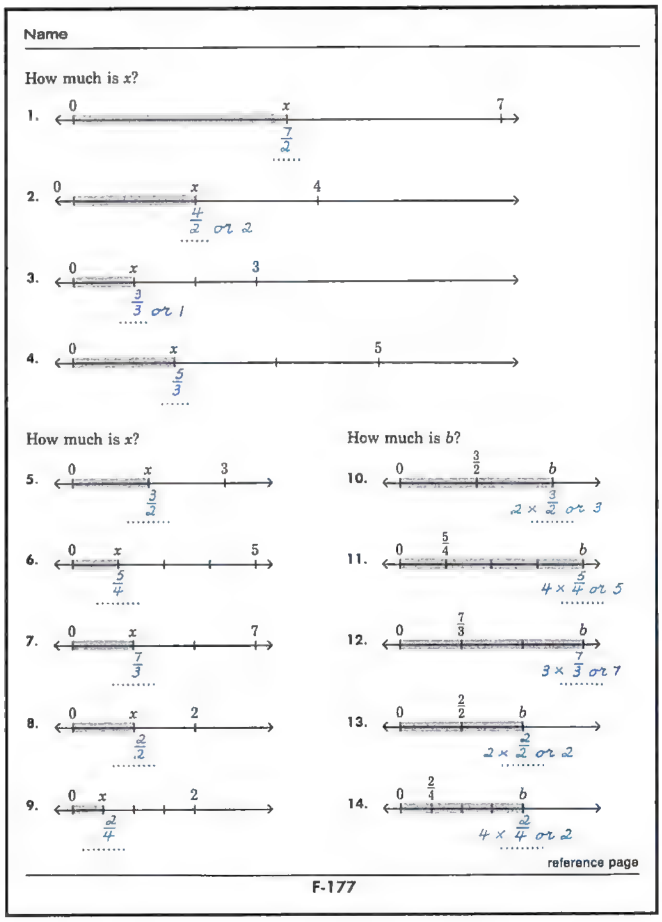
As the example is discussed, the pupils should be able to identify distance B as $\frac{1}{3}$; C as $\frac{2}{3}$, or $2 \times \frac{1}{3}$; and D as $\frac{3}{3}$, or $3 \times \frac{1}{3}$. They can also learn an easy way to locate any fractional number in the sequence $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \ldots$. In order to locate $\frac{4}{3}$, for example, on the number line, the pupil need not divide the segment between 0 and 4 into 3 equal parts. If he has already located $\frac{1}{3}$, he only needs to find the point that is 4 times as far from 0 as $\frac{1}{3}$ is. This point is the location of $4 \times \frac{1}{3}$, or $\frac{4}{3}$.

Do exercise 1 with the class. Have the children start with the first mark to the right of 0 and continue to the right, placing the fraction for the distance below each mark. Then let the class decide on the fraction for 0 ($\frac{0}{3}$). Ask the class to note that some of the fractional numbers on the number line are located at the same point as a whole number. Ask which ones these are ($\frac{0}{3}, \frac{3}{3}, \frac{6}{3}, \frac{9}{3}$, and $\frac{12}{3}$). Have the pupils write equations showing that these fractional numbers are whole numbers ($0 = \frac{0}{3}$, $1 = \frac{3}{3}$, $2 = \frac{6}{3}$, and so on).

Assign the remaining exercises for independent work.

● Page 181 provides experience in applying the concept of fractional numbers to comparison of weights.

Work exercises 1 through 4 with the class. Be sure the pupils realize that all the objects on each scale are alike. For example, all the balls in exercise 1 are the same size, shape, and weight. The pupils should see that the scale is balanced, so 6 balls weigh 1 pound. Ask what the weight of 1 ball is. The children should recognize that this is similar to thinking of a 1-region, or a 1-segment, divided into 6 equal parts. Each ball weighs $\frac{1}{6}$ pound. Follow this procedure in discussing the weight of an individual block in exercise 2, a book in exercise 3, and one box in exercise 4. Then have the pupils complete exercises 5 through 12 on their own. The teacher should note that the answers to the exercises can be given in either of two forms. For example, in exercise 5 the weight of 3 balls can be given as $3 \times \frac{1}{6}$ pound or as $\frac{3}{6}$ pound.



How much?

0, 4; a, b, c

The distance from 0 to a is $4 \div 3$ or $\frac{4}{3}$.

from 0 to b is $2 \times 4 \div 3$ or $2 \times \frac{4}{3}$.

from 0 to c is 3 $\times 4 \div 3$ or 3 $\times$ $\frac{4}{3}$.

0, 4; $\frac{4}{3}$, $2 \times \frac{4}{3}$, $3 \times \frac{4}{3}$

$\frac{4}{3}$

$2 \times \frac{4}{3}$

$3 \times \frac{4}{3}$

What is the distance from 0 to each point marked?

1. 0, 5; a $\frac{5}{3}$; b $2 \times \frac{5}{3}$; c $3 \times \frac{5}{3}$ or 5
2. 0, 3; a $\frac{3}{4}$; b $2 \times \frac{3}{4}$; c $3 \times \frac{3}{4}$; d $4 \times \frac{3}{4}$ or 3
3. 0, 2; a $\frac{2}{3}$; b $2 \times \frac{2}{3}$; c $3 \times \frac{2}{3}$ or 2
4. 0, 4; a $\frac{4}{4}$; b $2 \times \frac{4}{4}$; c $3 \times \frac{4}{4}$; d $4 \times \frac{4}{4}$ or 4

reference page

See pupil page suggestions.

Quotients of whole numbers are fractional numbers.

$1 \div 2$ or $\frac{1}{2}$ is a fractional number.

$2 \div 2$ or $\frac{2}{2}$ is a fractional number.

$3 \div 2$ or $\frac{3}{2}$ is a fractional number.

0, 1, 2, 3, 4; B, C, D, E

Are B, C, D, and E fractional numbers? How do you know? The lengths are quotients of two whole numbers. Answers may vary.

What is the distance from 0? Write a fraction.

1. 0, 1, 2, 3, 4: $\frac{0}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ $\frac{4}{3}$ $\frac{5}{3}$ $\frac{6}{3}$ $\frac{7}{3}$ $\frac{8}{3}$ $\frac{9}{3}$ $\frac{10}{3}$ $\frac{11}{3}$ $\frac{12}{3}$
2. 0, 1, 2, 3: $\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$ $\frac{5}{4}$ $\frac{6}{4}$ $\frac{7}{4}$ $\frac{8}{4}$ $\frac{9}{4}$ $\frac{10}{4}$ $\frac{11}{4}$ $\frac{12}{4}$
3. 0, 1, 2, 3: $\frac{0}{6}$ $\frac{1}{6}$ $\frac{2}{6}$ $\frac{3}{6}$ $\frac{4}{6}$ $\frac{5}{6}$ $\frac{6}{6}$ $\frac{7}{6}$ $\frac{8}{6}$ $\frac{9}{6}$ $\frac{10}{6}$ $\frac{11}{6}$ $\frac{12}{6}$ $\frac{13}{6}$ $\frac{14}{6}$ $\frac{15}{6}$ $\frac{16}{6}$ $\frac{17}{6}$ $\frac{18}{6}$
4. 0, 1, 2, 3: $\frac{0}{2}$ $\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$ $\frac{4}{2}$ $\frac{5}{2}$ $\frac{6}{2}$

reference page

Name

How much is c?

0, c, 1 — c is $2 \times \frac{1}{3}$ or $\frac{2}{3}$.

0, c, 2 — $\frac{2}{3}$ is $2 \times \frac{1}{3}$.

How much is c?

1. 0, c, 1; 0, c, 3 — c = $3 \times \frac{1}{6}$ or $\frac{3}{6}$
2. 0, c, 1; 0, c, 4 — c = $4 \times \frac{1}{5}$ or $\frac{4}{5}$
3. 0, c, 1; 0, c, 3 — c = $3 \times \frac{1}{4}$ or $\frac{3}{4}$
4. 0, c, 1; 0, c, 4 — c = $4 \times \frac{1}{8}$ or $\frac{4}{8}$

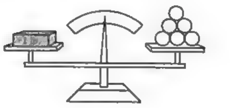
5. $6 \times \frac{1}{4} = \frac{6}{4}$ 6. 7 $\times \frac{1}{9} = \frac{7}{9}$ 7. $13 \times \frac{1}{5} = \frac{13}{5}$

reference page

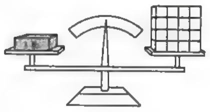
Name

What is the weight of one object?

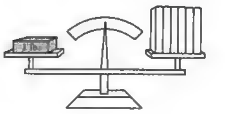
1. One ball $\frac{1}{6}$ pound



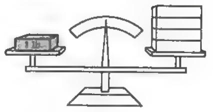
2. One block $\frac{1}{16}$ pound



3. One book $\frac{1}{6}$ pound

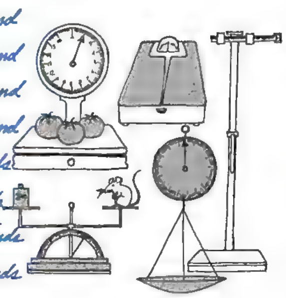


4. One box $\frac{1}{4}$ pound



Using your information from exercises 1, 2, 3, and 4, answer these questions.

5. How much do 3 balls weigh? $\frac{3}{6}$ pound
6. How much do 9 blocks weigh? $\frac{9}{16}$ pound
7. How much do 6 books weigh? $\frac{6}{6}$ pound
8. How much do 3 boxes weigh? $\frac{3}{4}$ pound
9. How much do 12 balls weigh? $\frac{12}{6}$ pounds
10. How much do 17 blocks weigh? $\frac{17}{16}$ pounds
11. How much do 12 books weigh? $\frac{12}{6}$ pounds
12. How much do 17 boxes weigh? $\frac{17}{4}$ pounds

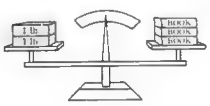


● Page 182 continues with comparison of weights. Work through exercise 1 with the class. Have the pupils identify the objects that balance on the scale (3 books and 2 pounds). Then ask how much 1 book weighs. Help the pupils to relate this to the idea of a 2-segment, or a 2-region, divided into 3 equal parts. When the pupils understand the procedure to be followed, assign exercises 2 through 9 for independent work.

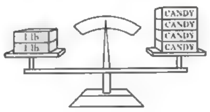
● Pages 183 and 184 describe other applications of fractional numbers. Use both pages for discussion. The pupils may disagree about the solutions on these pages being the same or different. If the children see the solutions as the end result, they are the same. But if the children see the solutions as the procedures involved, the solutions are different. Both views are correct.

What is the weight of one object?

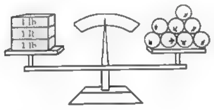
1. One book $\frac{2}{3}$ pound



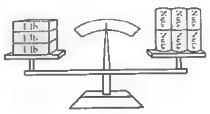
2. One box of candy $\frac{2}{4}$ pound



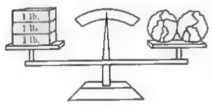
3. One grapefruit $\frac{3}{8}$ pound



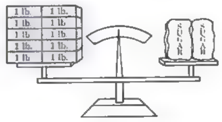
4. One can of nuts $\frac{3}{6}$ pound



5. One cabbage $\frac{3}{2}$ pounds



6. One bag of sugar $\frac{10}{2}$ pounds



7. If 5 apples weigh 2 pounds, how much does 1 apple weigh? $\frac{2}{5}$ pound

8. If 9 boxes weigh 1 pound, how much does 1 box weigh? $\frac{1}{9}$ pound

9. If 4 hammers weigh 10 pounds, how much does 1 hammer weigh? $\frac{10}{4}$ pounds

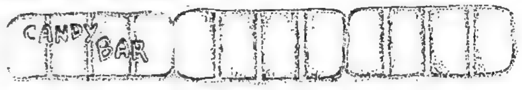
F-182

Name

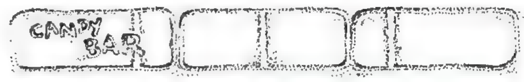
For Class Discussion

Sandy, Paula, Marlene, and Liz were hiking. Soon the girls were all hungry. Sandy had 3 candy bars which they decided to share equally. "But how can we divide them up?" asked Marlene. "There are four of us and only three candy bars."

Paula drew a plan in the sand.



Liz drew a plan in the sand.



How would Paula divide the candy? Paula would divide each bar into 4 pieces.*

How would Liz divide the candy? Liz would divide the total amount into 4 equal parts.*

Were the two girls' solutions the same or different? See pupil page suggestions

How much candy does each girl get? $\frac{3}{4}$ bar

If Sandy had 5 candy bars, how much candy would each girl get? $\frac{5}{4}$ bar

If Sandy had 2 candy bars, how much candy would each girl get? $\frac{2}{4}$ bar

Henry wanted 4 quarts of water. He found a 3-quart pan and a 5-quart pan. How could he measure 4 quarts using the two pans?

Let pupils demonstrate answers with rice, water, or sand.

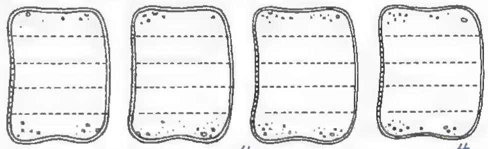
*Answers may vary.

F-183

For Class Discussion

Eric was making sandwiches for the Saturday meeting of the Jokers Club. "Adam, Benjy, Harv, and Marty will be there," he thought. So Eric made four sandwiches. At the meeting, he remembered that he hadn't made a sandwich for himself. So he decided to cut each sandwich into 5 parts. He gave each boy one part from each sandwich.

Benjy
Harv
Marty
Adam
Eric



How much does each boy get? $\frac{4}{5}$ How do you know? $5 \times \frac{4}{5}$ is 4.

"These are good," said Benjy, "but if I had cut them up, we wouldn't have so many little pieces."

How would Benjy have solved the problem? Benjy would have divided the total amount into 5 equal parts.

Were the two boys' solutions the same or different? See pupil page suggestions.

How much does each boy get? Each boy gets $\frac{4}{5}$ of a sandwich.

If a new member had come to the meeting, how might Eric have divided his 4 sandwiches among the 6 boys? Eric would have divided each sandwich into 6 parts.

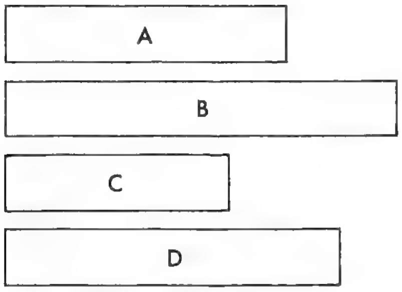
How would Benjy have done it? Benjy would have divided the total amount into 6 equal parts.

F-184

Supplemental Experiences

■ Cut out 4 cards measuring 3″ × 3″. Label one of these cards $\frac{7}{4}$.

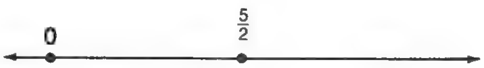
Then place on the flannel board a 3-inch by 15-inch region, a 3-inch by 21-inch region, a 3-inch by 12-inch region, and a 3-inch by 18-inch region. Label these regions A, B, C, and D respectively.



Give one pupil the card labeled $\frac{7}{4}$. Have this pupil compare his $\frac{7}{4}$-card with the regions on the flannel board and tell which of them is a 7-region. He should remember that four $\frac{7}{4}$-regions are a 7-region. The 7-region, therefore, is the region that is 4 times as large as the $\frac{7}{4}$-card. This is the one labeled C in the illustration. Take a second 3-inch by 3-inch card and label it $\frac{3}{5}$. Then ask another pupil to find the region on the flannel board that is a 3-region (A). Label the third and fourth cards $\frac{11}{6}$ and $\frac{5}{7}$. The pupil who selects the $\frac{11}{6}$-card is to find the 11-region; the pupil who selects the $\frac{5}{7}$-card is to find the 5-region.

■ On the chalkboard draw a number line like the one below.

Ask a pupil to locate the point for 9. Have him explain how he did this. He may say that 9 is 4 times as far from 0 as $\frac{9}{4}$ is. Thus he will mark off four $\frac{9}{4}$-segments to find 9. Continue the activity with similar exercises. For example, have a pupil locate 5 on this number line:



Have another pupil locate 6 on this number line:

Have 4 located on this number line:

■ Here is a quiz the teachers may wish to use.

SUGGESTED QUIZ

What is the distance from 0 to x?

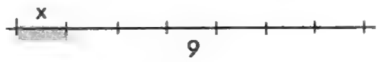
1. 

$x = \frac{7}{3}$

2. 

$x = \frac{3}{5}$

Write the number for the length of the x-segment. Then write an equation to show the number of x-segments that equal the given segment.

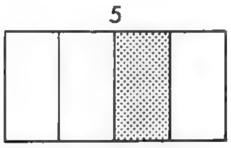
3. 

x is $\frac{9}{7}$
$7 \times \frac{9}{7} = 9$

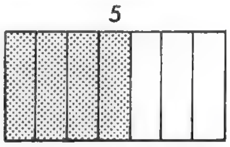
4. 

x is $\frac{3}{2}$
$2 \times \frac{3}{2} = 3$

The number 5 is assigned to each region below. What is the number of the shaded part?

5. 

$\frac{5}{4}$

6. 

$4 \times \frac{5}{7}$

Write another name for each fractional number.

7. $4 \times \frac{1}{6} = \frac{4}{6}$ 8. $9 \times \frac{1}{7} = \frac{9}{7}$ 9. $\frac{5}{8} = 5 \times \frac{1}{8}$

UNIT 14
FRACTIONAL NUMBERS: ADDITION AND SUBTRACTION
Pages 185 Through 196

OBJECTIVE
To compute sums and differences of fractional numbers with the same denominators.

The pupil reviews fractional numbers as quotients. Using models of fractional numbers, he learns that $\frac{a}{b} + \frac{b}{c} = \frac{a+b}{c}$ and that $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$. He also practices computing sums and differences of some fractional numbers.

See Key Topics in Mathematics for the Intermediate Teacher: The Set of Fractional Numbers.

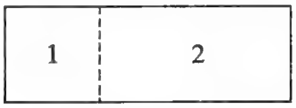
KEY IDEAS
$\frac{3}{5} + \frac{4}{5} = \frac{3+4}{5} = \frac{7}{5}$.
$\frac{7}{5} - \frac{3}{5} = \frac{7-3}{5} = \frac{4}{5}$.

KEY IDEA
$\frac{3}{5} + \frac{4}{5} = \frac{3+4}{5} = \frac{7}{5}$.

Scope
To compute sums of some fractional numbers.

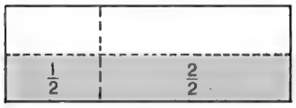
Fundamentals
Since we define $\frac{b}{a}$ as the quotient *b divided by a*, it is desirable to see how this definition helps us to understand computation with fractions.

A good model for addition of fractional numbers can be made by using regions. The region below, for example, illustrates $1 + 2 = 3$.



$1 + 2 = 3$

This region can be divided to show $\frac{1}{2} + \frac{2}{2} = \frac{3}{2}$ Continue to read this as "1 ÷ 2 plus 2 ÷ 2 equals 3 ÷ 2."



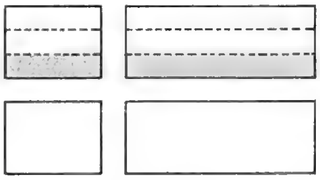
$\frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$

It is evident that other examples can be illustrated the same way. Thus, in the general case, the model shows that $\frac{a+b}{2}$ equals $\frac{a}{2} + \frac{b}{2}$. Similarly, the regions can be divided horizontally into any number of equal parts, so the model shows that $\frac{a+b}{c}$ is equal to $\frac{a}{c} + \frac{b}{c}$ no matter what numbers *a*, *b*, and *c* represent (providing *c* is not 0).

Readiness for Understanding
Knowledge of fractional numbers.

Developmental Experiences
for flannel board
two tagboard strips (6″ × 8″)
two tagboard strips (6″ × 16″)
felt-tip pen

► Prepare two pieces of tagboard 6 by 8 inches and two pieces 6 by 16 inches to adhere to the flannel board. Use a felt-tip pen to divide one piece of each size into 3 equal parts, as shown. Shade or color one of the parts of each of these two pieces.

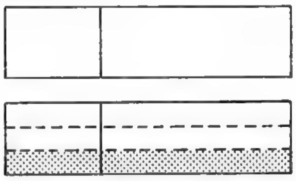


First place the two plain pieces of tagboard on the flannel board. Then point to the smaller piece and the larger piece in turn, and say, "If this is 5, what number is this?" The pupils may see that the larger is 10, or 2 times as much as the smaller. You can place the smaller piece on top of the larger to make the relationship obvious. Continue assigning different numbers, such as 2, 6, 4, and 1, to the smaller piece and asking pupils to tell the corresponding number for the larger piece.

After this quick review, place the two pieces together end to end.

Say, "If the smaller region is 5, the larger region is 10, and together they are 5 + 10." Then ask a pupil to tell the sum shown by the two regions if the smaller is 3. Though the computed sum 9 is correct, encourage the pupils to use 3 + 6. Continue by assigning numbers such as 4, 6, 1, 7, and 10 to the smaller region.

Now place the divided regions on the board below the undivided ones.



Pointing to the smaller divided region, ask, "If this region is 2, what number is the shaded part?" The pupils should see that the shaded part is 2 *divided by* 3. Then ask pupils to give the numbers for the larger region and its shaded part (4, 4 ÷ 3). Continue by letting pupils assign numbers to the smaller region and conduct the activity themselves. They may, for example, assign 5 to the smaller region. Then the small shaded part is 5 ÷ 3, the larger region is 10, and the large shaded part is 10 ÷ 3.

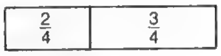
Finally ask the pupils to give the sum for the shaded parts. For example, if the smaller region is 2, the larger region is 4, and the shaded parts are $\frac{2}{3}$ and $\frac{4}{3}$. The sum for the shaded parts then is 2-*divided-by*-3 plus 4-*divided-by*-3. Write the sum on the chalkboard

as $\frac{2}{3}+\frac{4}{3}$. Then ask for the sum of the smaller and larger regions together (2 + 4). Ask a pupil to compare this combined region with the part of it that is shaded. Some pupils may realize that the total shaded part is 2 + 4 *divided by* 3. Write this on the chalkboard as $\frac{2+4}{3}$. Ask whether $\frac{2}{3}+\frac{4}{3}$ and $\frac{2+4}{3}$ are the same number. On the chalkboard write the following:

$$\frac{2+4}{3} = \frac{2}{3}+\frac{4}{3}$$
$$\frac{2}{3}+\frac{4}{3} = \frac{2+4}{3}$$

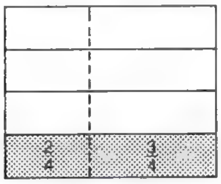
Repeat the activity, assigning different numbers to the regions.

▶ On the chalkboard draw a 4 by 8 inch rectangle and a 4 by 12 inch rectangle, as shown. Label them $\frac{2}{4}$ and $\frac{3}{4}$ respectively.



Ask several pupils to draw regions for the number 2. One pupil may draw a rectangle the same length as the rectangle for $\frac{2}{4}$ but 4 times as high. Another pupil may draw a rectangle the same height as the one for $\frac{2}{4}$ but 4 times as long. The common idea is that $\frac{2}{4}$ is $\frac{1}{4}$ of 2. In the same way, have pupils draw regions for the number 3.

Extend the rectangles for $\frac{2}{4}$ and $\frac{3}{4}$ to show 2 and 3 by increasing the dimensions to 16 by 8 inches and 16 by 12 inches. Shade the original rectangles for $\frac{2}{4}$ and $\frac{3}{4}$, and below the figure write $\frac{2}{4}+\frac{3}{4}$.



$\frac{2}{4}+\frac{3}{4}$

Let a pupil point out 2, 3, $\frac{2}{4}$, $\frac{3}{4}$, and $\frac{2}{4}+\frac{3}{4}$ in the diagram. Ask him to label the 2-region and the 3-region. Then write $\frac{2+3}{4}$ on the chalkboard, and let a pupil point out 2 + 3 and $\frac{2+3}{4}$ in the diagram. Ask pupils to explain how they know that $\frac{2}{4}+\frac{3}{4}$ and $\frac{2+3}{4}$ are the same number. Then ask a pupil for the computed sum of 2 + 3. Have the pupil use the computed sum in writing the number for the entire shaded region. If the number for the entire region is 5, the shaded part is $\frac{5}{4}$.

$$\frac{2}{4}+\frac{3}{4}=\frac{2+3}{4}=\frac{5}{4}$$

Erase all the labels and write $\frac{4}{4}+\frac{6}{4}$. Ask pupils to show 4, 6, $\frac{4}{4}$, and $\frac{6}{4}$ in the diagram and to label the diagram with these numbers. Then ask a pupil to compute the sum $\frac{4}{4}+\frac{6}{4}$. Some pupils may immediately say that $\frac{4}{4}+\frac{6}{4}$ is $\frac{10}{4}$. This is fine.

$$\frac{4}{4}+\frac{6}{4}=\frac{4+6}{4}=\frac{10}{4}$$

Continue the activity by assigning other numbers, such as $\frac{6}{4}$ and $\frac{9}{4}$, $\frac{8}{4}$ and $\frac{12}{4}$, and $\frac{12}{4}$ and $\frac{18}{4}$, to the shaded regions. In each case, have pupils label the parts of the diagram and compute the sum.

Pages 185 through 190

● Use pages 185 and 186 for discussion. Let the pupils discuss the illustrations. For example, the third illustration on page 185 is a (3 + 2)-region because it combines a 3-region and a 2-region. The meaning of the illustrations is clearer if fractional numbers are read as quotients of whole numbers. For example, the fourth illustration on page 185 is read, "The shaded part is 3 + 2 *divided by* 4." Ask pupils to explain how they know that the fifth illustration shows 3 *divided by* 4. As each sentence is read, let different pupils explain the reasoning. Ask the pupils to compare the fourth illustration with the last one on page 185. They should see that the shaded parts are the same: $\frac{3}{4}+\frac{2}{4}$ is $\frac{3+2}{4}$.

Use the top of page 186 for discussion. Ask why the top three parts are labeled $\frac{5}{3}$, $\frac{2}{3}$, and $\frac{5+2}{3}$. Encourage the pupils to tell how they know that $\frac{5}{3}+\frac{2}{3}$ is $\frac{5+2}{3}$ and $\frac{4}{3}+\frac{3}{3}$ is $\frac{4+3}{3}$ (the regions show this). Discuss the three questions at the bottom of the page after the children have had a chance to study them.

● Discuss the illustrations and computation at the top of page 187. In each case ask the pupils to give the number for the entire region shown (6 + 2, or 8; 4 + 2, or 6). Assign the exercises for practice in computing sums. As soon as the pupils have finished, discuss the reasoning they used in working the exercises. All fractional numbers shown by the model should be read as quotients. For example, 6-*divided-by*-3 plus 2-*divided-by*-3 is 6 + 2 *divided by* 3, or 8 *divided by* 3.

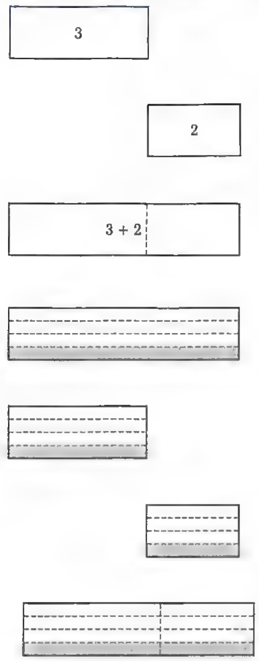
● The illustration at the top of page 188 uses a different model to show addition of fractional numbers. Be sure the children see that the third segment is 5 + 2, the sum of the first two segments. The shaded part of each segment is one of the 3 equal parts of that segment.

Assign the exercises for independent work. Let the class discuss their results.

Name

UNIT 14 FRACTIONAL NUMBERS: ADDITION AND SUBTRACTION

See pupil page suggestions.



This is a 3-region.

This is a 2-region.

Together they make a 3 + 2-region.

The shaded part is $\frac{3+2}{4}$.

The shaded part is $\frac{3}{4}$.

The shaded part is $\frac{2}{4}$.

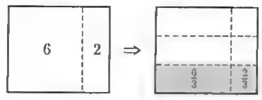
The shaded part is $\left(\frac{3}{4}+\frac{2}{4}\right)$.

We can see that $\frac{3}{4}+\frac{2}{4}$ is $\frac{3+2}{4}$. $\quad \frac{3}{4}+\frac{2}{4}=\frac{3+2}{4}=\frac{5}{4}$

reference page

Name

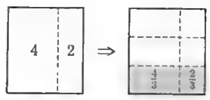
Compute $\frac{6}{3}+\frac{2}{3}$.



$\frac{6}{3}+\frac{2}{3}=\frac{6+2}{3}$

$=\frac{8}{3}$

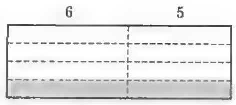
Compute $\frac{4}{3}+\frac{2}{3}$.



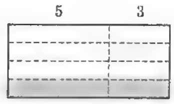
$\frac{4}{3}+\frac{2}{3}=\frac{4+2}{3}=\frac{6}{3}$

See pupil page suggestions.

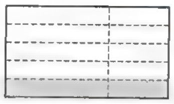
Compute.

1. 

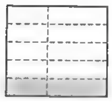
$\frac{6}{4}+\frac{5}{4}=$ $\frac{11}{4}$

2. 

$\frac{5}{4}+\frac{3}{4}=$ $\frac{8}{4}$

3. 

$\frac{5}{5}+\frac{3}{5}=$ $\frac{8}{5}$

4. 

$\frac{2}{5}+\frac{3}{5}=$ $\frac{5}{5}$

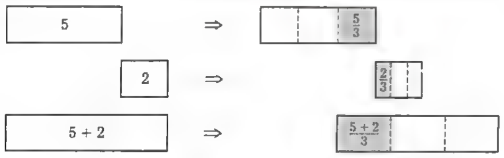
5. $\frac{5}{6}+\frac{1}{6}=$ $\frac{6}{6}$ 6. $\frac{2}{6}+\frac{2}{6}=$ $\frac{4}{6}$ 7. $\frac{2}{2}+\frac{1}{2}=$ $\frac{3}{2}$

8. $\frac{11}{7}+\frac{11}{7}=$ $\frac{22}{7}$ 9. $\frac{1}{3}+\frac{1}{3}=$ $\frac{2}{3}$ 10. $\frac{2}{4}+\frac{9}{4}=$ $\frac{11}{4}$

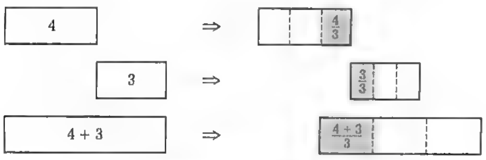
reference page

For Class Discussion *See pupil page suggestions*

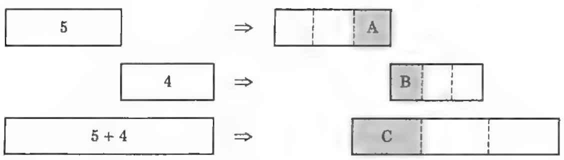
Each region has been divided into equal parts. How much is each part?



Does $\frac{5}{3}+\frac{2}{3}=\frac{5+2}{3}$? *Yes*



Does $\frac{4}{3}+\frac{3}{3}=\frac{4+3}{3}$? *Yes*

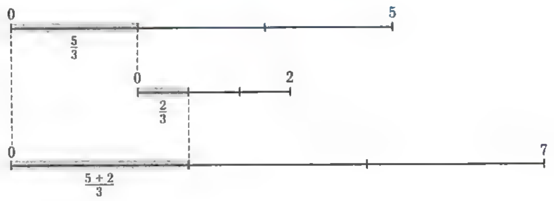


How much is—

1. part A? $\frac{5}{3}$ 2. part B? $\frac{4}{3}$ 3. part C? $\frac{5+4}{3}$

reference page

Does $\frac{5}{3}+\frac{2}{3}=\frac{5+2}{3}$? *See pupil page suggestions.*



Compute.

1. $\frac{4}{3}+\frac{6}{3}=$ $\frac{10}{3}$ 2. $\frac{7}{8}+\frac{6}{8}=$ $\frac{13}{8}$ 3. $\frac{4}{6}+\frac{7}{6}=$ $\frac{11}{6}$

4. $\frac{2}{5}+\frac{2}{5}=$ $\frac{4}{5}$ 5. $\frac{1}{3}+\frac{2}{3}=$ $\frac{3}{3}$ 6. $\frac{3}{7}+\frac{2}{7}=$ $\frac{5}{7}$

7. $\frac{5}{8}+\frac{6}{8}=$ $\frac{11}{8}$ 8. $\frac{1}{9}+\frac{5}{9}=$ $\frac{6}{9}$ 9. $\frac{3}{5}+\frac{7}{5}=$ $\frac{10}{5}$

10. $\frac{3}{12}+\frac{8}{12}=$ $\frac{11}{12}$ 11. $\frac{3}{6}+\frac{4}{6}=$ $\frac{7}{6}$ 12. $\frac{6}{10}+\frac{15}{10}=$ $\frac{21}{10}$

13. $\frac{7}{9}+\frac{8}{9}=$ $\frac{15}{9}$ 14. $\frac{9}{15}+\frac{8}{15}=$ $\frac{17}{15}$ 15. $\frac{4}{11}+\frac{5}{11}=$ $\frac{9}{11}$

reference page

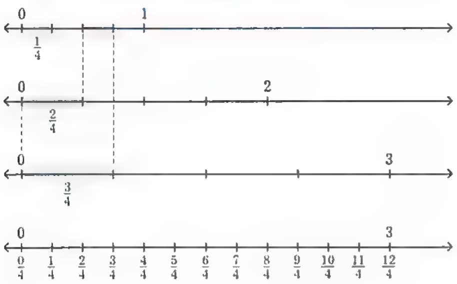
● Discuss the number line at the top of page 189. Compare each segment with the corresponding distance on the number line. Remember that each of the 4 equal parts of 1 is $\frac{1}{4}$.

Assign the exercises for practice in computation.

● Use the examples at the top of page 190 to introduce missing addends in sums of fractional numbers. The exercises may be used as a quick oral activity before they are assigned for independent work.

Name

Does $\frac{2}{4} + \frac{1}{4} = \frac{3}{4}$? *See pupil page suggestions.*



Compute.

1. $\frac{2}{8} + \frac{4}{8} =$ $\frac{6}{8}$
2. $\frac{3}{4} + \frac{2}{4} =$ $\frac{5}{4}$
3. $\frac{5}{10} + \frac{2}{10} =$ $\frac{7}{10}$
4. $\frac{6}{5} + \frac{2}{5} =$ $\frac{8}{5}$
5. $\frac{5}{6} + \frac{3}{6} =$ $\frac{8}{6}$
6. $\frac{3}{8} + \frac{4}{8} =$ $\frac{7}{8}$
7. $\frac{2}{9} + \frac{6}{9} =$ $\frac{8}{9}$
8. $\frac{4}{5} + \frac{6}{5} =$ $\frac{10}{5}$
9. $\frac{2}{3} + \frac{5}{3} =$ $\frac{7}{3}$
10. $\frac{7}{4} + \frac{9}{4} =$ $\frac{16}{4}$
11. $\frac{6}{8} + \frac{9}{8} =$ $\frac{15}{8}$
12. $\frac{2}{7} + \frac{6}{7} =$ $\frac{8}{7}$
13. $\frac{7}{3} + \frac{11}{3} =$ $\frac{18}{3}$
14. $\frac{12}{5} + \frac{2}{5} =$ $\frac{14}{5}$
15. $\frac{8}{4} + \frac{8}{4} =$ $\frac{16}{4}$
16. $\frac{7}{6} + \frac{4}{6} =$ $\frac{11}{6}$
17. $\frac{8}{3} + \frac{5}{3} =$ $\frac{13}{3}$
18. $\frac{12}{9} + \frac{15}{9} =$ $\frac{27}{9}$

reference page

G-189

Compute.

$\frac{7}{5} + \diamond = \frac{11}{5}$ $\qquad$ $\diamond + \frac{3}{2} = \frac{8}{2}$

$\frac{7}{5} + \diamond = \frac{7+4}{5}$ $\qquad$ $\diamond = \frac{5}{2}$

$\diamond = \frac{4}{5}$

1. $\frac{2}{12} + \diamond = \frac{5}{12}$ $\quad \diamond = \frac{3}{12}$
2. $\diamond + \frac{8}{6} = \frac{15}{6}$ $\quad \diamond = \frac{7}{6}$
3. $\frac{2}{5} + \diamond = \frac{6}{5}$ $\quad \diamond = \frac{4}{5}$
4. $\diamond + \frac{4}{2} = \frac{7}{2}$ $\quad \diamond = \frac{3}{2}$
5. $\frac{2}{3} + \diamond = \frac{4}{3}$ $\quad \diamond = \frac{2}{3}$
6. $\diamond + \frac{2}{6} = \frac{6}{6}$ $\quad \diamond = \frac{4}{6}$
7. $\frac{2}{8} + \diamond = \frac{3}{8}$ $\quad \diamond = \frac{1}{8}$
8. $\diamond + \frac{3}{12} = \frac{10}{12}$ $\quad \diamond = \frac{7}{12}$
9. $\frac{3}{5} + \diamond = \frac{12}{5}$ $\quad \diamond = \frac{9}{5}$
10. $\diamond + \frac{2}{3} = \frac{18}{3}$ $\quad \diamond = \frac{16}{3}$
11. $\diamond + \frac{0}{4} = \frac{3}{4}$ $\quad \diamond = \frac{3}{4}$
12. $\diamond + \frac{8}{7} = \frac{8}{7}$ $\quad \diamond = \frac{0}{7}$
13. $\frac{5}{9} + \diamond = \frac{10}{9}$ $\quad \diamond = \frac{5}{9}$
14. $\diamond + \frac{24}{5} = \frac{36}{5}$ $\quad \diamond = \frac{12}{5}$
15. $\frac{4}{11} + \diamond = \frac{9}{11}$ $\quad \diamond = \frac{5}{11}$
16. $\diamond + \frac{6}{8} = \frac{9}{8}$ $\quad \diamond = \frac{3}{8}$
17. $\frac{5}{7} + \diamond = \frac{11}{7}$ $\quad \diamond = \frac{6}{7}$
18. $\diamond + \frac{6}{3} = \frac{14}{3}$ $\quad \diamond = \frac{8}{3}$
19. $\frac{9}{6} + \diamond = \frac{18}{6}$ $\quad \diamond = \frac{9}{6}$
20. $\diamond + \frac{9}{6} = \frac{16}{6}$ $\quad \diamond = \frac{7}{6}$
21. $\frac{8}{3} + \diamond = \frac{8}{3}$ $\quad \diamond = \frac{0}{3}$
22. $\diamond + \frac{8}{13} = \frac{21}{13}$ $\quad \diamond = \frac{13}{13}$
23. $\frac{8}{7} + \diamond = \frac{15}{7}$ $\quad \diamond = \frac{7}{7}$
24. $\diamond + \frac{17}{10} = \frac{23}{10}$ $\quad \diamond = \frac{6}{10}$

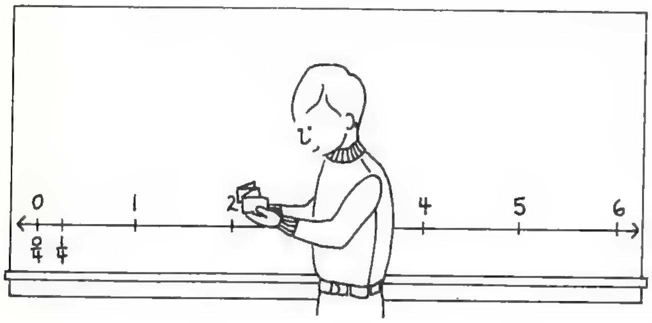
reference page

G-190

Supplemental Experience

■ Draw a number line about 50 inches long on the chalkboard. Mark and label a point for 0. Display strips of paper of the following sizes: 2″ × 8″, 2″ × 16″, 2″ × 24″, 2″ × 32″, 2″ × 40″, and 2″ × 48″. Pick up the 2 by 8 inch strip, and tell the class you are using it as a unit, so its length is 1. Have a pupil mark and label a point for 1 on the number line, using the given length for 1. Then ask pupils to find strips showing the lengths of 2, 3, 4, 5, and 6. Have each length marked and labeled on the number line. In each case, one end of the strip must be at 0 in order to use its length to mark a distance from 0.

Beneath the mark for 0, write $\frac{0}{4}$. Ask a pupil to use one of the paper strips to locate $\frac{1}{4}$ on the number line. He can do this by folding the 1-strip into 4 equal parts and marking the length of one of the parts on the number line.



Continue using the strips to locate $\frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}$, and $\frac{6}{4}$ on the number line.

Then let two pupils hold the strips for $\frac{2}{4}$ and $\frac{3}{4}$ end to end along the number line to find the computed sum $\frac{2}{4} + \frac{3}{4}$. Use other pairs of strips to check other sums on the number line in this manner.

KEY IDEA

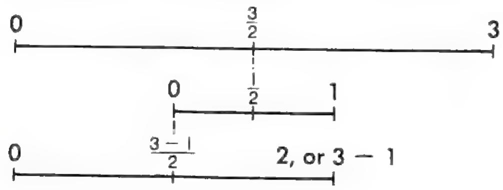
$\frac{7}{5} - \frac{3}{5} = \frac{7-3}{5} = \frac{4}{5}$.

Scope

To compute differences of some fractional numbers.

Fundamentals

The difference $\frac{3}{2} - \frac{1}{2}$ can be shown with line segments:



Since subtraction is the inverse of addition, differences of fractional numbers can be computed by using addition. For example, the difference $\frac{3}{2} - \frac{1}{2}$ is the missing addend in this equation.

$$\frac{1}{2} + \square = \frac{3}{2}$$

It can be seen that $\frac{2}{2}$, or $\frac{3-1}{2}$, is the missing addend.

$$\frac{1}{2} + \frac{3-1}{2} = \frac{3}{2}$$

The difference $\frac{3}{2} - \frac{1}{2}$ is $\frac{2}{2}$.

$$\frac{3}{2} - \frac{1}{2} = \frac{3-1}{2} = \frac{2}{2}$$

Regions also serve as models for subtraction of fractional numbers. No matter what numbers a, b, and c represent (providing c is not 0), $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

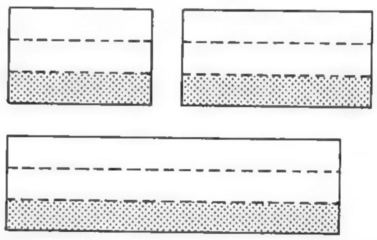
Readiness for Understanding

Knowledge of fractional numbers.

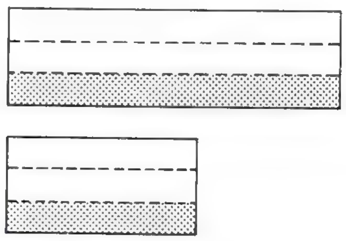
Developmental Experiences

for flannel board
3 tagboard strips (6″ × 9″,
6″ × 12″, 6″ × 21″)
felt-tip pen

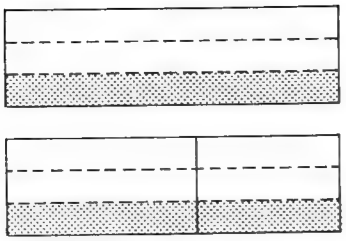
► Prepare tagboard rectangles 6 by 9 inches, 6 by 12 inches, and 6 by 21 inches. Use a felt-tip pen to partition the rectangles as shown. Shade or color the indicated part of each rectangle.



On the flannel board display the two larger rectangles as shown below. Tell the pupils that the larger one is 7 and the other is 4.



Ask someone to indicate where to place a rectangle that would show the difference between 7 and 4. The pupil may point to the space to the right of the 4-region. Put the other rectangle in place, and write 7 − 4 on the chalkboard. Review the idea that the difference 7 − 4 is the number that is added to 4 to give 7.



$4 + (7 - 4) = 7$

Ask pupils to give the number for the shaded part of the 7-region ($\frac{7}{3}$) and the number for the shaded part of the 4-region ($\frac{4}{3}$). Then ask a pupil to point to the region that shows the difference between $\frac{7}{3}$ and $\frac{4}{3}$. He should point to the shaded part of the $(7 - 4)$-region. Help the class to see that the number for this part is $\frac{7-4}{3}$.

On the chalkboard write the following:

$$\frac{7}{3} - \frac{4}{3} = \frac{7-4}{3}$$

Discuss whether $\frac{7-4}{3}$ and $\frac{7}{3} - \frac{4}{3}$ are the same number. The two expressions result from looking at the same diagram in two ways. We can first divide both 7 and 4 by 3 and then subtract the two quotients ($\frac{7}{3} - \frac{4}{3}$), or we can first subtract 4 from 7 and then divide the difference by 3 ($\frac{7-4}{3}$).

Now display the largest rectangle and the smallest rectangle on the flannel board. Assign the number 14 to the largest rectangle and 6 to the smallest. Ask a pupil to show the difference in the diagram and to write the difference on the chalkboard (the middle-sized rectangle is $14 - 6$). Then ask another pupil to tell the numbers for the shaded parts of the two rectangles on the flannel board ($\frac{14}{3}$ and $\frac{6}{3}$). Have him show the difference in the diagram and write the difference on the chalkboard. If he writes $\frac{14}{3} - \frac{6}{3}$, ask someone else to write the difference in another form ($\frac{14-6}{3}$).

Finally ask a pupil to write the computed difference for $14 - 6$. Then ask someone to write the computed difference for $\frac{14}{3} - \frac{6}{3}$.

$$14 - 6 = 8$$
$$\frac{14}{3} - \frac{6}{3} = \frac{14-6}{3} = \frac{8}{3}$$

Pages 191 through 196

● Use page 191 for discussion. Let some pupils explain the illustrations. For example, the third illustration shows the difference $7 - 5$, because, if this region is added to the 5-region, a 7-region will result.

$$5 + (7 - 5) = 7$$

The shaded part of the fourth illustration represents the number $\frac{7-5}{3}$, because it is one of the 3 equal parts of the $(7 - 5)$-region. The seventh illustration shows the difference of the shaded parts of the fifth and sixth illustrations ($\frac{7}{3} - \frac{5}{3}$). Ask the class to compare this difference with the shaded part of the fourth illustration. Are they the same? Write on the chalkboard the following:

$$\frac{7}{3} - \frac{5}{3} = \frac{7-5}{3}$$

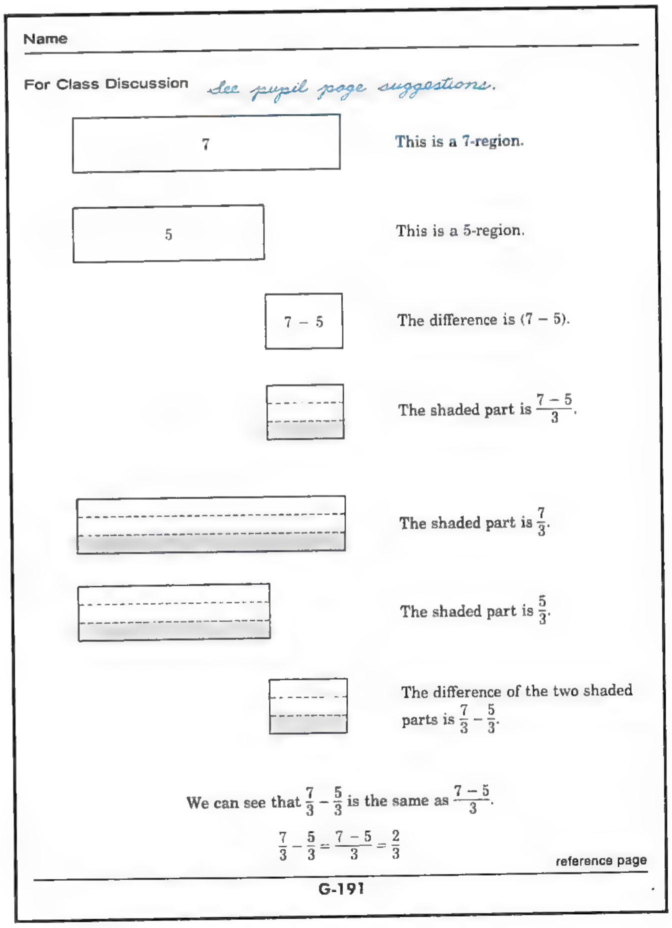
● Discuss computation of the differences shown at the top of page 192. Let the pupils complete the exercises. Then discuss the reasoning they used in their computations.

● The illustration at the top of page 193 is another model for subtraction of fractional numbers. Be sure the pupils notice the 5, 2, and $5 - 2$. They should see that $\frac{5-2}{4}$ is the difference between $\frac{5}{4}$ and $\frac{2}{4}$.

$$\frac{5}{4} - \frac{2}{4} = \frac{5-2}{4}$$

Assign the exercises and check the computation. Let the pupils discuss their reasoning if disagreements arise.

● Pages 194 and 195 can be used for further practice in computation. On each page let the pupils explain the illustrations at the top of the page.


Name

For Class Discussion *See pupil page suggestions.*

This is a 7-region.

This is a 5-region.

The difference is $(7 - 5)$.

The shaded part is $\frac{7-5}{3}$.

The shaded part is $\frac{7}{3}$.

The shaded part is $\frac{5}{3}$.

The difference of the two shaded parts is $\frac{7}{3} - \frac{5}{3}$.

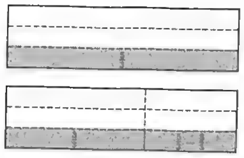
We can see that $\frac{7}{3} - \frac{5}{3}$ is the same as $\frac{7-5}{3}$.

$$\frac{7}{3} - \frac{5}{3} = \frac{7-5}{3} = \frac{2}{3}$$

reference page

G-191

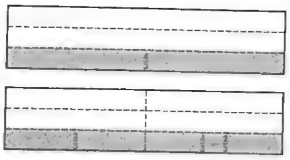
Compute $\frac{5}{3}-\frac{3}{3}$.



$\frac{5}{3}-\frac{3}{3}=\frac{5-3}{3}$

$=\frac{2}{3}$

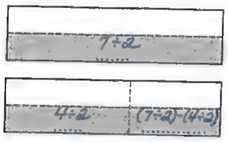
Compute $\frac{4}{3}-\frac{2}{3}$.



$\frac{4}{3}-\frac{2}{3}=$ $\frac{2}{3}$

Compute.

1.

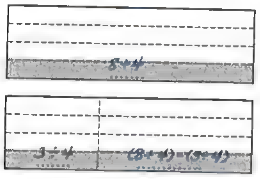


$\frac{7}{2}-\frac{4}{2}=$ $\frac{3}{2}$

2.

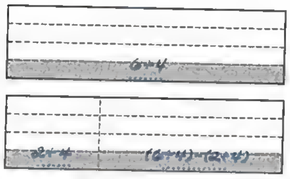
$\frac{3}{2}-\frac{1}{2}=$ $\frac{2}{2}$

3.



$\frac{8}{4}-\frac{3}{4}=$ $\frac{5}{4}$

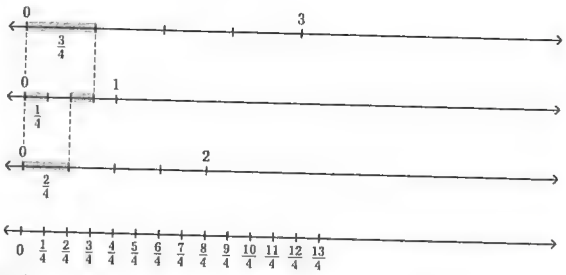
4.



$\frac{6}{4}-\frac{2}{4}=$ $\frac{4}{4}$

reference page

Does $\frac{3}{4}-\frac{1}{4}=\frac{2}{4}$?



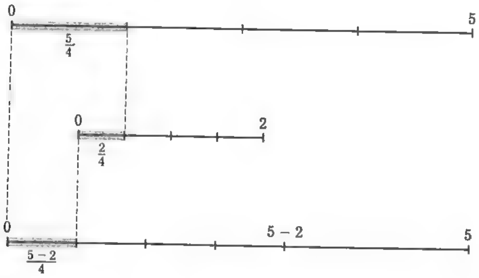
Compute.

1. $\frac{6}{8}-\frac{2}{8}=$ $\frac{4}{8}$
2. $\frac{8}{7}-\frac{5}{7}=$ $\frac{3}{7}$
3. $\frac{7}{5}-\frac{4}{5}=$ $\frac{3}{5}$
4. $\frac{5}{4}-\frac{5}{4}=$ $\frac{0}{4}$
5. $\frac{8}{6}-\frac{4}{6}=$ $\frac{4}{6}$
6. $\frac{7}{8}-\frac{4}{8}=$ $\frac{3}{8}$
7. $\frac{5}{5}-\frac{3}{5}=$ $\frac{2}{5}$
8. $\frac{8}{10}-\frac{4}{10}=$ $\frac{4}{10}$
9. $\frac{8}{9}-\frac{6}{9}=$ $\frac{2}{9}$
10. $\frac{9}{4}-\frac{1}{4}=$ $\frac{8}{4}$
11. $\frac{10}{4}-\frac{1}{4}=$ $\frac{9}{4}$
12. $\frac{10}{4}-\frac{9}{4}=$ $\frac{1}{4}$
13. $\frac{7}{3}-\frac{2}{3}=$ $\frac{5}{3}$
14. $\frac{7}{3}-\frac{5}{3}=$ $\frac{2}{3}$
15. $\frac{5}{3}-\frac{2}{3}=$ $\frac{3}{3}$
16. $\frac{5}{6}-\frac{3}{6}=$ $\frac{2}{6}$
17. $\frac{8}{6}-\frac{3}{6}=$ $\frac{5}{6}$
18. $\frac{5}{6}-\frac{2}{6}=$ $\frac{3}{6}$
19. $\frac{9}{8}-\frac{4}{8}=$ $\frac{5}{8}$
20. $\frac{9}{8}-\frac{5}{8}=$ $\frac{4}{8}$
21. $\frac{14}{8}-\frac{5}{8}=$ $\frac{9}{8}$
22. $\frac{13}{8}-\frac{4}{8}=$ $\frac{9}{8}$
23. $\frac{14}{5}-\frac{5}{5}=$ $\frac{9}{5}$
24. $\frac{18}{14}-\frac{9}{14}=$ $\frac{9}{14}$

reference page

Name

Is $\frac{5}{4}-\frac{2}{4}$ the same as $\frac{5-2}{4}$ or $\frac{3}{4}$?



Compute.

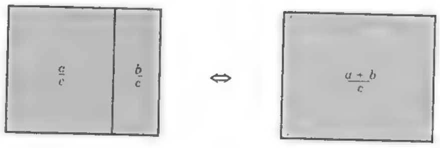
1. $\frac{7}{2}-\frac{4}{2}=$ $\frac{3}{2}$
2. $\frac{3}{2}-\frac{1}{2}=$ $\frac{2}{2}$
3. $\frac{6}{4}-\frac{2}{4}=$ $\frac{4}{4}$
4. $\frac{8}{4}-\frac{3}{4}=$ $\frac{5}{4}$
5. $\frac{12}{7}-\frac{8}{7}=$ $\frac{4}{7}$
6. $\frac{13}{6}-\frac{5}{6}=$ $\frac{8}{6}$
7. $\frac{11}{3}-\frac{10}{3}=$ $\frac{1}{3}$
8. $\frac{6}{7}-\frac{3}{7}=$ $\frac{3}{7}$
9. $\frac{4}{3}-\frac{3}{3}=$ $\frac{1}{3}$
10. $\frac{19}{6}-\frac{11}{6}=$ $\frac{8}{6}$
11. $\frac{4}{5}-\frac{3}{5}=$ $\frac{1}{5}$
12. $\frac{12}{9}-\frac{3}{9}=$ $\frac{9}{9}$
13. $\frac{16}{10}-\frac{9}{10}=$ $\frac{7}{10}$
14. $\frac{14}{11}-\frac{8}{11}=$ $\frac{6}{11}$
15. $\frac{14}{5}-\frac{2}{5}=$ $\frac{12}{5}$
16. $\frac{7}{8}-\frac{3}{8}=$ $\frac{4}{8}$

reference page

Name

$\frac{3}{4}+\frac{5}{4}=\frac{8}{4}$ $\qquad$ $\frac{8}{4}-\frac{3}{4}=\frac{5}{4}$

$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$ $\qquad$ $\frac{a+b}{c}-\frac{a}{c}=\frac{b}{c}$



Compute each sum or difference.

1. $\frac{5}{6}+\frac{2}{6}=$ $\frac{7}{6}$
2. $\frac{10}{2}+\frac{1}{2}=$ $\frac{11}{2}$
3. $\frac{9}{4}-\frac{7}{4}=$ $\frac{2}{4}$
4. $\frac{4}{6}+\frac{3}{6}=$ $\frac{7}{6}$
5. $\frac{4}{8}-\frac{3}{8}=$ $\frac{1}{8}$
6. $\frac{6}{16}-\frac{6}{16}=$ $\frac{0}{16}$
7. $\frac{3}{8}+\frac{19}{8}=$ $\frac{22}{8}$
8. $\frac{7}{2}+\frac{9}{2}=$ $\frac{16}{2}$
9. $\frac{9}{12}+\frac{6}{12}=$ $\frac{15}{12}$
10. $\frac{5}{7}-\frac{4}{7}=$ $\frac{1}{7}$
11. $\frac{3}{4}+\frac{10}{4}=$ $\frac{13}{4}$
12. $\frac{8}{11}+\frac{14}{11}=$ $\frac{22}{11}$
13. $\frac{19}{8}-\frac{12}{8}=$ $\frac{7}{8}$
14. $\frac{8}{10}+\frac{3}{10}=$ $\frac{11}{10}$
15. $\frac{7}{6}+\frac{17}{6}=$ $\frac{24}{6}$
16. $\frac{3}{2}+\frac{8}{2}=$ $\frac{11}{2}$
17. $\frac{1}{2}-\frac{0}{2}=$ $\frac{1}{2}$
18. $\frac{23}{3}+\frac{4}{3}=$ $\frac{27}{3}$
19. $\frac{9}{4}+\frac{7}{4}=$ $\frac{16}{4}$
20. $\frac{9}{3}-\frac{2}{3}=$ $\frac{7}{3}$
21. $\frac{6}{13}-\frac{2}{13}=$ $\frac{4}{13}$
22. $\frac{8}{12}+\frac{15}{12}=$ $\frac{23}{12}$
23. $\frac{27}{6}-\frac{15}{6}=$ $\frac{12}{6}$
24. $\frac{13}{9}+\frac{27}{9}=$ $\frac{40}{9}$
25. $\frac{12}{10}-\frac{8}{10}=$ $\frac{4}{10}$
26. $\frac{17}{3}-\frac{8}{3}=$ $\frac{9}{3}$
27. $\frac{15}{6}+\frac{8}{6}=$ $\frac{23}{6}$
28. $\frac{13}{4}-\frac{7}{4}=$ $\frac{6}{4}$

reference page

● Page 196 provides story exercises for pupils to solve. Read through the exercises with the class. Then assign them for independent work to be followed by class discussion. If pupils cannot agree on a solution, post the problem on the bulletin board to be discussed at a later time.

1. Each small panda bear weighs $\frac{3}{2}$ pounds. How much does the large panda bear weigh? $\frac{6}{2}$ or 3 pounds

2. Bill weighs twice as much as Jim. Jim weighs twice as much as Tom. Bill weighs 198 pounds.
   How much does Jim weigh? $\frac{198}{2}$ or 99 pounds
   How much does Tom weigh? $\frac{99}{2}$ or $49\frac{1}{2}$ pounds

3. Ted finished a race in $\frac{7}{12}$ of a minute. John took $\frac{2}{12}$ of a minute less to finish. How long did John take? $\frac{5}{12}$ of a minute

4. Mary baked 827 cupcakes for a school fair. If she packs 24 cupcakes in a box, how many full boxes will she have? 34 full boxes
   How many cupcakes will remain? 11 cupcakes

5. Bob's father built a swimming pool. The length of one side is 298 inches.
   What is the length of the side in feet? 24 feet 10 inches
   What is the length of the side in yards? 8 yards 10 inches

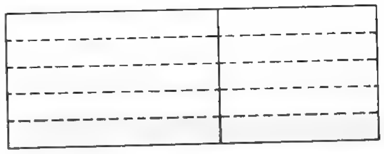
6. The new section of an expressway is 26,400 feet long.
   How many yards long is this section of the expressway? 8800 yards long
   How many miles long is this section of the expressway? 5 miles
   (Hint: 1 mile = 5280 feet)

If a hen and a half lays an egg and a half in a day and a half, how many eggs will one hen lay in a week? $\frac{14}{3}$ or $4\frac{2}{3}$ eggs

G-196

Supplemental Experience

■ On the chalkboard draw an illustration like the following:



Explain to the pupils that they can think of the rectangle on the left as representing a number larger than the rectangle on the right.

Divide the class into two teams. On the chalkboard write the sum or difference of two fractional numbers. For example:

$$\frac{8}{5} + \frac{6}{5} \text{ or } \frac{14}{5} - \frac{6}{5}$$

Ask the first pupil on one team to explain the sum and computed sum, or the difference and computed difference, by pointing out the correct parts of the picture. For example, he may point out parts of the diagram for 8, 6, 14, $\frac{8}{5}$, $\frac{6}{5}$, and $\frac{14}{5}$ as he says that 8-*divided-by*-5 plus 6-*divided-by*-5 is 14 *divided by* 5. If the pupil correctly identifies each part, he earns six points for his team. (There are six possible points for each exercise, one for each part correctly identified.) If he does not, give a member of the other team a chance to explain the same sum or difference. If the first pupil succeeds, write a different sum or difference for a pupil on the opposite team. Use sums and differences like the following: $\frac{4}{5}+\frac{7}{5}$, $\frac{8}{5}-\frac{3}{5}$, $\frac{12}{5}-\frac{10}{5}$, $\frac{7}{5}+\frac{8}{5}$, and $\frac{4}{5}+\frac{9}{5}$. The diagram can be changed so that there are, say, 6 parts instead of 5. Then sums and differences with 6 as the denominator must be used.

UNIT 15
GEOMETRY: LINES
Pages 197 Through 208

OBJECTIVE
To develop a perception of line.

The pupil learns that two points determine a line and that three or more points may be on one line. He explores the relationship of points of intersection to number of lines. He concludes that the greatest number of points of intersection for a given number of lines occurs when every line intersects every other line and no more than two lines intersect in the same point.

See Key Topics in Mathematics for the Intermediate Teacher: Geometry.

KEY IDEAS
Two points determine one line.
Two lines may, or may not, intersect.

CONCEPT
point of intersection

KEY IDEA
Two points determine one line.

Scope
To explore some properties of lines.

Fundamentals
Any two points determine one line. However, three or more points may, or may not, belong to one line.

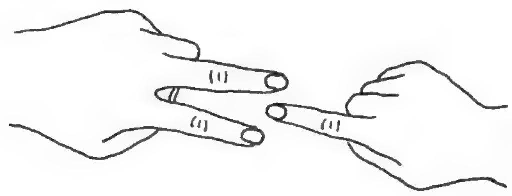
A line is without end, since there is always a point that belongs to the line but is beyond any previously chosen point.

In the diagram below, points A and C belong to the line. Point B lies beyond C, and point D lies beyond A. Both B and D belong to the line too.

Readiness for Understanding
Ability to perceive distances between points.

Developmental Experiences
► Show the pupils how you can form three points in a line using three of your fingertips.

Begin by holding up two fingers of one hand and sighting along the fingertips to see that they are in a line. Then ask, "Can I line up another fingertip with these two?" Show how another fingertip can be placed in line with the other two.



First place the third fingertip in line between the two fingers. Then place it in line on either side of the two fingers.

Ask each child to do this with you. Emphasize that the line of sight is a test of whether three objects are in a line.

When a child puts his third fingertip in line beyond the other two fingertips, it can be on either side of the two fingertips. Ask whether he can move it farther out and still have it in line. Ask him to try several different positions in which the third fingertip is in line with the other two.

► Ask two pupils to stand up. Then ask another pupil to stand between them. Have a fourth pupil, who will serve as "line-checker," see whether the third pupil is in line with the other two.

Joe Amy Mark

Now ask, "Is it possible to put someone else in line with these three children?" Suggest that another pupil get in line beyond the two original pupils. Ask, "Can we put someone else in line at the other end?" Suggest that another pupil get in line at the opposite end beyond the original pupils.

Jeff Joe Amy Mark Jon

Ask, "Can we put someone else in line?" Have several more pupils do so. Then ask the pupil who is the line-checker to see whether each pupil is in line.

Now ask, "Can we put another child in line beyond these? Can we keep on putting children in this line? Can all the children in the class be used in the line? If there were more children available, could they take places in the line too? How long could we continue adding children? After using all the children available, would we still have places left in the line?"

Ask the class, "How long is the line?" They should conclude that there is always another place in either direction.

Pages 197 through 203

● Use page 197 for class discussion after the development activities have been completed. In each exercise, let the pupils discuss which three points are in line. Some pupils may hold the book up and use the line of sight test. Pupils should write their conclusions so that they can check their ideas later.

Have each pupil fold a sheet of paper to make a model of a line. Then discuss the exercises again. This time the pupils should use the folded paper. See whether the same conclusions are reached using the folded-paper model of a line.

● Page 198 provides further experience in applying the folded-paper test to determine which points are all on one line. Have the pupils complete exercise 1 independently. Then discuss the points they chose. Some may give several answers to one question. Some may use the same points to answer several questions. For example, points N, L, H, G, D, and B are all on one line. Of course, any three, four, or five of these points are all on one line too.

As the class discusses exercise 2, they should conclude that the point in one corner of the ceiling and the point diagonally across the room in the corner of the floor are the points farthest apart in the room. Given two points that are farthest apart, there are many points in between.

● Use page 199 for discussion, and bring out the idea that any two points are always on one line. Two points determine a line. Possible answers for exercise 5 are: *P* and *Q*, *P* and *R*, *P* and *S*, *P* and *T*, *P* and *V*, *Q* and *R*, *Q* and *S*, *Q* and *T*, *Q* and *V*, *R* and *S*, *R* and *T*, *R* and *V*, *S* and *T*, *S* and *V*, *T* and *V*.

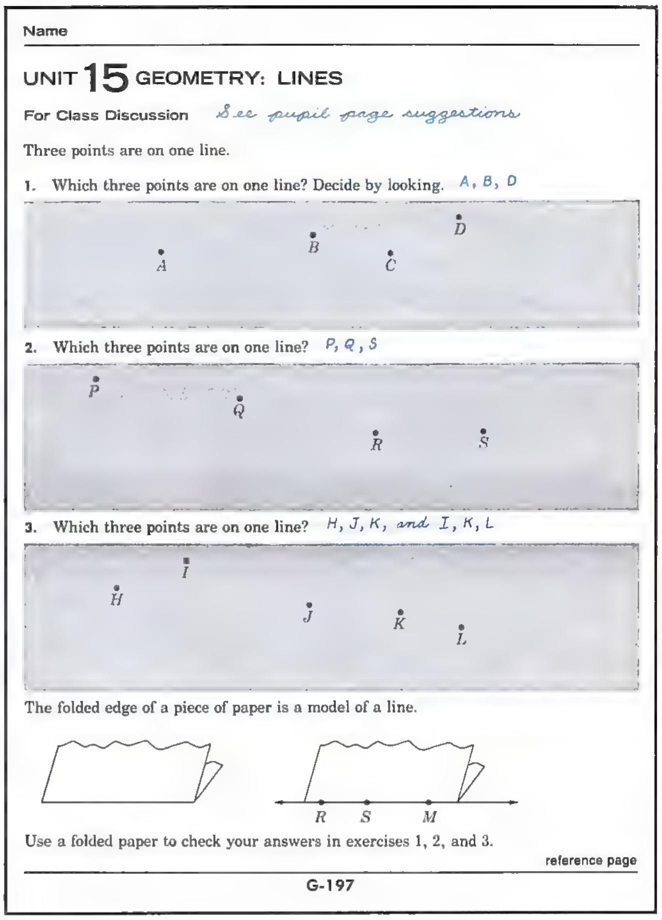
● Page 200 provides exercises involving the idea that two points determine a line. Some pupils will be able to visualize the lines through each pair of points. Others may need to draw lines through the points.

Use exercise 1 for class discussion. Then let the pupils proceed independently. In exercises 4 and 5 the children encounter the idea that more than two points determine a line when those points lie on the same line. Discuss these exercises with the class.

● Pages 201 and 202 provide practice in locating a given number of points to determine a specific number of lines. Some of the exercises cannot be done. Others can be done in only one way. Still others can be done in several ways.

Pupils may work together in groups of 2 or 3 to provide an interchange of ideas. When the class has finished, discuss the constructions. Let some pupils display their solutions for the more puzzling exercises.

As the activity with the ruler is discussed, pupils should become aware that 2 points determine 1 line. If more than 1 line passes through any 2 points, the ruler is not straight.



Name

UNIT 15 GEOMETRY: LINES

For Class Discussion *See pupil page suggestions*

Three points are on one line.

1. Which three points are on one line? Decide by looking. *A, B, D*

2. Which three points are on one line? *P, Q, S*

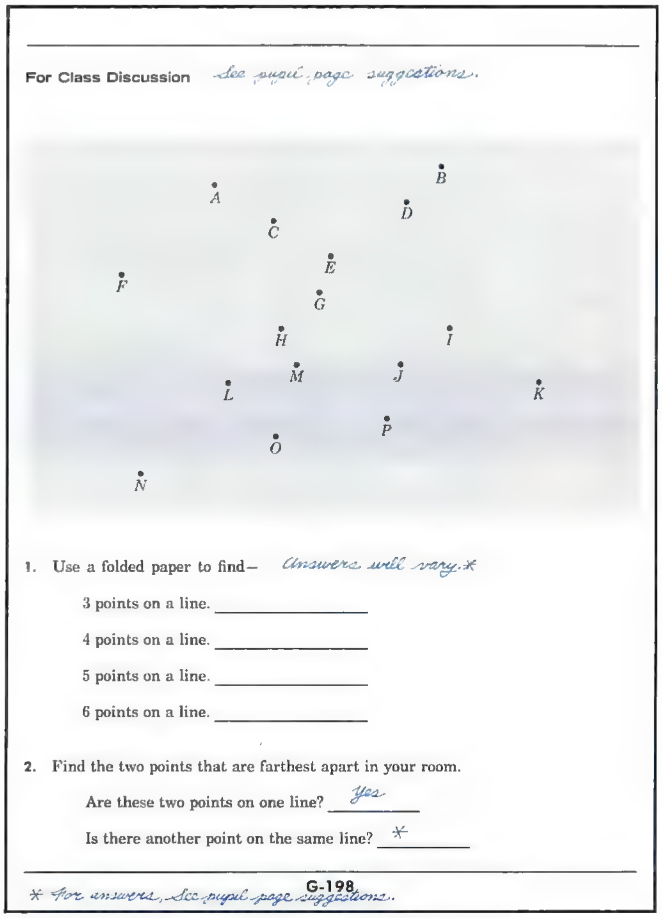
3. Which three points are on one line? *H, J, K, and I, K, L*

The folded edge of a piece of paper is a model of a line.

Use a folded paper to check your answers in exercises 1, 2, and 3.

reference page

G-197



For Class Discussion *See pupil page suggestions.*

1. Use a folded paper to find— *Answers will vary.**
   3 points on a line. ______
   4 points on a line. ______
   5 points on a line. ______
   6 points on a line. ______

2. Find the two points that are farthest apart in your room.
   Are these two points on one line? *Yes*
   Is there another point on the same line? ***

G-198

** For answers, see pupil page suggestions.*

Name

1. Which two points are on one line? (Decide by looking.) A and B, B and C, A and C

2. Which two points are on one line? D and E, E and F, D and F

3. Which two points are on one line? G and H, H and J, G and J

4. Which two points are on one line? K and L, K and M, K and N, L and M, L and N, M and N

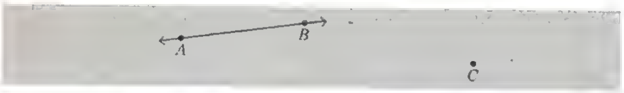
5. Which two points are on one line? For answers, see pupil page suggestions.

Use a folded paper to check your answers in exercises 1, 2, 3, 4, and 5.
What do you know about two points? See pupil page suggestions.

reference page

Two points determine one line.

1. How many lines are determined by points A, B, and C? 3 lines
(Hint: Line AB is one of them.)

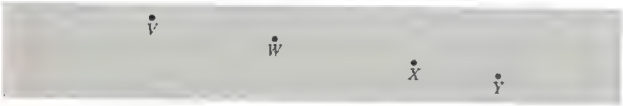


Do points A and B determine more than one line? no

2. How many lines are determined by D, E, and F? 3 lines

3. How many lines are determined by J, K, L, and M? 6 lines

4. How many lines are determined by V, W, X, and Y? 1 line



5. How many lines are determined by P, Q, R, S, and T? 5 lines

reference page

Name

Can it be done? Try to draw these. Position of dots may vary.

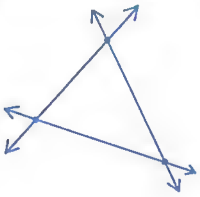
1. 2 points which determine 1 line.

2. 2 points which determine 2 lines.
Cannot be done

3. 3 points which determine exactly 1 line.

4. 3 points which determine exactly 2 lines.
Cannot be done

5. 3 points which determine exactly 3 lines.



6. 3 points which determine exactly 4 lines.
Cannot be done

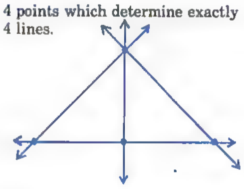
Can it be done? Try to draw these. Make drawings for the bulletin board to show your results.

1. 4 points which determine exactly 1 line.

2. 4 points which determine exactly 2 lines.
Cannot be done

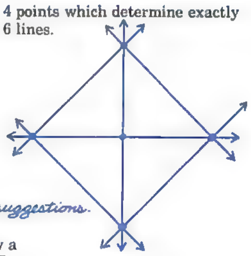
3. 4 points which determine exactly 3 lines.
Cannot be done

4. 4 points which determine exactly 4 lines.



5. 4 points which determine exactly 5 lines.
Cannot be done

6. 4 points which determine exactly 6 lines.



Is your ruler straight? See pupil page suggestions.

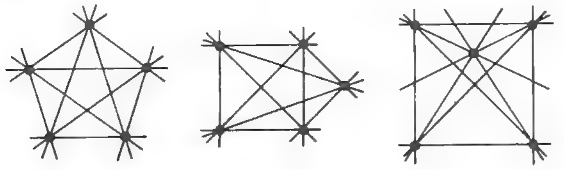
Draw two points. Use your ruler and draw a line through them. Turn your ruler over. Put the same edge along the points. Draw a line.

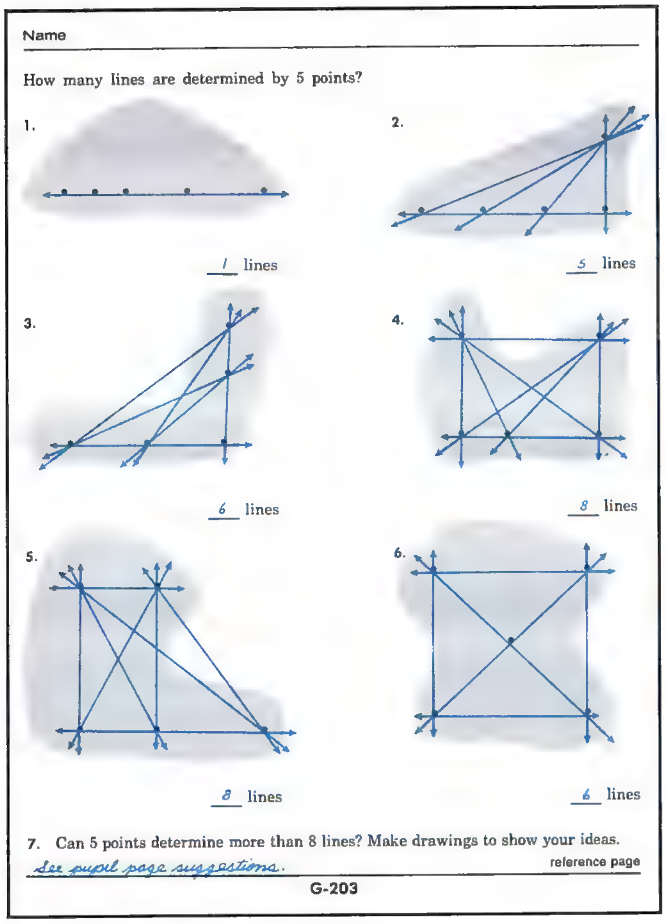
How many lines do you see?

Now do you know whether your ruler is straight?

● Use page 203 for practice in counting the number of lines determined by 5 points.

As exercise 7 is explored, the pupils should find that when no 3 of the 5 points are in a line, 10 lines are determined.





Name

How many lines are determined by 5 points?

1. 1 lines

2. 5 lines

3. 6 lines

4. 8 lines

5. 8 lines

6. 6 lines

7. Can 5 points determine more than 8 lines? Make drawings to show your ideas.

See pupil page suggestions.

reference page

G-203

KEY IDEA

Two lines may, or may not, intersect.

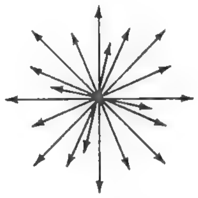
Scope

To explore the intersection of lines in space.

Fundamentals

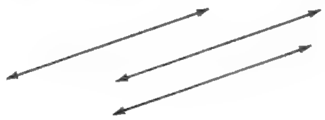
The pupil begins to develop a perception of line by working with lines, participating in activities, and using models of line segments. Exploring the intersection points of lines provides another way of developing this perception.

An unlimited number of lines may intersect at one point.

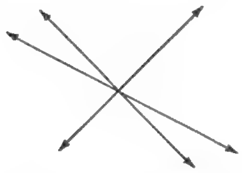


Three lines may intersect at 0, 1, 2, or 3 points.

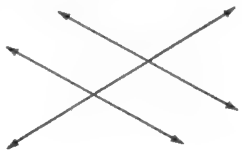
Three lines can intersect at 0 points.



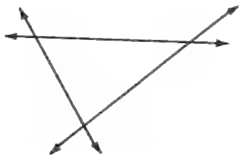
Three lines can intersect at 1 point.



Three lines can intersect at 2 points.



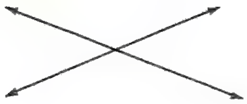
And three lines can intersect at 3 points.



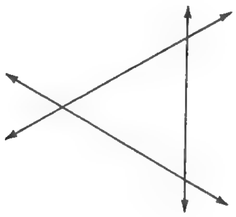
It is impossible for three lines to have more than three points of intersection.

The greatest number of points of intersection determined by a given set of lines occurs when every line in the set intersects every other line and no more than two lines intersect at any one point.

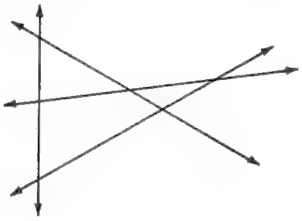
For a set of 2 lines, the greatest number of points of intersection is 1.



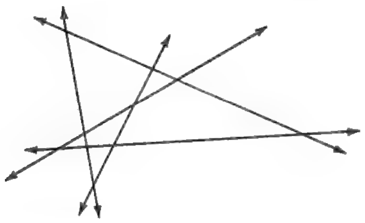
For a set of 3 lines, 1 + 2, or 3.



For a set of 4 lines, 1 + 2 + 3, or 6.



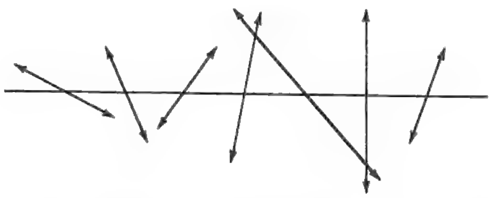
For a set of 5 lines, 1 + 2 + 3 + 4, or 10.



This information can be generalized. The greatest number of points of intersection possible for a set of n lines is

$$1 + 2 + 3 + 4 + \ldots + (n - 1).$$

The problem of determining the greatest possible number of intersection points of n lines can be solved in another way. Consider a set of eight lines in which each line intersects every other line at a unique point. If all of these lines are extended, each line will intersect every other line at a unique point.



Look at the points of intersection on one line, and note that this line intersects each of the other 7 lines, forming 7, or 8 − 1, points of intersection. In the set of 8 lines, each line has 8 − 1 points of intersection. The number of intersections for each line, 8 − 1, can be multiplied by the total number of lines, 8. However, in the product 8 × (8 − 1), or 8 (8 − 1), each point of intersection is counted twice, so the number of intersections is only $\frac{8(8-1)}{2}$. In general, the greatest number of intersection points determined by n lines is $\frac{n(n-1)}{2}$. However, this general formula is incidental; the step-by-step reasoning process that leads to understanding the generalization is much more important.

Readiness for Understanding

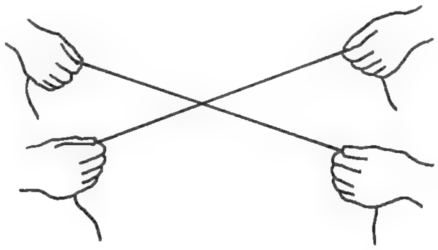
Knowledge of pattern.

Developmental Experiences

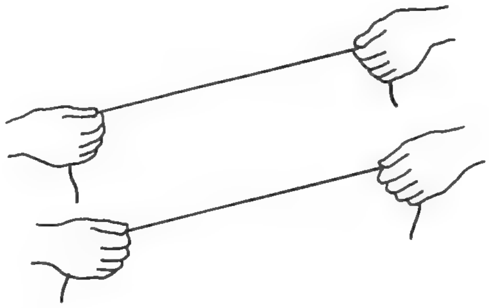
string

► The following activity shows that two lines may, or may not, intersect.

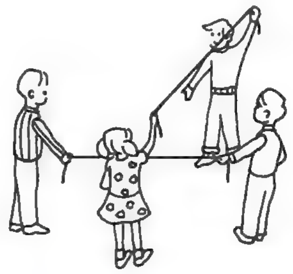
Explain that intersecting lines are lines that meet at a point. Then have four pupils hold two taut strings so that one touches the other. This is an example of two lines intersecting at a point.



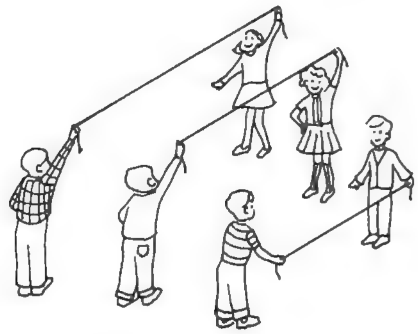
Tell the class that there are lines that never intersect. Have the four pupils hold the two strings next to each other so that they do not intersect. Ask the class to imagine that the lines represented by the strings go on without end.



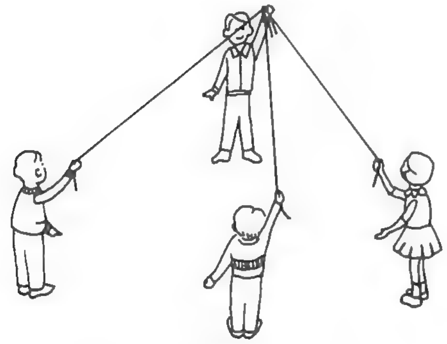
Here is another example of two lines that do not intersect.



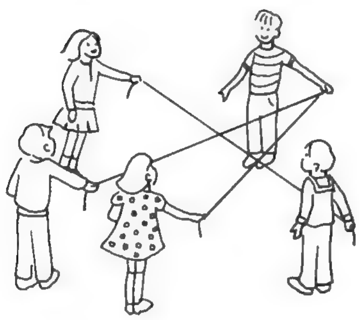
► Have several pupils use three lengths of string to show different arrangements of 3 lines. A few suggestions for this activity are pictured.



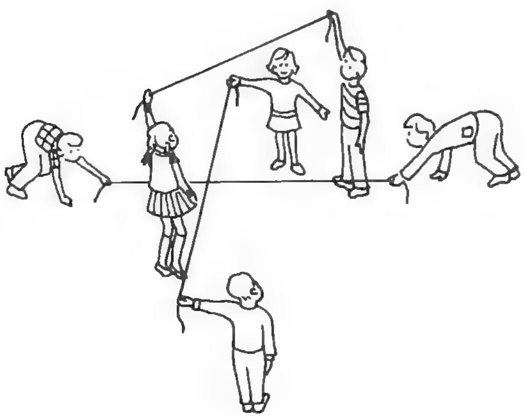
0 points of intersection



1 point of intersection



3 points of intersection



0 points of intersection

As each set of 3 lines is demonstrated, ask the pupils what they notice about the lines. If they observe that the lines intersect, ask how many points of intersection the lines have.

Pages 204 through 208

● Discuss page 204 to help the pupils understand the diagrams of intersecting and nonintersecting lines. Lines AB and BC meet each other because point B is on both lines. As the pupils lay a straightedge along line AD, they should see that lines AD and BC will not meet. They stay the same distance apart and are nonintersecting.

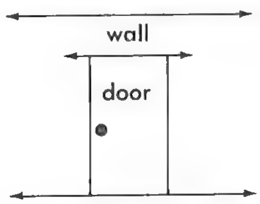
Have the pupils use two straightedges as they explore the other pairs of lines. If one straightedge is lined up with each pair of points, it is easier to decide whether the two lines determined by the two pairs of points intersect or not. During the discussion draw the following picture on the chalkboard.

The pupils should recognize that the two lines intersect even though the point of intersection is not shown. There are more points in either direction on each line. The lines do not stay the same distance apart.

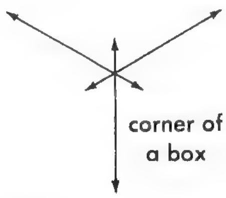
A pair of intersecting lines can be seen in any corner of the classroom. A pair of nonintersecting lines exist at the top and bottom of each wall, and also at the top of one wall and the bottom of an adjacent wall.

● Page 205 provides experience in recognizing points of intersection. As the pupils try to draw a given number of lines with a definite number of points of intersection, they will discover that some cannot be done. In the discussion that follows completion of the work, call attention to the greatest number of points of intersection possible for 3 lines, and for 4 lines. Ask the pupils to tell how many of the lines pass through each point of intersection when the number of intersection points is the greatest possible.

The class may enjoy working in small groups for the treasure hunt. Each group may submit a diagram for each solution.



3 lines
0 points of intersection

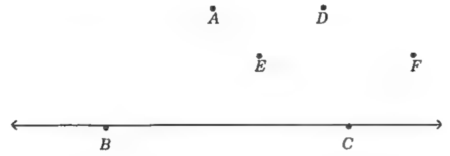


3 lines
1 point of intersection

Examples may be given in which the 3 or 4 lines do not all lie in the same plane.

For Class Discussion *See pupil page suggestions.*

Do lines AB and BC intersect? Do lines AD and BC intersect?



Two lines intersect or they do not intersect.

Lines AB and BC are intersecting lines.
At what point do they intersect?

Lines AD and BC are nonintersecting lines. They never intersect.
Lay your ruler along line AD to see this.

Do these lines intersect?

1. Lines AD and EF *No*
2. Lines DB and BC *Yes*
3. Lines DB and AE *Yes*
4. Lines EF and BC *No*
5. Lines AF and BC *Yes*
6. Lines DE and FC *Yes*

Describe a pair of intersecting lines in your classroom. *See pupil page suggestions.*
Describe a pair of nonintersecting lines in your classroom.
Can two lines intersect in 2 points? *Two lines cannot intersect in 2 points.*

reference page

G-204

Name

Lines intersect at points. *See pupil page suggestions.*

How many lines are shown? How many points of intersection?

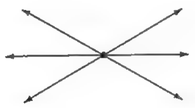
1.

2 lines
0 points of intersection

2.

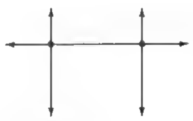
2 lines
1 points of intersection

3.



3 lines
1 points of intersection

4.



3 lines
2 points of intersection

Can it be done? Try to draw these on a sheet of paper.

5. 3 lines with 0, 3, and 4 points of intersection. *0 and 3 points – yes; 4 points – no*
6. 4 lines with 0, 1, 2, 3, 4, 5, 6, and 7 points of intersection. *0, 1, 3, 4, 5, and 6 points – yes; 2 and 7 points – no*
7. 1 line with 0 and 1 points of intersection. *0 points – yes; 1 point – no*

Classroom Treasure Hunt. *See pupil page suggestions.*
Find 3 lines with 0, 1, 2, and 3 points of intersection.
Find 4 lines with 0, 1, 3, 4, 5, and 6 points of intersection.

G-205

● Page 206 provides an exploration of the greatest possible number of intersection points for various numbers of lines. Tell the class that the second picture is the first picture with a new line added. The third picture is the second picture with a third line added. Let the pupils complete the investigation of the number of intersections and make a table like the one shown.

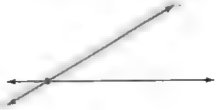
In discussing the results, let some pupils tell how they arrived at their conclusions. In essence, each new line crosses each of the previous lines. Some pupils may draw diagrams to show this idea.

Finally ask the pupils to compute the greatest number of intersection points possible for 8 lines, and for 10 lines.

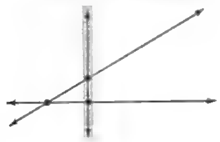
What is the greatest number of points of intersection?

1. 1 line, *0* points of intersection

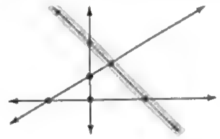
2. 2 lines, *1* points of intersection



3. 3 lines
1 + *2* points of intersection



4. 4 lines
1 + 2 + *3* points of intersection



Complete the table.

Number of lines	Greatest number of points of intersection
1 line	0
2 lines	1
3 lines	1 + *2*
4 lines	1 + 2 + *3*
5 lines	1 + 2 + 3 + *4*
6 lines	1 + 2 + *3* + *4* + *5*

G-206

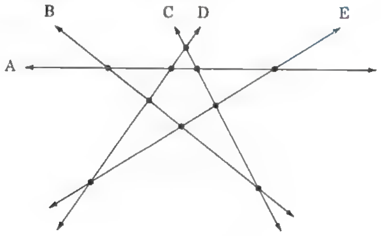
● Discussion of page 207 develops another way to look at the greatest possible number of intersection points of a given number of lines. The pupils will find that if there are 5 lines, each line contains 4 points of intersection. This might suggest that there are 5 × 4 points of intersection. However, a careful examination of this method of counting will reveal that each point is counted twice. For example, the intersection of lines A and B is counted once as an intersection point on A and again as an intersection point on B.

Have the pupils look at 3 and 4 intersecting lines in the same way. If there are 3 lines, there are 2 intersections on each line. If there are 4 lines, there are 3 intersections on each line.

Name

For Class Discussion *See pupil page suggestions*

How many points of intersection for 5 lines?



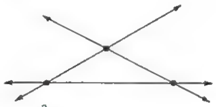
1. How many points of intersection on line A? *4*
   On line B? *4* On line C? *4* On line D? *4* On line E? *4*

2. How many points of intersection on each line? *4*
   There are 5 lines. This is a total of 5 × *4* points of intersection.

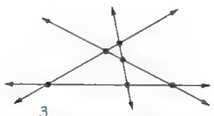
3. Count the points of intersection. *10* What is wrong?

The greatest number of points of intersection for 5 lines is
$(5 \times 4) \div 2$ or $\frac{5 \times 4}{2}$.

What is the greatest number of points of intersection—
for 3 lines? *3* for 4 lines? *6*



$\frac{3 \times 2}{2}$ points of intersection



$\frac{4 \times 3}{2}$ points of intersection

reference page

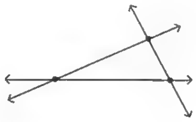
G-207

● Page 208 provides a chance for the pupils to use their ideas about the greatest number of intersection points. In the table the number of points is always given in at least one way. Patterns and computation are clues to other ways of expressing the same number.

Some pupils may want to compute the greatest number of points of intersection for 1001 lines: $1 + 2 + 3 + \ldots + 1000 = ?$

Some children may enjoy trying to arrange 10 pennies on 10 lines with 3 pennies on each line.

What is the greatest number of points of intersection? *See pupil page suggestions.*



3 lines

1 + 2 or $\frac{3 \times 2}{2}$ points of intersection

Complete the table.

Number of Lines	Number of Points of Intersection		
3	3	1 + 2	$\frac{3 \times 2}{2}$
4	6	1 + 2 + *3*	$\frac{4 \times 3}{2}$
5	*10*	1 + 2 + 3 + 4	$\frac{5 \times 4}{2}$
6	*15*	1 + 2 + 3 + 4 + 5	$\frac{6 \times 5}{2}$
7	21	*1 + 2 + 3 + 4 + 5 + 6*	$\frac{7 \times 6}{2}$
8	*28*	1 + 2 + 3 + 4 + 5 + 6 + 7	$\frac{8 \times 7}{2}$
9	*36*	*1 + 2 + 3 + 4 + 5 + 6 + 7 + 8*	$\frac{9 \times 8}{2}$
10	*45*	1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9	$\frac{10 \times 9}{2}$
11	55	*1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10*	$\frac{11 \times 10}{2}$

G-208

UNIT 16
EXACT DIVISION: FRACTIONS AND MIXED FRACTIONS
Pages 209 Through 232

OBJECTIVE

To continue to explore fractional numbers and to introduce the exact-division algorism and mixed fractions.

By using pictures of regions, the pupil learns that the fractional number $\frac{14}{3}$ is the mixed fraction $4 + \frac{2}{3}$. His knowledge of fractional numbers is used to extend the division algorism. He uses the exact-division algorism and learns to compute mixed fractions.

See Key Topics in Mathematics for the Intermediate Teacher: The Set of Fractional Numbers.

KEY IDEAS

$\frac{11}{6} = \frac{6}{6} + \frac{5}{6} = 1 + \frac{5}{6}$.
$\frac{11}{4} = \frac{4 \times 2}{4} + \frac{3}{4}$.
$\frac{7}{3} = 2\frac{1}{3}$.

KEY IDEA

$\frac{11}{6} = \frac{6}{6} + \frac{5}{6} = 1 + \frac{5}{6}$.

Scope

To explore fractional numbers and to introduce the mixed-fraction form.

Fundamentals

Some fractional numbers are whole numbers and some are not. For example, $16 \div 4$ and $12 \div 4$ are whole numbers; $15 \div 4$ is not a whole number. When we examine what we know about $15 \div 4$, we find that $15 \div 4$ is more than $12 \div 4$ but less than $16 \div 4$ ($\frac{12}{4} + \frac{3}{4} = \frac{15}{4}$ and $\frac{15}{4} + \frac{1}{4} = \frac{16}{4}$).

Since $\frac{12}{4} + \frac{3}{4}$ is $\frac{15}{4}$, we can write as follows:

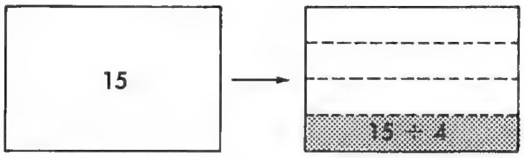
$$\frac{15}{4} = 3 + \frac{3}{4}$$

Similarly, we can write:

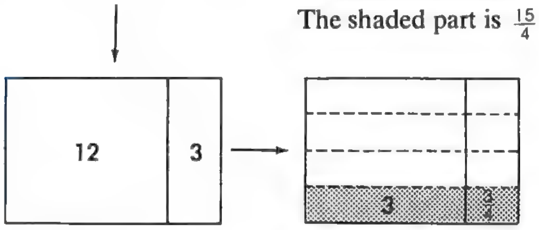
$$\frac{15}{4} = \frac{8}{4} + \frac{7}{4} = 2 + \frac{7}{4}$$
$$\frac{15}{4} = \frac{4}{4} + \frac{11}{4} = 1 + \frac{11}{4}$$

Each of the mixed fractions, $3 + \frac{3}{4}$, $2 + \frac{7}{4}$, and $1 + \frac{11}{4}$, is $15 \div 4$.

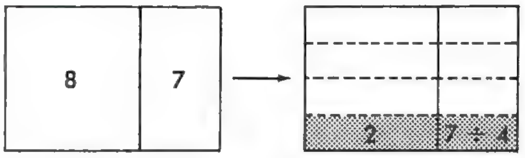
These mixed fractions for $15 \div 4$ can be illustrated by the model for the quotient $15 \div 4$.



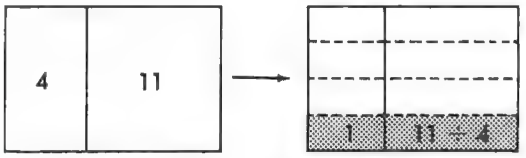
The shaded part is $\frac{15}{4}$



The shaded part is $3 + \frac{3}{4}$

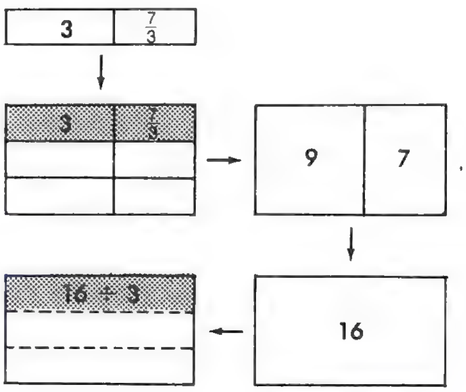


The shaded part is $2 + \frac{7}{4}$



The shaded part is $1 + \frac{11}{4}$

Similarly, a quotient for the mixed fraction $3 + \frac{7}{3}$ can be illustrated using this model.



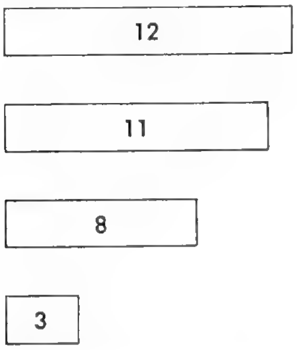
Readiness for Understanding

Understanding that $\square \div 4$ is $\frac{\square}{4}$.

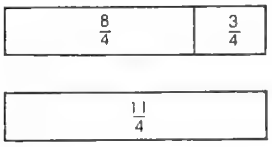
Developmental Experiences

for each child
construction paper

▶ Have each pupil make and label four strips of paper (1 inch by 12 inches, 1 inch by 11 inches, 1 inch by 8 inches, and 1 inch by 3 inches).



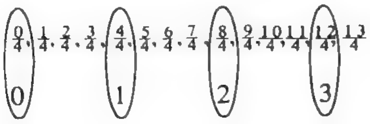
Ask the pupils how they might divide their strips into 4 equal parts (fold in half; fold in half again). When they have discussed this, have them divide the 12-strip into 4 equal parts. Then ask a pupil, "How much is each part?" (12 *divided by* 4) Have the pupils label each part $\frac{12}{4}$. Follow this procedure with the 11-strip and the 8-strip. Then ask a pupil to compare $\frac{11}{4}$ with $\frac{8}{4}$ ($\frac{11}{4}$ is greater than $\frac{8}{4}$). Then have a pupil compare $\frac{11}{4}$ with $\frac{12}{4}$ ($\frac{11}{4}$ is less than $\frac{12}{4}$). Write these comparisons on the chalkboard. Next, write on the chalkboard $\frac{8}{4} + \frac{3}{4} = \frac{11}{4}$ and ask the pupils to show this with their strips. One way to do this is by dividing the 3-strip into 4 parts and showing $\frac{3}{4}$, $\frac{8}{4}$, and $\frac{11}{4}$ as indicated below. Allow some time to think and time for discussion.



When the children have finished, show the following numbers on an overhead projector, a wall chart, or chalkboard.

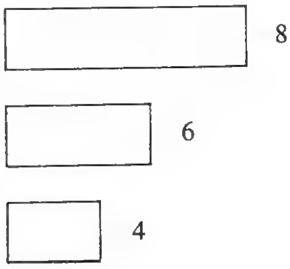
$\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \frac{8}{4}, \frac{9}{4}, \frac{10}{4}, \frac{11}{4}, \frac{12}{4}, \frac{13}{4}$

Ask the class to identify which fractions are whole numbers and show this as indicated below.

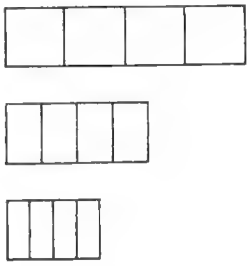


Then point out that $\frac{11}{4}$ is more than 2 and less than 3.

Ask a pupil to compare $6 \div 4$ with 1 and 2 ($6 \div 4$ is more than 1 and less than 2). Then ask him to help you show this. Sketch the following regions on the chalkboard. Ask another pupil how to use the regions to compare $\frac{6}{4}$ with 1 and 2.



He may suggest dividing each region into four equal parts and comparing the parts. Ask him to come to the chalkboard and show his method.



Ask another pupil about $3 \div 4$ (it is more than 0 but less than 1). Ask another pupil about $\frac{13}{4}$ ($\frac{13}{4}$ is more than 3 but less than 4). Then ask the pupils to try $\frac{127}{4}$. You may want the pupils to keep their ideas secret until everyone has located $\frac{127}{4}$ between two consecutive whole numbers.

Pages 209 through 211

● Read and discuss the examples at the top of page 209. Ask someone why the shaded regions show that $\frac{11}{6}$ is less than 2 ($\frac{12}{6}$ is 2 and the 12-*divided-by*-6-region is the larger region). Allow the pupils to complete and discuss exercises 1, 2, and 3.

● Guide the class in a discussion of the example at the top of page 210. Then assign the exercises for independent work. When the class has finished, have them discuss their results.

● Discuss the material at the top of page 211 with the class. Work exercises 1 through 3 with the class. Then assign the rest of the exercises for independent work, followed by discussion.

Name

UNIT 16 EXACT DIVISION FRACTIONS AND MIXED FRACTIONS

For Class Discussion *See pupil page suggestions.*

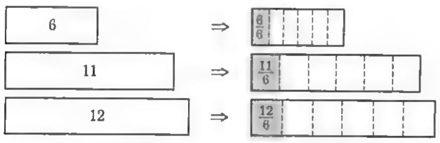
Some fractional numbers are whole numbers.

$\frac{12}{6} = 2$ $\qquad \frac{6}{6} = 1$

$\frac{35}{7} = 5$ $\qquad \frac{169}{13} = 13$

Some fractional numbers are not whole numbers.

$\frac{11}{6}$ is greater than 1 and less than 2. How do you know?



Complete each table.

1.

Fraction	$\frac{12}{6}$	$\frac{13}{6}$	$\frac{14}{6}$	$\frac{15}{6}$	$\frac{16}{6}$	$\frac{17}{6}$	$\frac{18}{6}$
Whole number?	2	None	None	None	*None*	*None*	*3*

2.

Fraction	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{9}{3}$	$\frac{12}{3}$	$\frac{20}{3}$	$\frac{45}{3}$
Whole number?	*None*	*None*	*2*	*3*	*4*	*None*	*15*

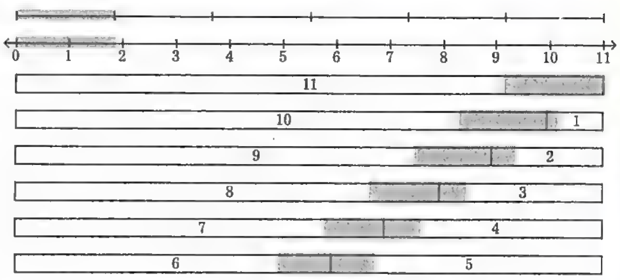
3.

Fraction	$\frac{10}{5}$	$\frac{40}{8}$	$\frac{21}{3}$	$\frac{14}{3}$	$\frac{36}{12}$	$\frac{48}{16}$	$\frac{54}{12}$
Whole number?	*2*	*5*	*7*	*None*	*3*	*3*	*None*

reference page

G-209

The shaded distance is $\frac{11}{6}$. How do you know?



$\frac{11}{6} = \frac{11}{6} + \frac{0}{6}$ $\qquad \frac{11}{6} = \frac{8}{6} + \frac{3}{6}$

$\frac{11}{6} = \frac{10}{6} + \frac{1}{6}$ $\qquad \frac{11}{6} = \frac{7}{6} + \frac{4}{6}$

$\frac{11}{6} = \frac{9}{6} + \frac{2}{6}$ $\qquad \frac{11}{6} = \frac{6}{6} + \frac{5}{6}$

$\frac{11}{6} = 1 + \frac{5}{6}$ How do you know?
$1 = \frac{6}{6}$; $\frac{11}{6} = 1 + \frac{5}{6}$

Is $1 + \frac{5}{6}$ greater than 2? How do you know?
No, $1 + \frac{5}{6} = \frac{11}{6}$; $2 = \frac{12}{6}$

Is $1 + \frac{5}{6}$ greater than $\frac{12}{6}$? Explain.
No, $1 + \frac{5}{6} = \frac{11}{6}$

Is $1 + \frac{5}{6}$ less than $\frac{6}{6}$? Explain.
No, $1 + \frac{5}{6} = \frac{11}{6}$; $\frac{11}{6}$ is more than $\frac{6}{6}$.

Complete.

$\frac{11}{6} = \frac{6}{6} + \frac{5}{6} = 1 + \frac{5}{6}$

1. $\frac{10}{6} =$ *$\frac{6}{6}$* $+ \frac{4}{6} = 1 + \frac{4}{6}$

2. $\frac{13}{5} = \frac{10}{5} +$ *$\frac{3}{5}$* $= 2 +$ *$\frac{3}{5}$*

3. $\frac{14}{5} = \frac{10}{5} +$ *$\frac{4}{5}$* $= 2 +$ *$\frac{4}{5}$*

4. $\frac{15}{5} = \frac{10}{5} +$ *$\frac{5}{5}$* $= 2 + 1$

5. $\frac{16}{5} = \frac{15}{5} +$ *$\frac{1}{5}$* $= 3 +$ *$\frac{1}{5}$*

reference page

G-210

Name

For Class Discussion

These are mixed fractions.

$1 + \frac{5}{6}$ $\qquad 2 + \frac{7}{5}$ $\qquad 5 + \frac{11}{5}$

$1 + \frac{4}{6}$ $\qquad 7 + \frac{19}{3}$ $\qquad 6 + \frac{3}{7}$

$2 + \frac{0}{5}$ $\qquad 0 + \frac{7}{2}$ $\qquad 11 + \frac{2}{8}$

Can any fraction be written as a mixed fraction?

$\frac{11}{6} = 1 + \frac{5}{6}$ $\qquad \frac{4}{7} = 0 + \frac{4}{7}$ $\qquad \frac{36}{5} = 5 + \frac{11}{5}$

$\frac{10}{6} = 1 + \frac{4}{6}$ $\qquad \frac{5}{8} =$ *$0 + \frac{5}{8}$* $\qquad \frac{45}{7} =$ *$6 + \frac{3}{7}$*

$\frac{21}{5} = 4 + \frac{1}{5}$ $\qquad \frac{15}{3} =$ *$5 + \frac{0}{3}$* $\qquad \frac{21}{4} = 2 +$ *$\frac{13}{4}$*

Write each sum as a mixed fraction.

$\frac{6}{6} + \frac{5}{6} = 1 + \frac{5}{6}$

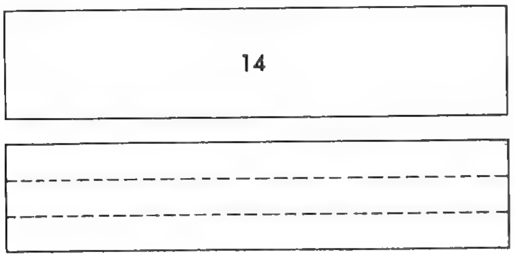
1. $\frac{6}{6} + \frac{4}{6} =$ *$1 + \frac{4}{6}$*
2. $\frac{5}{5} + \frac{4}{5} =$ *$1 + \frac{4}{5}$*
3. $\frac{10}{5} + \frac{4}{5} =$ *$2 + \frac{4}{5}$*
4. $\frac{12}{6} + \frac{5}{6} =$ *$2 + \frac{5}{6}$*
5. $\frac{12}{6} + \frac{4}{6} =$ *$2 + \frac{4}{6}$*
6. $\frac{20}{5} + \frac{3}{5} =$ *$4 + \frac{3}{5}$*
7. $\frac{30}{6} + \frac{5}{6} =$ *$5 + \frac{5}{6}$*
8. $\frac{15}{3} + \frac{7}{3} =$ *$5 + \frac{7}{3}$, $7 + \frac{1}{3}$ or $6 + \frac{4}{3}$*
9. $\frac{21}{7} + \frac{10}{7} =$ *$3 + \frac{10}{7}$ or $4 + \frac{3}{7}$*

reference page

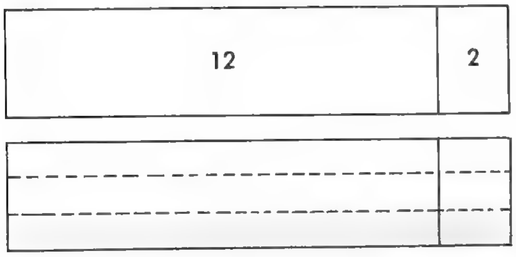
G-211

Developmental Experiences

▶ On the chalkboard draw and label a 14-region as indicated below. Then beside this 14-region draw another 14-region divided into 3 equal parts, as shown.



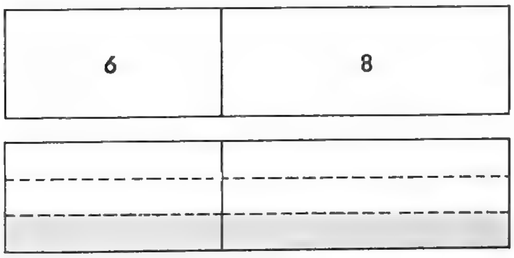
Ask a pupil to tell how much each of the parts is ($14 \div 3$). Shade and label the part. Below the first region draw another 14-region and divide it roughly into two parts, representing 12 and 2. Label the first part 12 and ask a pupil to label the other part (2). Then draw another 14-region next to this region. Divide it into 12- and 2-regions, and then divide it into 3 equal parts as indicated. Do not label, but shade or otherwise indicate the $12 \div 3$ region.



Ask a pupil to tell how much this part is (it is $12 \div 3$ because it is 1 of 3 equal parts of the 12-region). Label the $(12 \div 3)$-region. Next indicate the $(2 \div 3)$-region, and ask a pupil to tell how much this part is (it is a $(2 \div 3)$-region because it is 1 of 3 equal parts of 2). Then shade and label the $(2 \div 3)$-region. Now ask a pupil to compare the shaded parts of both 14-regions (they are the same). On one end of the chalkboard, write the following:

$$\frac{14}{3} = \frac{12}{3} + \frac{2}{3}$$
$$\frac{14}{3} = 4 + \frac{2}{3}$$

Erase the vertical dividing lines from the two bottom regions. Divide the regions again to indicate 6, 8, $6 \div 3$, and $8 \div 3$, as indicated below.



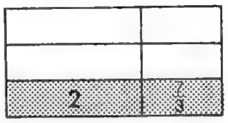
Shade as illustrated. Ask how much each of the shaded parts is. Then label the parts and ask a pupil to compare the shaded parts of both 14-regions (they are the same).

$$\frac{14}{3} = \frac{6}{3} + \frac{8}{3}$$
$$\frac{14}{3} = 2 + \frac{8}{3}$$

Continue the activity. Divide the regions to show that $\frac{14}{3} = \frac{3}{3} + \frac{11}{3}$.

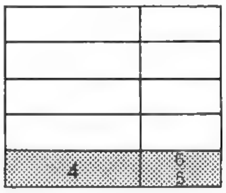
$$\frac{14}{3} = \frac{3}{3} + \frac{11}{3}$$
$$\frac{14}{3} = 1 + \frac{11}{3}$$

When this is completed, draw the following:



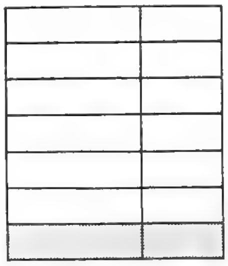
Ask a pupil to tell how much the whole region is ($6 + 7 = 13$, because 3×2 is 6 and $3 \times \frac{7}{3}$ is 7). Then ask a pupil to complete the equation $2 + \frac{7}{3} = \frac{}{3}$ ($2 + \frac{7}{3} = \frac{13}{3}$, because $2 + \frac{7}{3}$ is 1 of 3 equal parts of 13). Continue to use this technique for several other whole numbers and fractions with denominators of 3. Use the same picture; change only the numbers.

Then change the picture and label the parts as indicated below.



Ask a pupil to tell how much the whole region is ($20 + 6$, or 26, because $5 \times 4 = 20$ and $5 \times \frac{6}{5} = 6$). Then ask a pupil to complete the equation $4 + \frac{6}{5} = \frac{?}{5}$ ($\frac{26}{5}$). Continue using this technique for other whole numbers and fractions with denominators of 5. Use the same picture; change only the numbers. Try letting the children choose the numbers; for example, $10 + \frac{3}{5} = \frac{?}{5}$ ($\frac{53}{5}$).

Now change the picture again.



Continue the activity, letting the children assign whole numbers and fractions with denominators of 7 to the shaded parts. Ask them each time how much the whole region is. Then have them write a quotient for the mixed fraction.

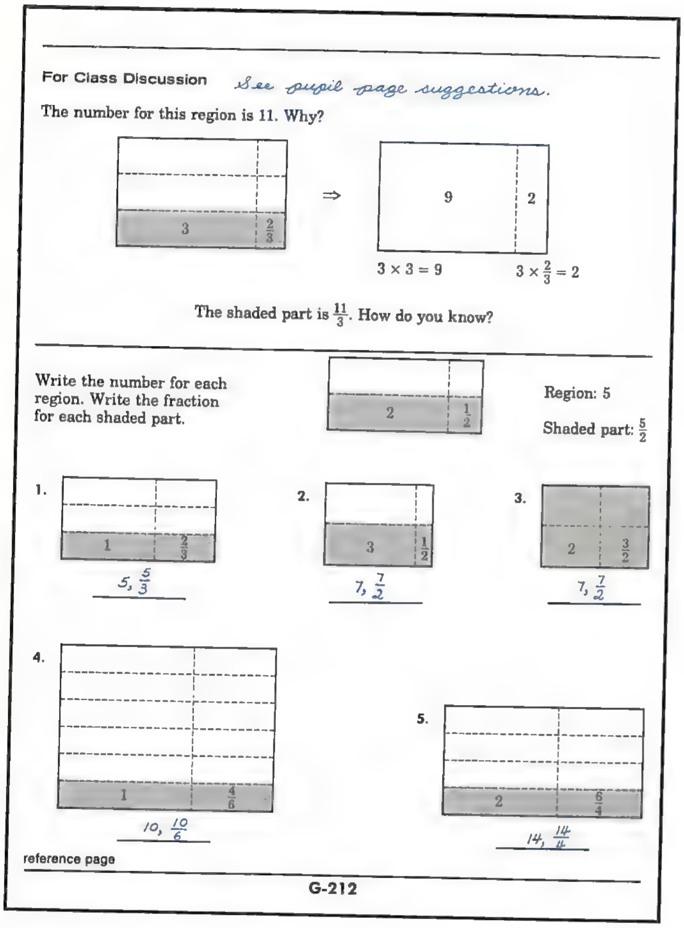
Pages 212 and 213

● Lead a class discussion of the example at the top of page 212. Encourage the pupils to explain why the region is 11 and the shaded part is 11 ÷ 3 (the shaded part is one of three equal parts of 11).

Work the example and the first exercise. Let the class provide the reasoning and answers. Have the pupils complete the remaining exercises independently, and then let them tell their answers and the reasons for their answers.

● As the class discusses the example at the top of page 213, the pupils may see that the model shows that $3 \times 2 = 6$ and $3 \times \triangle = 1$; $\triangle = \frac{1}{3}$. The model also shows that the shaded part is $7 \div 3$, so $\frac{7}{3} = 2 + \frac{1}{3}$. The pupils will probably not describe their ideas in this language—accept correct ideas regardless of how they describe them.

Present each exercise in sequence by asking a pupil to find the fraction and mixed fraction for the shaded region. Then ask him to give his reasoning. Allow time for class discussion.



For Class Discussion *See pupil page suggestions.*

The number for this region is 11. Why?

$3 \times 3 = 9$ $3 \times \frac{2}{3} = 2$

The shaded part is $\frac{11}{3}$. How do you know?

Write the number for each region. Write the fraction for each shaded part.

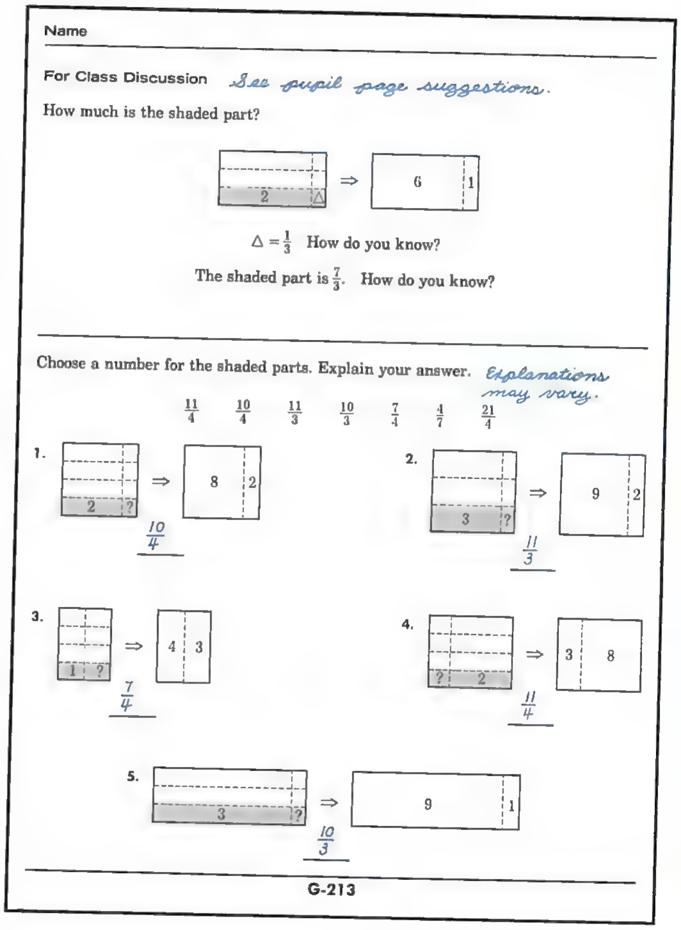
Region: 5

Shaded part: $\frac{5}{2}$

1. $5, \frac{5}{3}$
2. $7, \frac{7}{2}$
3. $7, \frac{7}{2}$
4. $10, \frac{10}{6}$
5. $14, \frac{14}{4}$

reference page

G-212



Name

For Class Discussion *See pupil page suggestions.*

How much is the shaded part?

$\triangle = \frac{1}{3}$ How do you know?

The shaded part is $\frac{7}{3}$. How do you know?

Choose a number for the shaded parts. Explain your answer. *Explanations may vary.*

$\frac{11}{4}$ $\frac{10}{4}$ $\frac{11}{3}$ $\frac{10}{3}$ $\frac{7}{4}$ $\frac{4}{7}$ $\frac{21}{4}$

1. $\frac{10}{4}$
2. $\frac{11}{3}$
3. $\frac{7}{4}$
4. $\frac{11}{4}$
5. $\frac{10}{3}$

G-213

KEY IDEA

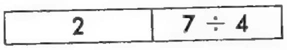
$\frac{11}{4} = \frac{4 \times 2}{4} + \frac{3}{4}$.

Scope

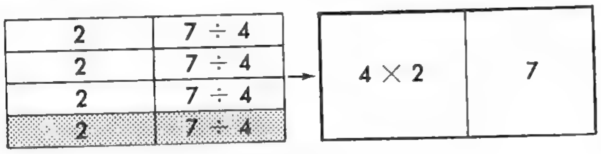
To continue to explore fractional numbers and to use the quotient-remainder algorism to compute mixed fractions.

Fundamentals

The quotient form of a mixed fraction is illustrated using a model for the mixed fraction.

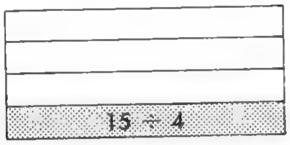


Since one part is $7 \div 4$, four of these parts is the whole number 7.



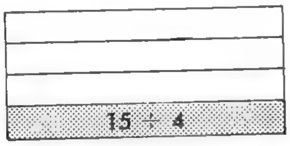
$4 \times 2 = 8$ and $4 \times (7 \div 4) = 7$

The model shows that $2 + \frac{7}{4}$ is one of the four equal parts of 15.

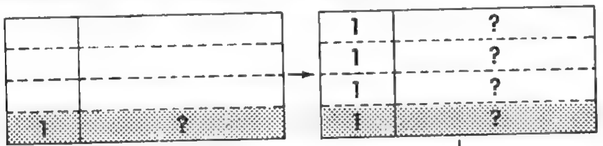


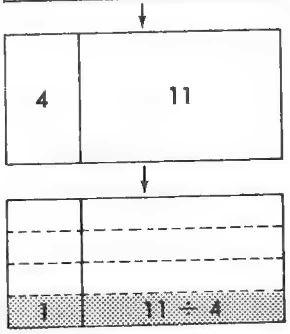
$2 + \frac{7}{4} = \frac{15}{4}$

A mixed fraction for the quotient $15 \div 4$ is illustrated using the model.



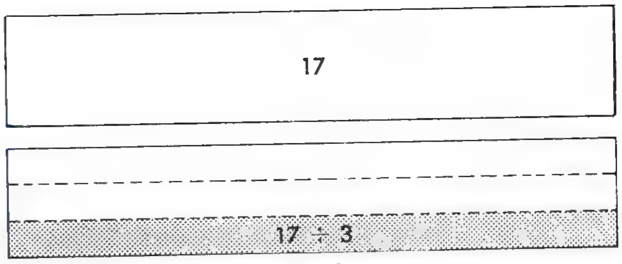
To obtain the mixed fraction $1 + \frac{\square}{4}$, the 15-region is partitioned and labeled as indicated.





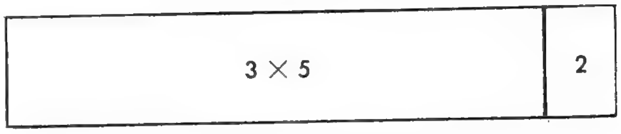
$\frac{15}{4} = 1 + \frac{11}{4}$

Now the quotient-remainder algorism is introduced with the pictures. The algorism results in the fractional part of the mixed fraction being less than one.

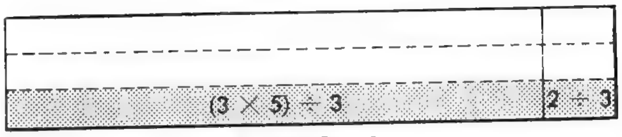


3) 17
15 | 5
2 | 5

$17 = 3 \times 5 + 2$



$17 = 3 \times 5 + 2$



$\frac{17}{3} = \frac{3 \cdot 5}{3} + \frac{2}{3}$

$\frac{17}{3} = 5 + \frac{2}{3}$

Readiness for Understanding

Ability to use the quotient-remainder algorism.

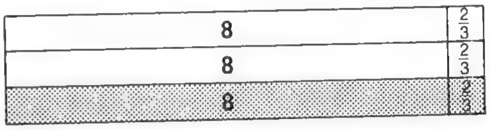
Developmental Experiences

for each child

$\frac{1}{4}$-inch or $\frac{1}{2}$-inch graph paper

► On the chalkboard draw a model for $8 + \frac{2}{3}$.

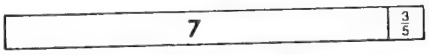
Then add to the drawing, as indicated below.



Ask a pupil to tell how much the whole region is $(24 + 2 = 26)$. If he needs help to see the 2, ask how much $3 \times \frac{2}{3}$ is. Shade part of the region as already shown, and ask the pupils how much the shaded part is (the whole region is 26, so $8 + \frac{2}{3}$, the shaded part, is $26 \div 3$, or $\frac{26}{3}$). Write this on the chalkboard:

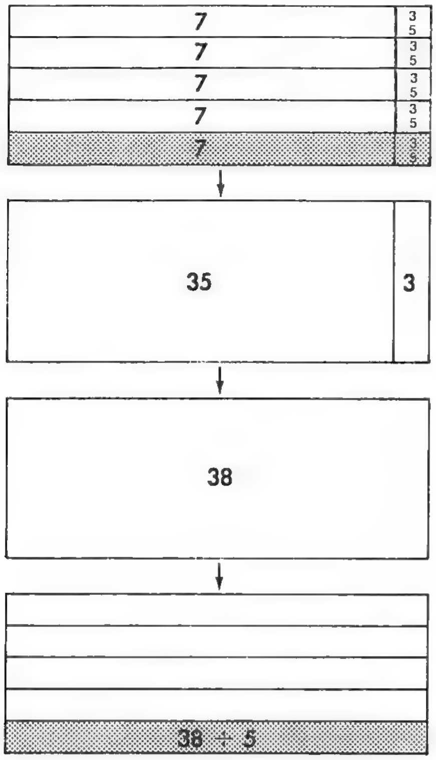
$$8 + \frac{2}{3} = \frac{26}{3}$$

Then draw a model for $7 + \frac{3}{5}$.



Lead a class discussion on how to find a fraction for $7 + \frac{3}{5}$. The discussion should bring out that it will be helpful to expand the figure to get a whole-number region. The discussion should deal with how to get

a whole-number region and why it is needed (7 is a whole number, so that is no problem; but $\frac{3}{5}$ is not a whole number—5 of them are needed to get the whole number 3).

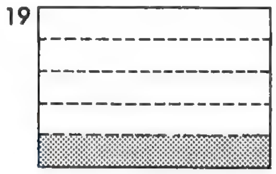


The procedure is illustrated above. Once it is known that the whole region is 38, then it is clear that $7 + \frac{3}{5}$ is 38 ÷ 5. Write this on the chalkboard:

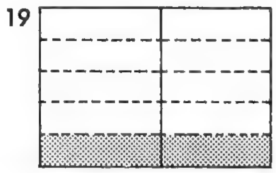
$$7 + \frac{3}{5} = \frac{38}{5}$$

Let the class discuss what was done. Then repeat this activity for $2 + \frac{3}{7}$, $5 + \frac{3}{4}$, and $4 + \frac{1}{2}$. Allow the pupils to work independently and then discuss their work.

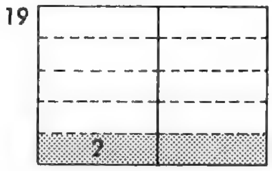
► Show a model for 19 ÷ 5 on the chalkboard.

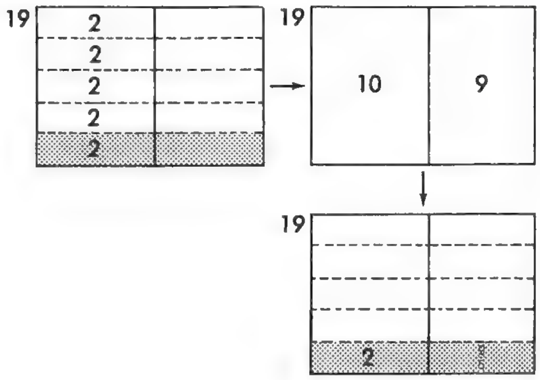


Ask a pupil to tell how much the shaded part is (19 ÷ 5). Then partition the region as indicated below, and ask the class to find a mixed fraction for 19 ÷ 5.



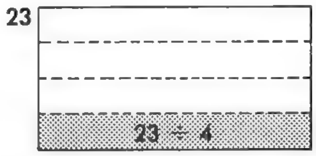
Ask the pupils to try some numbers. Allow some time for this; then have the pupils discuss their results. Suppose someone guesses that 2 will work. Try it.



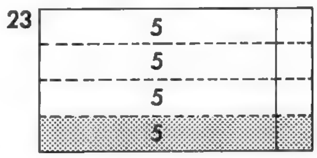


Two does work, but then try 3 and 4, in turn. Tell the pupils to draw their own pictures. They should see for themselves that 3 will help them find a mixed fraction for 19 ÷ 5, but 4 will not.

Have the pupils work with you to find a mixed fraction for 23 ÷ 4.



Suppose a pupil says that 5 works.

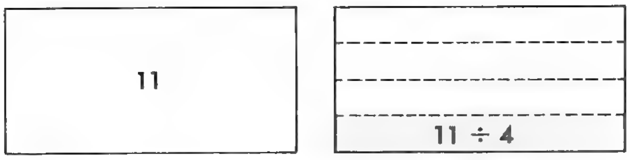


Label the figure as shown, and ask the pupil how much is left (3). Then ask the pupil how much each small part is (3 ÷ 4). Label the small parts $\frac{3}{4}$ and write $\frac{23}{4} = 5 + \frac{3}{4}$.

Next allow the class to work independently to find mixed fractions for such fractions as $\frac{25}{4}$, $\frac{23}{3}$, $\frac{7}{3}$, and $\frac{9}{2}$. (Their answers for $\frac{25}{4}$ could include $4 + \frac{9}{4}$ and $5 + \frac{5}{4}$.) The pupils will have an easier time drawing their pictures if they use $\frac{1}{4}$-inch or $\frac{1}{2}$-inch graph paper.

Then ask the class to find mixed fractions for $\frac{25}{4}$, $\frac{9}{2}$, $\frac{10}{3}$, and $\frac{26}{8}$ so that the fraction part of the mixed fraction names a number less than 1. For example, the desired mixed fraction for $\frac{19}{5}$ is $3 + \frac{4}{5}$; $2 + \frac{9}{5}$ does not satisfy the new condition because $\frac{9}{5}$ is greater than 1.

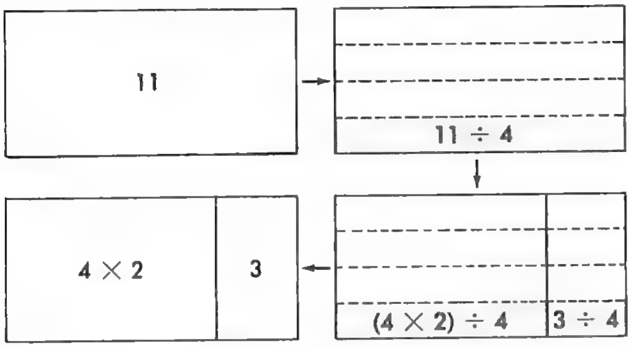
► On the chalkboard draw two 11-regions, one of them divided into 4 equal parts.



Have a pupil help you complete the quotient-remainder algorism as follows.

$$\begin{array}{r|l} 4)\overline{11} & \\ 8 & 2 \\ \hline 3 & 2 \end{array} \qquad 11 = 4 \times 2 + 3$$

Then draw two more 11-regions, partitioned to show $(4 \times 2) + 3$ and $\frac{4\times 2}{4} + \frac{3}{4}$, and labeled as indicated below.

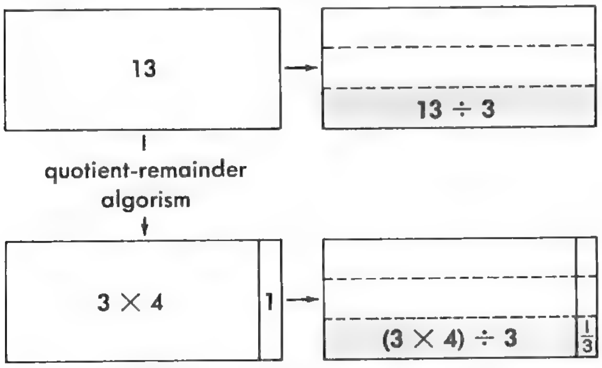


Ask a pupil to compare the $(11 \div 4)$-region and the $\{(4 \times 2) \div 4 + (3 \div 4)\}$-region (they are the same). Write this on the chalkboard:

$$\frac{11}{4} = \frac{4 \times 2}{4} + \frac{3}{4}$$
$$\frac{11}{4} = 2 + \frac{3}{4}$$

Then ask the pupils to discuss why $\frac{4\times 2}{4}$ is 2, and point out that $\frac{3}{4}$ is less than 1.

On the chalkboard draw two 13-regions, one of them divided into 3 equal parts. Ask a pupil to use the quotient-remainder algorism to solve $13 = 3q + r$. When he has the remainder 1, draw and label two more 13-regions as shown. Write the equations $13 = (3 \times 4) + 1$ and $\frac{13}{3} = \frac{3 \times 4}{3} + \frac{1}{3}$ under the regions.

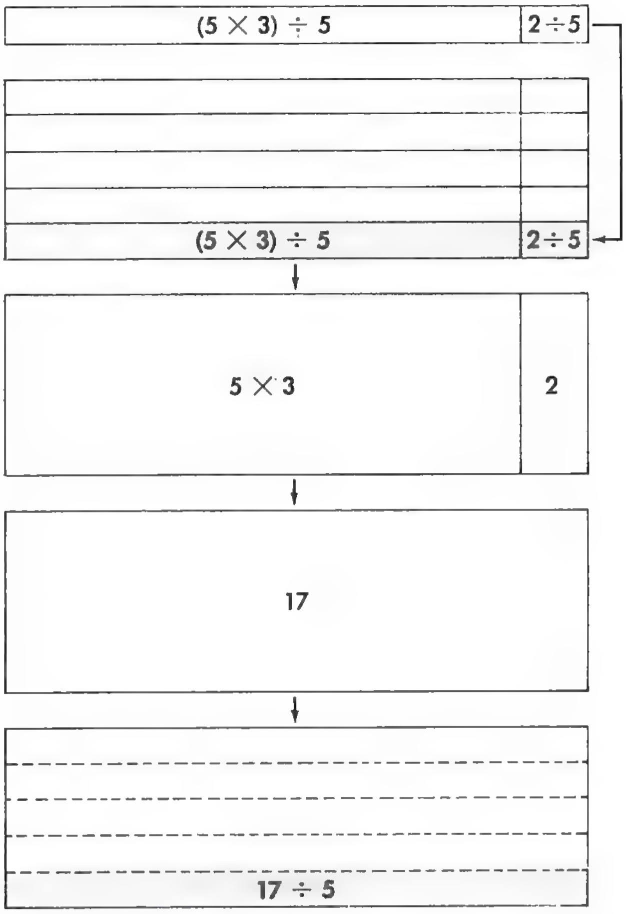


$13 = (3 \times 4) + 1$ $\qquad$ $\frac{13}{3} = \frac{3 \times 4}{3} + \frac{1}{3}$

Ask the pupils for a mixed fraction for $\frac{13}{3}$ $(4 + \frac{1}{3})$. Then ask the pupils to find a mixed fraction for $\frac{17}{5}$ without using pictures.

$$\begin{array}{r|l} 5)\overline{17} & \\ 15 & 3 \\ \hline 2 & 3 \end{array} \qquad \begin{aligned} 17 &= 5 \times 3 + 2 \\ \frac{17}{5} &= \frac{5 \times 3}{5} + \frac{2}{5} \\ \frac{17}{5} &= 3 + \frac{2}{5} \end{aligned}$$

With the pupils, check this result as follows.

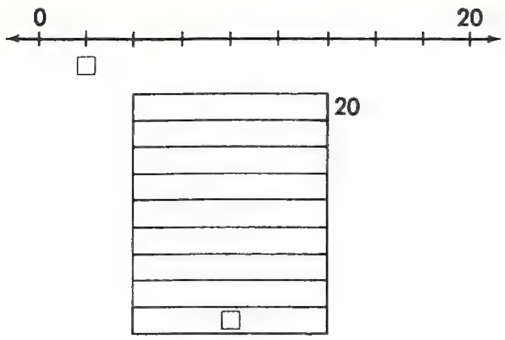


Work several more examples with the class using the quotient-remainder algorism to compute the mixed fraction. For example,

$$\frac{13}{4} = \frac{3 \times 4}{4} + \frac{1}{4} \qquad \frac{39}{5} = \frac{5 \times 7}{5} + \frac{4}{5}$$
$$\frac{17}{2} = \frac{8 \times 2}{2} + \frac{1}{2} \qquad \frac{46}{8} = \frac{8 \times 5}{8} + \frac{6}{8}$$

Pages 214 through 217

● Work the example and exercises 1, 2, 7, and 13 on page 214 with the class. Some pupils may wish to draw a picture for $9 \times \square = 20$.



Assign the remaining exercises for independent work followed by class discussion.

● To introduce page 215, ask the pupils to answer the question "How much is the shaded part?" Write $4 \times \square = 11$ on the chalkboard and ask the pupils if $\square$ is a whole number (no). Continue the discussion referring to the statement in the middle of the page.

● Discuss with the class the material at the top of page 216, using exercise 1 as a guide for discussion. If desired, have a pupil complete a division algorism for $11 \div 4$ on the chalkboard. Then discuss the other exercises and have pupils illustrate them with regions and the quotient-remainder division algorism.

What fractional number is $\square$? Write $\square$ as a quotient $\frac{a}{b}$.

$9 \times \square = 20$

$\square = \frac{20}{9}$

1. $\square \times 4 = 5$ $\square =$ *5/4*
2. $4 \times \square = 5$ $\square =$ *5/4*
3. $\square \times 6 = 5$ $\square =$ *5/6*
4. $4 \times \square = 13$ $\square =$ *13/4*
5. $\square \times 5 = 7$ $\square =$ *7/5*
6. $9 \times \square = 11$ $\square =$ *11/9*

What whole number is $\square$?

7. $\frac{10}{6} \times \square = 10$ $\square =$ *6*
8. $(30 \div 6) \times \square = 30$ $\square =$ *6*
9. $\frac{15}{4} \times \square = 15$ $\square =$ *4*
10. $(50 \div 6) \times \square = 50$ $\square =$ *6*
11. $\frac{40}{6} \times \square = 40$ $\square =$ *6*
12. $(60 \div 6) \times \square = 60$ $\square =$ *6*

Compute.

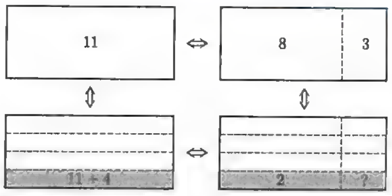
13. $5 \times \frac{6}{5} =$ *6*
14. $4 \times \frac{5}{4} =$ *5*
15. $\frac{4}{3} \times 3 =$ *4*
16. $6 \times \frac{5}{6} =$ *5*
17. $9 \times \frac{5}{9} =$ *5*
18. $\frac{2}{3} \times 3 =$ *2*
19. $8 \times \frac{3}{8} =$ *3*
20. $6 \times \frac{7}{6} =$ *7*
21. $\frac{4}{5} \times 5 =$ *4*
22. $10 \times \frac{3}{10} =$ *3*
23. $11 \times \frac{5}{11} =$ *5*
24. $\frac{5}{7} \times 7 =$ *5*

G-214

Name

For Class Discussion *See pupil page suggestions.*

Fractional numbers are quotients of whole numbers.



How much is the shaded part?

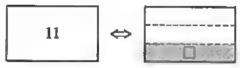
The shaded part is greater than 2 and less than 3.

Before we had the set of fractional numbers, we could say only that the shaded part is greater than 2 and less than 3. Now we know that the shaded part is the fractional number $\frac{11}{4}$ or $2 + \frac{3}{4}$.

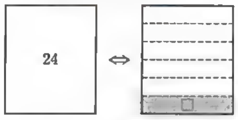
If $4 \times \square = 11$, then $\square = \frac{11}{4}$ or $2 + \frac{3}{4}$.

Write the fractional number.

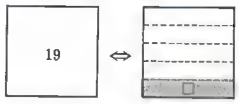
1. $3 \times \square = 11$ $\square =$ *11/3 or 3 + 2/3*



2. $6 \times \square = 24$ $\square =$ *24/6 or 4*



3. $5 \times \square = 19$ $\square =$ *19/5 or 3 + 4/5*



reference page

G-215

For Class Discussion

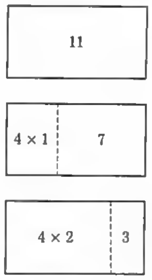
How does the quotient-remainder equation help us compute mixed fractions?

$4 \times \square = 11$

$\square = \frac{11}{4}$

Quotient-Remainder Division $11 = 4q + r$

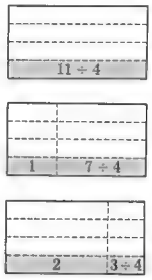
Exact Division $11 = 4q$



$11 = (4 \times 0) + 11$, $q = 0, r = 11$

$11 = (4 \times 1) + 7$, $q = 1, r = 7$

$11 = (4 \times 2) + 3$, $q = 2, r = 3$



$q = \frac{11}{4}$

$q = 1 + \frac{7}{4}$

$q =$ *2* $+ \frac{3}{4}$

How can we compute mixed fractions?

1. $\frac{11}{4} = ?$

$11 = 4q + r$

$11 = (4 \times 2) + 3$

$\frac{11}{4} = \frac{4 \times 2}{4} + \frac{3}{4}$

$\frac{11}{4} =$ *2* $+ \frac{3}{4}$

2. $\frac{19}{3} = ?$

$19 = 3q + r$

$19 = (3 \times 6) + 1$

$\frac{19}{3} = \frac{3 \times 6}{3} + \frac{1}{3}$

$\frac{19}{3} =$ *6* $+ \frac{1}{3}$

3. $\frac{14}{5} = ?$

$14 = 5q + r$

$14 = (5 \times 2) + 4$

$\frac{14}{5} = \frac{5 \times 2}{5} + \frac{4}{5}$

$\frac{14}{5} =$ *2* $+$ *4/5*

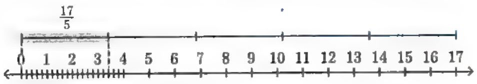
reference page

G-216

● Discuss the example at the top of page 217 with the class. Point out that $3\frac{2}{5}$ is a short way of writing $3 + \frac{2}{5}$. Assign exercise 1 for independent work followed by class discussion. Use the same procedure for exercises 2, 3, and 4. Next ask pupils to come to the chalkboard and do exercises 5, 6, 7, and 8. Have the class discuss these exercises. Then assign the remaining exercises for independent work.

Name

What is the mixed fraction for $\frac{17}{5}$? *See pupil page suggestions.*



$17 = 5q + r$

$17 = (5 \times 3) + 2$

$\frac{17}{5} = \frac{5 \times 3}{5} + \frac{2}{5}$

$\frac{17}{5} = 3 + \frac{2}{5}$

$3 + \frac{2}{5}$ can be written $3\frac{2}{5}$.

$5\overline{)17}$ 15 | 3, 2 | 3

Write the mixed fraction.

1. $15 = (4 \times 3) + 3$ — $\frac{15}{4} = 3\frac{3}{4}$
2. $19 = (2 \times 9) + 1$ — $\frac{19}{2} = 9\frac{1}{2}$
3. $24 = (5 \times 4) + 4$ — $\frac{24}{5} = 4\frac{4}{5}$
4. $18 = (7 \times 2) + 4$ — $\frac{18}{7} = 2\frac{4}{7}$
5. $10 = (8 \times 1) + 2$ — $\frac{10}{8} = 1\frac{2}{8}$
6. $13 = (5 \times 2) + 3$ — $\frac{13}{5} = 2\frac{3}{5}$
7. $27 = (7 \times 3) + 6$ — $\frac{27}{7} = 3\frac{6}{7}$
8. $31 = (2 \times 15) + 1$ — $\frac{31}{2} = 15\frac{1}{2}$
9. $32 = (3 \times 15) + 2$ — $\frac{32}{3} = 10\frac{2}{3}$
10. $29 = (4 \times 7) + 1$ — $\frac{29}{4} = 7\frac{1}{4}$
11. $35 = (6 \times 5) + 5$ — $\frac{35}{6} = 5\frac{5}{6}$
12. $42 = (8 \times 5) + 2$ — $\frac{42}{8} = 5\frac{2}{8}$

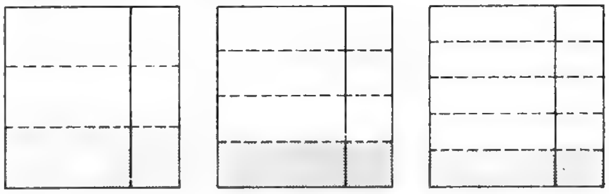
reference page

G-217

Supplemental Experience

■ The game of Quo-Algo.

On the chalkboard draw three regions of the same size and shape—divided and shaded as indicated below.

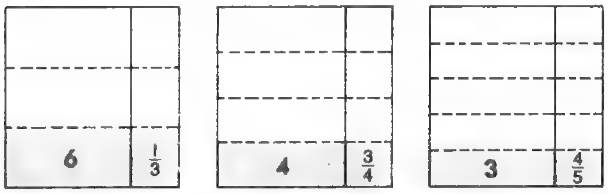


To introduce the game, write the following numbers at the top of the chalkboard:

GIVEN NUMBERS
7, 11, 13, 14, 17, 19, 23,
26, 29, 31, 34, 37, 38, 41

Explain the rules of the game. The player chooses a number from this list. He assigns that number to each of the three regions and finds a mixed fraction for each shaded part. The fraction part of the mixed fraction must name a number less than 1.

Divide the class into two teams. To start, a team chooses a given number, say 19, and crosses it off. Then 3 players from this team go to the board and show the mixed fraction for each shaded region.



Allow them to check their work by consulting with each other and with the rest of their team. When they have finished, they return to their seats.

Each team may select a scorer to keep a list on the chalkboard. If the three mixed fractions are all correct, the scorer records their whole-number parts in a team list. In this case, 6, 4, and 3 are put in the team list. If any of the mixed fractions is incorrect, no new numbers are put in the team list.

Next the other team chooses a given number and crosses it off. Three players from that team find the three mixed fractions, and their scorer writes the whole-number parts in the team list. The game continues in this way, alternating play.

Points are earned by obtaining three consecutive numbers in the team list. When a team gets three consecutive numbers, it scores an Algo and wins a point. Whenever a team scores an Algo, its team list is erased, and it must begin forming its next Algo.

The game is over when all the given numbers have been crossed out. Determine the winner, restore the original given numbers, and have the game begin again. Let the losing team play first.

After several games have been played, change the rules so that five consecutive numbers, instead of three, are needed to score an Algo. Later the class might like to require more than five consecutive numbers for

an Algo. (If you need to lengthen the list, use 43, 46, 47, 49, 53, 58, 59, and 61.)

The pupils may want to play this game for several class periods, and it may be used often during the year. A team can contain as few as two pupils if teachers wish to use Quo-Algo as a small group activity.

KEY IDEA

$\frac{7}{3} = 2\frac{1}{3}$.

Scope

To extend the quotient-remainder algorism.

Fundamentals

Using the quotient-remainder algorism, we find the greatest partial quotient and the least remainder. For example:

```
23) 270
    230 | 10
     40 |
     23 |  1
     17 | 11
```

$$270 = 11 \times 23 + 17$$

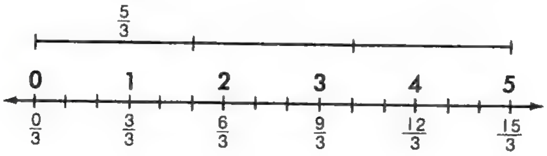
Now that we have fractional numbers, we can extend the algorism to compute a mixed fraction for the quotient, $\frac{270}{23}$.

```
23) 270
    230 | 10
     40 |
     23 |  1
     17 |
     17 |      17/23
      0 | 11 + 17/23
```

Then $\frac{270}{23} = 11 + \frac{17}{23}$

Notice that the algorism is extended by using the knowledge that $23 \times \frac{17}{23} = 17$.

The number line is used to illustrate fractions and mixed fractions. The illustration below shows that $\frac{5}{3}$ is $1 + \frac{2}{3}$ (read 5 thirds is 1 plus 2 thirds or $5 \div 3$ is $\{1 + (2 \div 3)\}$).



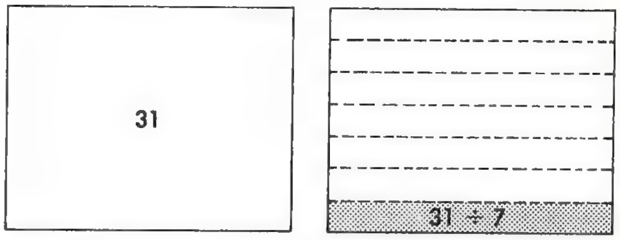
Readiness for Understanding

Knowledge of quotients and the division algorism. Understanding that $\square \times \frac{\triangle}{\square} = \triangle$.

Developmental Experiences

piece of string (6′ long)

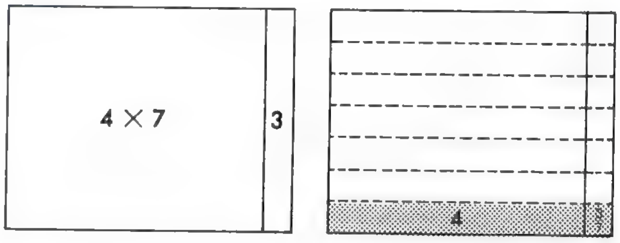
► On the chalkboard draw and label the following illustrations.



Then ask a pupil to help you use the quotient-remainder algorism to find the mixed fraction for $31 \div 7$.

```
7) 31
   28 | 4
    3 | 4          31 = 4 × 7 + 3
```

Then illustrate the quotient $\frac{31}{7}$ as indicated below.



When the illustration is finished, write $\frac{31}{7} = 4 + \frac{3}{7}$. Next have the pupils discuss the question, "Why does the quotient-remainder algorism help us find a mixed fraction with a fractional part less than one?" (because the remainder is always less than the divisor)

Then ask a pupil to help you complete this quotient-remainder algorism.

```
13) 310
    260 | 20
     50 |
     39 |  3
     11 | 23
```

At this point, we usually write $310 = 13 \times 23 + 11$. Have the class discuss the reason they can go no further (11 is less than the divisor, 13). Beside this example have a pupil help you complete another quotient-remainder algorism.

```
13) 310               3) 35
    260 | 20             30 | 10
     50 |                 5 |
     39 |  3              3 |  1
     11 | 23              2 | 11
```

Again discuss with the class the reason they can go no further (2 is less than the divisor, 3).

Now lead a class discussion on the question "How could we go further with this division algorism?"

The discussion should bring out the idea that you need some number, $\square$, such that $3 \times \square$ is 2. We have such a number: $\square = \frac{2}{3}$.

Have a pupil complete the algorism as indicated.

$$
\begin{array}{r|l}
3)\ 35 & \\
30 & 10 \\ \hline
5 & \\
3 & 1 \\ \hline
2 & \\
2 & \quad \tfrac{2}{3} \\ \hline
0 & 11 + \tfrac{2}{3}
\end{array}
\qquad \tfrac{35}{3} = 11 + \tfrac{2}{3}
$$

Then write $\frac{35}{3} = 11\frac{2}{3}$.

Have another pupil complete the algorism for $310 \div 13$. He will need a number, $\square$, such that $13 \times \square$ is 11; $\square = \frac{11}{13}$.

$$
\begin{array}{r|l}
13)\ 310 & \\
260 & 20 \\ \hline
50 & \\
39 & 3 \\ \hline
11 & \\
11 & \quad \tfrac{11}{13} \\ \hline
0 & 23 + \tfrac{11}{13}
\end{array}
\qquad \tfrac{310}{13} = 23\tfrac{11}{13}
$$

Then ask the pupils to compute mixed fractions for these fractions.

$$\tfrac{200}{6},\ \tfrac{70}{8},\ \tfrac{30}{8},\ \tfrac{30}{7},\ \tfrac{400}{6},\ \tfrac{450}{10},\ \tfrac{650}{12}$$

Have some pupils work at the chalkboard and the others work at their desks.

Pages 218 through 223

● Lead the class in a discussion of the material at the top of page 218. Point out that using the exact-division algorism and the model for $14 \div 3$ we can answer the question $14 \div 3 = 4 + ?$. Work the example with the class. Ask why we use the partial quotient $\frac{2}{3}$ (because $3 \times \frac{2}{3}$ is 2 and $2 - 2 = 0$). Ask the class to help a pupil do exercise 1, using the exact-division algorism. Then use the same procedure in letting the class work and discuss each of the remaining exercises.

● Discuss the material at the top of page 219, pointing out that the exact-division algorism is an extension of the quotient-remainder algorism. Note that every exact-division algorism is completed by using the idea that $a \times \frac{b}{a} = b$. Then through class discussion, have the pupils work and discuss the exercises.

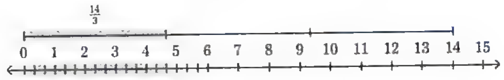
● Have the pupils check and discuss the examples at the top of page 220 to see if they are correct. Ask some pupils to come to the board and do exercises 1 and 2. Then assign the other exercises for independent work followed by discussion.

● Use your judgment regarding a discussion of the examples on page 221. Assign the exercises for independent work.

● Pages 222 and 223 provide practice in computing mixed fractions. Assign a reasonable number of exercises for independent work.

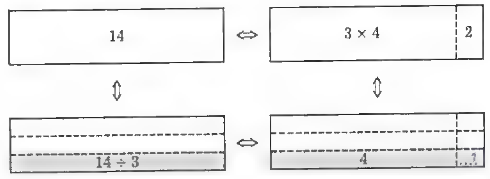
For Class Discussion *See pupil page suggestions.*

$\frac{14}{3} = 4 + ?$



$\frac{14}{3}$ is greater than 4 and less than 5.

$14 = (4 \times 3) + 2$



$$
\begin{array}{r|l}
3\overline{)14} & \\
12 & 4 \\ \hline
2 & \\
2 & \tfrac{2}{3} \Leftarrow (3 \times \tfrac{2}{3} = 2) \\ \hline
0 & 4 + \tfrac{2}{3}
\end{array}
$$

$\frac{14}{3} = 4 + \underline{\tfrac{2}{3}}$

Complete.

1. $\frac{16}{3} = 5 + \underline{\tfrac{1}{3}}$
2. $\frac{13}{3} = 4 + \underline{\tfrac{1}{3}}$
3. $\frac{10}{3} = 3 + \underline{\tfrac{1}{3}}$
4. $\frac{7}{3} = 2 + \underline{\tfrac{1}{3}}$
5. $\frac{4}{3} = 1 + \underline{\tfrac{1}{3}}$
6. $\frac{67}{3} = \underline{22} + \frac{1}{3}$

reference page

G-218

Name

For Class Discussion

The quotient-remainder algorism and exact division.

Quotient-Remainder Algorism	Exact Division
$\begin{array}{r\|l} 2\overline{)9} & \\ 8 & 4 \\ \hline 1 & 4 \end{array}$ $9 = (2 \times 4) + 1$	$\begin{array}{r\|l} 2\overline{)9} & \\ 8 & 4 \\ \hline 1 & \\ 1 & \tfrac{1}{2} \Leftarrow (2 \times \tfrac{1}{2} = 1) \\ \hline 0 & 4 + \tfrac{1}{2} \end{array}$ $\frac{9}{2} = 4\frac{1}{2}$
$\begin{array}{r\|l} 4\overline{)27} & \\ 24 & 6 \\ \hline 3 & 6 \end{array}$ $27 = (4 \times 6) + 3$	$\begin{array}{r\|l} 4\overline{)27} & \\ 24 & 6 \\ \hline 3 & \\ 3 & \tfrac{3}{4} \Leftarrow \underline{4 \times \tfrac{3}{4} = 3} \\ \hline 0 & 6 + \tfrac{3}{4} \end{array}$ $\frac{27}{4} = 6\frac{3}{4}$

Complete.

1. $\begin{array}{r|l} 5\overline{)18} & \\ 15 & 3 \\ \hline 3 & \\ 3 & \tfrac{3}{5} \\ \hline 0 & 3 + \tfrac{3}{5} \end{array}$ $\quad \frac{18}{5} = \underline{3\tfrac{3}{5}}$

2. $\begin{array}{r|l} 6\overline{)29} & \\ 24 & 4 \\ \hline 5 & \\ 5 & \tfrac{5}{6} \\ \hline 0 & 4 + \tfrac{5}{6} \end{array}$ $\quad \frac{29}{6} = \underline{4\tfrac{5}{6}}$

3. $\begin{array}{r|l} 13\overline{)43} & \\ 39 & 3 \\ \hline 4 & \\ 4 & \tfrac{4}{13} \\ \hline 0 & 3 + \tfrac{4}{13} \end{array}$ $\quad \frac{43}{13} = \underline{3\tfrac{4}{13}}$

reference page

G-219

Use the exact division algorism to compute the missing fractional number.

$\frac{7}{3} = 2 + ?$

$$\begin{array}{r|l l}
3\,\overline{)7} & & \\
6 & 2 & \Leftarrow (2 \times 3 = 6) \\
1 & & \\
1 & \frac{1}{3} & \Leftarrow \left(\frac{1}{3} \times 3 = 1\right) \\
\hline
0 & 2 + \frac{1}{3} &
\end{array}$$

$\frac{23}{6} = 3 + ?$

$$\begin{array}{r|l l}
6\,\overline{)23} & & \\
18 & 3 & \Leftarrow (3 \times 6 = 18) \\
5 & & \\
5 & \frac{5}{6} & \Leftarrow \left(\frac{5}{6} \times 6 = 5\right) \\
\hline
0 & 3 + \frac{5}{6} &
\end{array}$$

1. $\frac{10}{4} = 2 + \underline{\frac{2}{4}}$

$$\begin{array}{r|l l}
4\,\overline{)10} & & \\
8 & 2 & \Leftarrow (2 \times 4 = 8) \\
2 & & \\
2 & \frac{2}{4} & \Leftarrow \left(\frac{2}{4} \times 4 = 2\right) \\
\hline
0 & 2 + \frac{2}{4} &
\end{array}$$

2. $\frac{13}{2} = 6 + \underline{\frac{1}{2}}$

$$\begin{array}{r|l l}
2\,\overline{)13} & & \\
12 & 6 & \Leftarrow (6 \times 2 = 12) \\
1 & & \\
1 & \frac{1}{2} & \Leftarrow \left(\frac{1}{2} \times 2 = 1\right) \\
\hline
0 & 6 + \frac{1}{2} &
\end{array}$$

3. $\frac{17}{5} = 3 + \underline{\frac{2}{5}}$

4. $\frac{41}{7} = 5 + \underline{\frac{6}{7}}$

5. $\frac{21}{6} = 3 + \underline{\frac{3}{6}}$

6. $\frac{30}{7} = 4 + \underline{\frac{2}{7}}$

7. $\frac{32}{5} = 6 + \underline{\frac{2}{5}}$

8. $\frac{75}{8} = 9 + \underline{\frac{3}{8}}$

Write the mixed fraction.

1. $\frac{95}{2} = \underline{47\frac{1}{2}}$

2. $\frac{54}{3} = \underline{18\frac{0}{3}}$

3. $\frac{73}{6} = \underline{12\frac{1}{6}}$

4. $\frac{145}{7} = \underline{20\frac{5}{7}}$

5. $\frac{904}{9} = \underline{100\frac{4}{9}}$

6. $\frac{609}{2} = \underline{304\frac{1}{2}}$

7. $\frac{904}{30} = \underline{30\frac{4}{30}}$

8. $\frac{336}{20} = \underline{16\frac{16}{20}}$

9. $\frac{243}{12} = \underline{20\frac{3}{12}}$

10. $\frac{348}{11} = \underline{31\frac{7}{11}}$

Name

What is the mixed fraction?

$\frac{78}{4} = ?$

$$\begin{array}{r|l}
4\,\overline{)78} & \\
40 & 10 \\
38 & \\
36 & 9 \\
2 & \\
2 & \frac{2}{4} \\
\hline
0 & 19 + \frac{2}{4}
\end{array}$$

$\frac{78}{4} = 19\frac{2}{4}$

$\frac{82}{3} = ?$

$$\begin{array}{r|l}
3\,\overline{)82} & \\
60 & 20 \\
22 & \\
21 & 7 \\
1 & \\
1 & \frac{1}{3} \\
\hline
0 & 27 + \frac{1}{3}
\end{array}$$

$\frac{82}{3} = 27\frac{1}{3}$

Write the mixed fraction.

1. $\frac{27}{5} = \underline{5\frac{2}{5}}$

2. $\frac{80}{9} = \underline{8\frac{8}{9}}$

3. $\frac{45}{9} = \underline{5\frac{0}{9}}$

4. $\frac{34}{4} = \underline{8\frac{2}{4}}$

5. $\frac{56}{6} = \underline{9\frac{2}{6}}$

6. $\frac{56}{8} = \underline{7\frac{0}{8}}$

7. If a butcher cuts a 7-pound roast in half, how much does each part weigh? Write a fraction for your answer. $\underline{\frac{7}{2}\text{ pounds}}$

Name

Write the mixed fraction.

1. $\frac{62}{3} = \underline{20\frac{2}{3}}$

2. $\frac{81}{4} = \underline{20\frac{1}{4}}$

3. $\frac{407}{14} = \underline{29\frac{1}{14}}$

4. $\frac{520}{40} = \underline{13\frac{0}{40}}$

5. $\frac{425}{13} = \underline{32\frac{9}{13}}$

6. $\frac{850}{26} = \underline{32\frac{18}{26}}$

7. $\frac{625}{100} = \underline{6\frac{25}{100}}$

8. $\frac{461}{100} = \underline{4\frac{61}{100}}$

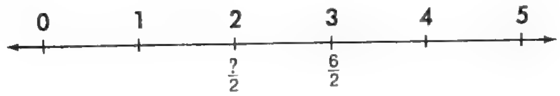
9. $\frac{842}{200} = \underline{4\frac{42}{200}}$

10. $\frac{678}{17} = \underline{39\frac{15}{17}}$

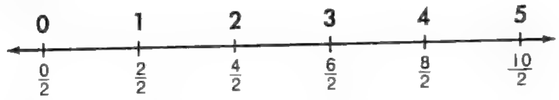
Developmental Experiences

▶ Carefully draw the number line on the chalkboard, using units about 1 foot long. Label as indicated.

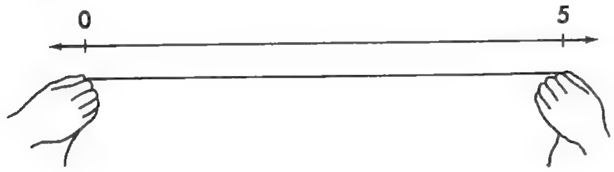
Below the 3, write $\frac{6}{2}$. Below the 2, write $\frac{?}{2}$ as indicated.



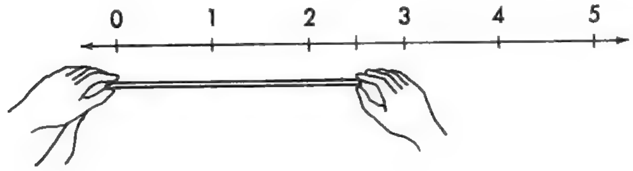
Ask a pupil to complete the quotient ($\frac{4}{2}$). Then have other pupils provide quotients for the remaining whole numbers.



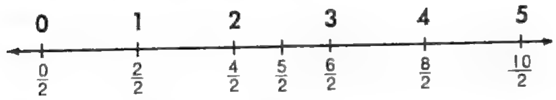
Next ask a pupil to help you measure a length of string 5 units long.



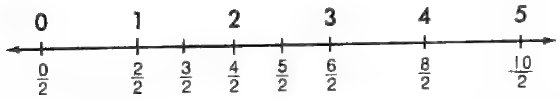
Then halve the distance from 0 to 5 by carefully folding this piece of string. Using this length, locate and mark the point for 5 ÷ 2 on the number line.



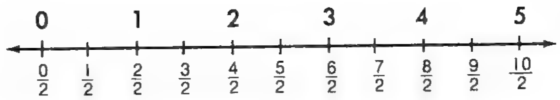
Have the pupil label the point $\frac{5}{2}$.



Repeat this procedure. Ask another pupil to locate $\frac{3}{2}$ on the number line, and label it.

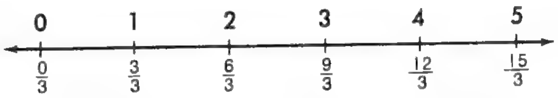


Then mark the point halfway between 3 and 4, and ask a pupil to decide how to label it. Continue in this way until each of the quotients indicated below has been shown by a pupil.

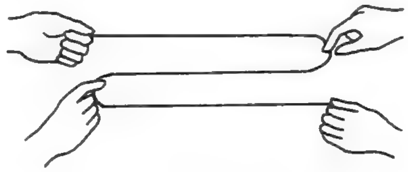


Then ask a pupil to state the quotient for the mixed fraction $1 + \frac{1}{2}$ ($\frac{3}{2}$). Now ask him to come to the board and show the distance 1 and the distance $\frac{1}{2}$. Also have him show the distance $\frac{3}{2}$ (from 0 to $\frac{3}{2}$). Similarly, ask pupils to tell the quotients for $4\frac{1}{2}$, 5, and $2\frac{1}{2}$. Then ask them to tell the mixed fractions for $\frac{4}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, and $\frac{3}{2}$.

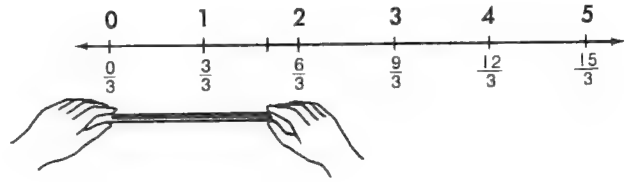
Now draw the number line again, using units about 1 foot long. This time, label the lengths as indicated below.

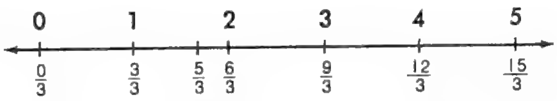


Ask a pupil to help you measure a length of string 5 units long. Then fold this string into 3 equal parts.

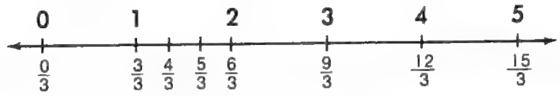


Using this length, locate $\frac{5}{3}$ on the number line. Ask the pupil to label the point.

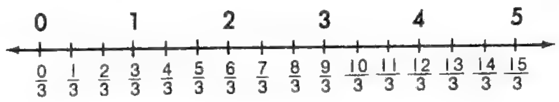




Repeat this procedure with another pupil to measure a 4-unit length of string and locate 4 ÷ 3.



Then divide the remaining intervals into thirds, and have the pupils tell the quotient for each mark.



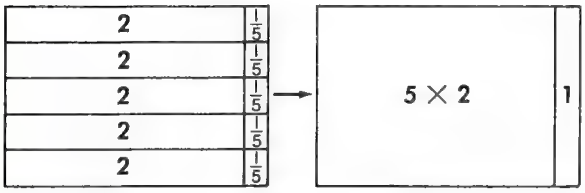
Next ask a pupil to tell the quotient for $4 + \frac{1}{3}$. Ask him to come to the board to show the distances 4, $\frac{1}{3}$, and $\frac{13}{3}$. Similarly, have pupils tell mixed fractions for other quotients and vice-versa.

Pages 224 through 232

● Discuss exercises 1 and 2 on page 224 with the class. In exercise 1, have the pupils indicate the distances 2 and $\frac{2}{3}$ which make up the mixed fraction $2 + \frac{2}{3}$. Then assign exercises 3 through 6 for independent work followed by discussion.

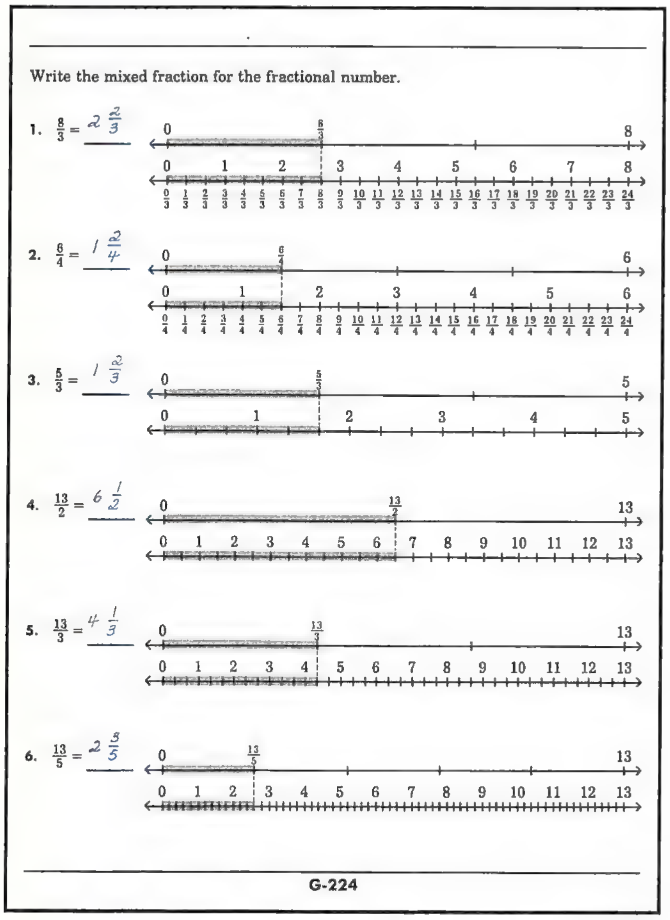
● Discuss the example and exercises 1 and 2 on page 225 with the class. Then assign the remaining exercises for independent work followed by discussion.

● Discuss the example at the top of page 226 with the class. It may be helpful to copy these figures on the chalkboard to review the procedure.



$$2\tfrac{1}{5} = \frac{5 \times 2}{5} + \frac{1}{5}$$

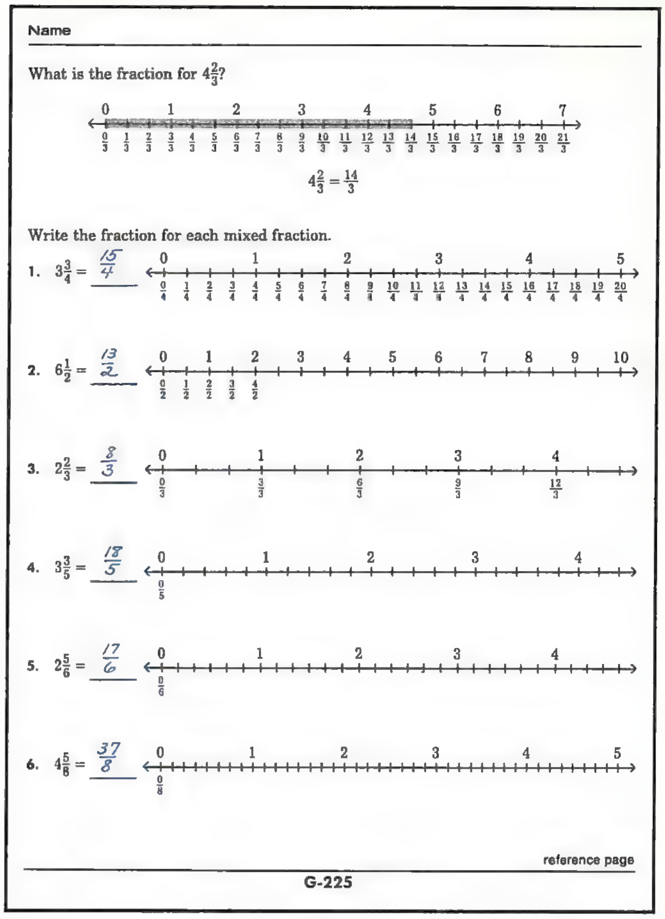
Then work exercises 1 through 6. Assign exercises 7 through 10 for independent work followed by discussion.



Write the mixed fraction for the fractional number.

1. $\frac{8}{3} = 2\frac{2}{3}$
2. $\frac{6}{4} = 1\frac{2}{4}$
3. $\frac{5}{3} = 1\frac{2}{3}$
4. $\frac{13}{2} = 6\frac{1}{2}$
5. $\frac{13}{3} = 4\frac{1}{3}$
6. $\frac{13}{5} = 2\frac{3}{5}$

G-224



Name

What is the fraction for $4\frac{2}{3}$?

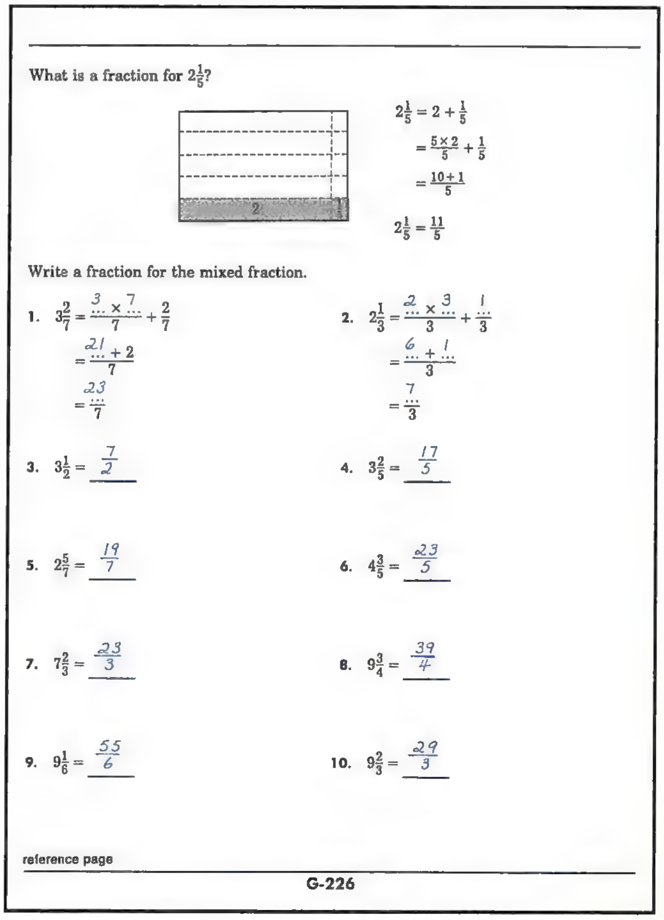
$4\frac{2}{3} = \frac{14}{3}$

Write the fraction for each mixed fraction.

1. $3\frac{3}{4} = \frac{15}{4}$
2. $6\frac{1}{2} = \frac{13}{2}$
3. $2\frac{2}{3} = \frac{8}{3}$
4. $3\frac{3}{5} = \frac{18}{5}$
5. $2\frac{5}{6} = \frac{17}{6}$
6. $4\frac{5}{8} = \frac{37}{8}$

reference page

G-225



What is a fraction for $2\frac{1}{5}$?

$2\frac{1}{5} = 2 + \frac{1}{5}$

$= \frac{5 \times 2}{5} + \frac{1}{5}$

$= \frac{10+1}{5}$

$2\frac{1}{5} = \frac{11}{5}$

Write a fraction for the mixed fraction.

1. $3\frac{2}{7} = \frac{3 \times 7}{7} + \frac{2}{7} = \frac{21 + 2}{7} = \frac{23}{7}$
2. $2\frac{1}{3} = \frac{2 \times 3}{3} + \frac{1}{3} = \frac{6 + 1}{3} = \frac{7}{3}$
3. $3\frac{1}{2} = \frac{7}{2}$
4. $3\frac{2}{5} = \frac{17}{5}$
5. $2\frac{5}{7} = \frac{19}{7}$
6. $4\frac{3}{5} = \frac{23}{5}$
7. $7\frac{2}{3} = \frac{23}{3}$
8. $9\frac{3}{4} = \frac{39}{4}$
9. $9\frac{1}{6} = \frac{55}{6}$
10. $9\frac{2}{3} = \frac{29}{3}$

reference page

G-226

● Discuss with the class the examples at the top of page 227. Let pupils explain how the models show that 5 is $\frac{15}{3}$ and that 5 is $\frac{20}{4}$. As exercise 1 is discussed, draw the following region on the chalkboard.

7
7
7
7
7

Write 7= . Have a pupil complete the equation and explain his reasoning ($7 = \frac{5 \times 7}{5}$ because 7 is one of 5 equal parts of 5×7).

Similarly, draw regions for exercises 2 and 3. Then let the pupils complete the remaining exercises independently.

● Page 228 provides practice in writing fractions for mixed fractions. Work the example at the top of the page and exercises 1 and 2 with the class, and assign exercises 3 through 7 for independent work followed by discussion.

● Pages 229 and 230 provide practice in writing fractions for mixed fractions. Assign the pages for independent work. After the pages have been completed, discuss any exercises which caused problems for the pupils.

● The exercises on page 231 provide another look at division. Work the examples at the top of the page with the class. Then assign exercises 1 through 3 for independent work followed by discussion. Assign the remaining exercises for independent work.

● Page 232 provides the pupils an opportunity to use and interpret pictures in solving problems. With the class, work and discuss each exercise in turn.

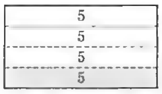
Name

$5 = \frac{?}{3}$

$5 = \frac{3 \times 5}{3}$

$5 = \frac{15}{3}$

$5 = \frac{?}{4}$



$5 = \frac{4 \times 5}{4}$

$5 = \frac{20}{4}$

Write a fraction for the whole number.

1. $7 = \frac{35}{5}$ 2. $7 = \frac{42}{6}$ 3. $7 = \frac{49}{7}$

4. $7 = \frac{56}{8}$ 5. $11 = \frac{99}{9}$ 6. $11 = \frac{110}{10}$

7. $11 = \frac{121}{11}$ 8. $11 = \frac{132}{12}$ 9. $16 = \frac{48}{3}$

10. $16 = \frac{80}{5}$ 11. $16 = \frac{96}{6}$ 12. $16 = \frac{112}{7}$

G-227

What is the fraction for $5\frac{3}{8}$?

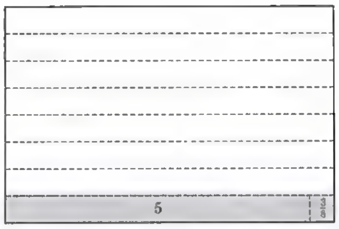
$5\frac{3}{8} = 5 + \frac{3}{8}$

$= \frac{8 \times 5}{8} + \frac{3}{8}$

$= \frac{40}{8} + \frac{3}{8}$

$= \frac{40 + 3}{8}$

$5\frac{3}{8} = \frac{43}{8}$



Write the fraction for each mixed fraction.

1. $2\frac{7}{10} = \frac{27}{10}$ 2. $4\frac{8}{9} = \frac{44}{9}$ 3. $10\frac{2}{3} = \frac{32}{3}$

4. $9\frac{3}{6} = \frac{57}{6}$ 5. $8\frac{2}{4} = \frac{34}{4}$ 6. $7\frac{5}{7} = \frac{54}{7}$

7. Ellen shared her collection of 56 leaves equally with 3 other girls. How many leaves does each girl have? 14 leaves

G-228

Name

Write the fraction for each mixed fraction.

1. $7\frac{3}{14} = \frac{101}{14}$ 2. $6\frac{5}{19} = \frac{119}{19}$ 3. $3\frac{7}{11} = \frac{40}{11}$

4. $2\frac{15}{18} = \frac{51}{18}$ 5. $40\frac{2}{3} = \frac{122}{3}$ 6. $21\frac{1}{2} = \frac{43}{2}$

7. $15\frac{2}{3} = \frac{47}{3}$ 8. $15\frac{3}{10} = \frac{153}{10}$ 9. $27\frac{2}{5} = \frac{137}{5}$

Name

What number is b?

6 | b
2 | 3

$b = 20$ Why? Because $(6 \times 3) + 2$ is 20

What fraction is $\frac{b}{a}$?

a | b
0 | $3 + \frac{2}{6}$

$\frac{b}{a} = \frac{20}{6}$ Why? Because $3 + \frac{2}{6}$ is $\frac{20}{6}$

1. 4 | b ; 3 | 2 ; $b =$ 11

2. 5 | b ; 3 | 3 ; $b =$ 18

3. a | b ; 0 | $8 + \frac{1}{2}$; $\frac{b}{a} = \frac{17}{2}$

4. 8 | b ; 2 | 10 ; $b =$ 82

5. 2 | b ; 1 | 20 ; $b =$ 41

6. a | b ; 0 | $4 + \frac{5}{6}$; $\frac{b}{a} = \frac{29}{6}$

7. 8 | b ; 6 | 9 ; $b =$ 78

8. 10 | b ; 9 | 8 ; $b =$ 89

9. a | b ; 0 | $10 + \frac{3}{11}$; $\frac{b}{a} = \frac{113}{11}$

10. a | b ; 0 | $3 + \frac{6}{7}$; $\frac{b}{a} = \frac{27}{7}$

11. a | b ; 0 | $9 + \frac{2}{3}$; $\frac{b}{a} = \frac{29}{3}$

12. a | b ; 0 | $8 + \frac{10}{20}$; $\frac{b}{a} = \frac{170}{20}$

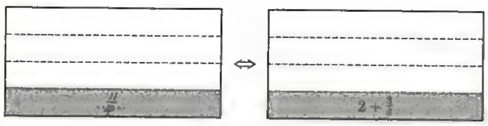
reference page

Write the fraction for each mixed fraction.

1. $3\frac{7}{9} = \frac{34}{9}$ 2. $8\frac{3}{5} = \frac{43}{5}$ 3. $10\frac{5}{8} = \frac{85}{8}$

4. $11\frac{1}{2} = \frac{23}{2}$ 5. $13\frac{2}{3} = \frac{41}{3}$ 6. $8\frac{2}{16} = \frac{130}{16}$

7. $32\frac{3}{4} = \frac{131}{4}$ 8. $14\frac{3}{10} = \frac{143}{10}$ 9. $7\frac{5}{8} = \frac{61}{8}$

10. 

For Class Discussion

1. $13\frac{3}{4}$ ounces of soap cost 29 cents.

13	$\frac{3}{4}$	25¢ 1¢ 1¢ 1¢ 1¢
13	$\frac{3}{4}$	25¢ 1¢ 1¢ 1¢ 1¢
13	$\frac{3}{4}$	25¢ 1¢ 1¢ 1¢ 1¢
13	$\frac{3}{4}$	25¢ 1¢ 1¢ 1¢ 1¢

55 ounces cost 116 cents. How do you know? $4 \times 13\frac{3}{4} = 55$, $4 \times 29¢ = 116¢$

How much does one ounce cost? $2\frac{6}{55}$ ¢

2. $2\frac{1}{2}$ pounds of sugar cost 23 cents.

2	$\frac{1}{2}$	10¢ 10¢ 1¢ 1¢ 1¢
2	$\frac{1}{2}$	10¢ 10¢ 1¢ 1¢ 1¢

How much does 5 pounds cost? 46¢

How much does 1 pound cost? $9\frac{1}{5}$ ¢

How do you know? $46 \div 5 = 9\frac{1}{5}$

3. $1\frac{1}{2}$ pounds of bread cost 35¢.

How much does one pound cost? $23\frac{1}{3}$ ¢

Draw a picture to show how you know.

1	$\frac{1}{2}$	10 10 10 1 1 1 1 1
1	$\frac{1}{2}$	10 10 10 1 1 1 1 1

3 lb. cost 70¢ so 1 lb. costs $\frac{70}{3}$ ¢ or $23\frac{1}{3}$ ¢

Supplemental Experiences

■ Prepare some 4 by 8 inch cards to fit in a pocket chart. On each card write a quotient or a mixed fraction. The following are suggested.

$\frac{8}{3}$	$8 \div 3$	$2\frac{2}{3}$
$\frac{3}{8}$	$3 \div 8$	$2\frac{2}{8}$
$\frac{15}{2}$	$15 \div 2$	$7\frac{1}{2}$
$\frac{2}{15}$	$2 \div 15$	$15\frac{1}{2}$
$\frac{17}{8}$	$17 \div 8$	$2\frac{1}{8}$
$\frac{8}{17}$	$8 \div 17$	$2\frac{1}{17}$

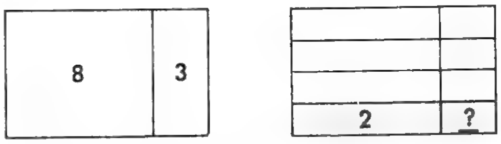
Arrange the cards at random in the pocket chart. Ask a pupil to select a quotient or a mixed fraction from the chart and hold it for his classmates to see. Then have another pupil find another card that names the same number and stand beside the first child. Let a third pupil try to find another card that names the same number. Some numbers are named on three cards; some, on only two; and some, on only one. Let the class decide whether the proper selections have been made.

■ The following is a suggested quiz.

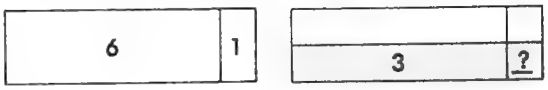
SUGGESTED QUIZ

Complete.

1.



2.



3. $4\overline{)9}$

8 | 2
1
1 | $\frac{1}{4}$
0 | $2 + \frac{1}{4}$

$\frac{9}{4} = 2 + \underline{?}$

$\frac{9}{4} = 2 + \frac{1}{4}$

4. $5\overline{)33}$

30 | 6
3
3 | $\frac{3}{5}$
0

$\frac{33}{5} = 6 + \underline{?}$

$\frac{33}{5} = 6 + \frac{3}{5}$

Copy and compute the mixed fraction.

5. $7\overline{)45}$ $\frac{45}{7} = \_\_\_$ $\frac{45}{7} = 6 + \frac{3}{7}$

6. $2\overline{)21}$ $\frac{21}{2} = \_\_\_$ $\frac{21}{2} = 10 + \frac{1}{2}$

7. $9\overline{)32}$ $\frac{32}{9} = \_\_\_$ $\frac{32}{9} = 3 + \frac{5}{9}$

8. $\frac{38}{8} = \frac{32}{8} + \frac{?}{8}$ $\frac{38}{8} = \frac{32}{8} + \frac{6}{8}$

$\frac{38}{8} = 4 + \frac{?}{8}$ $\frac{38}{8} = 4 + \frac{6}{8}$

9. $\frac{37}{5} = \frac{?}{5} + \frac{2}{5}$ $\frac{37}{5} = \frac{35}{5} + \frac{2}{5}$

10. $\frac{5 \times 7}{7} + \frac{3}{7} = 5 + \underline{?}$ $\frac{5 \times 7}{7} + \frac{3}{7} = 5 + \frac{3}{7}$

UNIT 17
DIVISION AND MULTIPLICATION
Pages 233 Through 266

OBJECTIVE
To practice division and multiplication.

The pupil practices the division algorism, using the greatest partial quotient and least remainder as replacements in the general division equation, $b = aq + r$. He computes $aq + r$ to check the accuracy of his division. He reviews mixed fractions and learns to check multiplication by dividing the product by one of the factors.

See Key Topics in Mathematics for the Intermediate Teacher: Multiplication and Division of Whole Numbers.

KEY IDEAS
Quotient times divisor equals dividend.

Division undoes multiplication; multiplication undoes division.

CONCEPTS
dividend
divisor

KEY IDEA
Quotient times divisor equals dividend.

Scope
To use multiplication to check division.

Fundamentals
Although many different partial quotients can be used in working a given division exercise, choosing greater partial quotients is more efficient. For example, procedures A and B below both result in the greatest partial quotient, but B does so in fewer steps.

A)

17) 457	
170	10
287	
170	10
117	
85	5
32	
17	1
15	26

B)

17) 457	
340	20
117	
102	6
15	26

The greatest partial quotient (26) is the number of 17's in 457.

The accuracy of division can be checked by multiplication, computing with the equation $b = aq + r$. In the preceding example, this equation becomes $457 = (17 \times 26) + 15$, which can be verified by computation.

The quotient-remainder algorism is extended to the exact-division algorism, in which the remainder is zero. Exact division is used to compute mixed fractions for quotients.

The development of skill in using the division algorism entails not only a thorough understanding of why the algorism works, but also extensive practice until use of the algorism is automatic. It is important that the child's program be a blend of exploration and practice.

Readiness for Understanding
Knowledge of division and multiplication.

Developmental Experiences
▶ Write the following on the chalkboard:

$$598 = 5q + r$$

Sally

5) 598	
500	100
88	
50	10
28	
25	5
3	115

$598 = (5 \times 115) + 3$

Ned

5) 598	
500	100
98	
90	10
8	
5	1
3	111

$598 = (5 \times 111) + 3$

Tom

5) 598	
500	100
98	
50	10
48	
45	9
3	119

$$598 = (5 \times 119) + 3$$

Point out that each pupil got a different result. Ask the pupils how they could check the results. If necessary, review computation with the quotient-remainder algorism. For example:

$$19 = (3 \times 6) + 1$$
Check: $3 \times 6 = 18$ and $18 + 1 = 19$

Then suggest that three pupils check the results using the multiplication algorism. Their work may look like this:

115	111	119
× 5	× 5	× 5
25	5	45
50	50	50
500	500	500
575	555	595
+ 3	+ 3	+ 3
578	558	598

The pupils should see that Tom's answer is correct. Ask them to find the errors in Sally's and Ned's work.

Next write other quotient-remainder exercises on the chalkboard. For example:

```
6) 799 |
   600 | 100
    99 |
    60 |  10
    39 |
    36 |   6
     3 | 116
```

$799 = (6 \times 116) + 3$

```
3) 1078 |
    900 | 300
    178 |
    150 |  50
     28 |
     27 |   9
      1 | 369
```

$1078 = (3 \times 369) + 1$

```
4) 958 |
   800 | 200
   158 |
   120 |  30
    38 |
    36 |   9
     2 | 239
```

$958 = (4 \times 239) + 2$

Direct the pupils to check each exercise. Then have them find the errors.

► Write the following on the chalkboard:

28) 768	28) 531	28) 1432	28) 756
28) 7631	28) 1853	28) 473	28) 5314
28) 1275	28) 1001	28) 6035	28) 1670

Point out that all the divisors are the same. Tell the class you want them to help build a table of multiples of 28, from 1×28 through 9×28. Put the table on the chalkboard. Have the pupils use addition to develop the table.

1×28	28
2×28	
3×28	
4×28	
5×28	
6×28	
7×28	
8×28	
9×28	

Then have a child come to the chalkboard and check the last entry by computing 9×28.

Ask some pupils to suggest ways of using the information in the table to give the standard numeral for each of the following products:

30×28	(840)
70×28	(1960)
300×28	(8400)
900×28	(25,200)
50×28	(1400)

Now use the table to work the first exercise on the chalkboard with the class.

```
28) 768 |
    560 | 20
    208 |
    196 |  7
     12 | 27
```

Inspect the table and note that 20×28 is 560.
Inspecting the table again, note that 196 is the greatest multiple of 28 less than 208.

Help the pupils see that by using the table they can complete the exercise in two steps. Follow the same procedure in working the second exercise with the class. Then have the pupils use the table to complete the remaining exercises independently.

► Write the following exercise on the chalkboard:

```
ab) cccab |
    cccxx | 300
       ax |
       a7 |   1
        0 | 301
```

Explain to the class that they are to replace the letters with digits so that the division is correct. Tell them that the *a's* each represent the same digit, the *b's* each represent the same digit, and the *c's* each represent the same digit, but that the *x's* may represent different digits. Allow a few minutes for questions and reflections on possible approaches, without giving any clues. When a pupil offers a suggestion, accept it, test it within the framework of the exercise, and allow the pupils to discuss its correctness or incorrectness.

These are some of the suggestions the pupils may offer:

The *x* above the 7 is a 7 because $7 - 7$ equals 0.

The *x's* below the *ab* are zeros because $300 \times ab$ must end in two zeros.

The *b* must be 7 because $b - 0$ equals 7.

After these suggestions the problem looks like this:

```
a7) ccca7 |
    ccc00 | 300
       a7 |
       a7 |   1
        0 | 301
```

If there are no more suggestions from the class, leave the problem on the chalkboard for some other time. It is likely that someone will suggest that *c* is 1, since 3×7 is 21.

```
a7) 111a7 |
    11100 | 300
       a7 |
       a7 |   1
        0 | 301
```

Now someone may observe that $3 \times a7$ equals 111, and that by testing some of the nine possible digits, the correct digit for *a* may be found.

$$\begin{array}{r} \square 7 \\ \times \ \ 3 \\ \hline 111 \end{array}$$

Once this digit is found, have a pupil write a quotient-remainder equation showing the solution.

$$11{,}137 = (37 \times 301) + 0$$

or

$$11{,}137 = (301 \times 37) + 0$$

Then have another child check the result by multiplying.

$$\begin{array}{r} 301 \\ \times\ 37 \\ \hline 7 \\ 2100 \\ 30 \\ 9000 \\ \hline 11{,}137 \end{array}$$

You may wish to put the following division exercise on the chalkboard and have the pupils try to solve it.

```
de) fffde |
    fffxx | 300
       xx |
       x4 |   1
        0 | 301
```

Pages 233 through 250

● Pages 233 through 235 provide practice in using the quotient-remainder algorism. Discuss with the class the example at the top of page 233. Introduce the words *dividend* and *divisor*. Explain that these words are standard terms used to refer to certain parts of division exercises. Have the students make a chart like the one below. Post the chart in the classroom so the pupils can refer to it.

dividend	=	divisor	×	greatest partial quotient	+	remainder
b	=	a	×	q	+	r
359	=	17	×	21	+	2

Have the pupils identify the dividend and the divisor in some of the exercises on pages 233 through 235. The students should then be able to complete the exercises independently. The number of steps used by the pupils in completing the exercises will vary. Be sure the children understand that though there are two ways of checking each division exercise, they need to use only one.

Name

UNIT 17 DIVISION AND MULTIPLICATION

See pupil page suggestions.

In the set of whole numbers, division is quotient-remainder division.

The division equation is $b = aq + r$.

```
         b
a → 17)359
       340 | 20
        19 |
        17 |  1
  r →    2 | 21 ← q
```

359 = (17 × 21) + 2

Does 359 equal (17 × 21) + 2?
Check either way:

$$\begin{array}{r} 21 \\ \times 17 \\ \hline 7 \\ 140 \\ 210 \\ \hline 357 \\ +2 \\ \hline 359 \end{array} \qquad \begin{array}{r} 17 \\ \times 21 \\ \hline 17 \\ 140 \\ 200 \\ \hline 357 \\ +2 \\ \hline 359 \end{array}$$

Compute. Check your results. *Partial quotient may vary.*

1. 21)826 — 420 | 20; 406; 210 | 10; 196; 189 | 9; 7 | 39
Check: 39 × 21: 9, 30, 180, 600, 819, +7, 826
826 = (21 × 39) + 7

2. 19)395 — 380 | 20; 15 | 20
Check: 20 × 19: 180, 200, 380, 15, 395
395 = (19 × 20) + 15

3. 15)357 — 300 | 20; 57; 45 | 3; 12 | 23
Check: 23 × 15: 15, 100, 30, 200, 345, +12, 357
357 = (15 × 23) + 12

4. 13)223 — 130 | 10; 93; 91 | 7; 2 | 17
Check: 17 × 13: 21, 30, 70, 100, 221, +2, 223
223 = (13 × 17) + 2

reference page

G-233

Compute. Check your results. *Partial quotients may vary.*

1. 16)547 — 16)547; 480 | 30; 67; 64 | 4; 3 | 34
Check: 34 × 16: 24, 180, 40, 300, 544, +3, 547
547 = (16 × 34) + 3

2. 18)957 — 18)957; 900 | 50; 57; 54 | 3; 3 | 53
Check: 53 × 18: 24, 400, 30, 500, 954, +3, 957
957 = (18 × 53) + 3

3. 23)1039 — 23)1039; 920 | 40; 119; 115 | 5; 4 | 45
Check: 45 × 23: 15, 120, 100, 800, 1035, +4, 1039
1039 = (23 × 45) + 4

4. 17)801 — 17)801; 680 | 40; 121; 119 | 7; 2 | 47
Check: 47 × 17: 49, 280, 70, 400, 799, +2, 801
801 = (17 × 47) + 2

5. 12)496 — 12)496; 480 | 40; 16; 12 | 1; 4 | 41
Check: 41 × 12: 2, 80, 10, 400, 492, +4, 496
496 = (12 × 41) + 4

6. 31)448 — 31)448; 310 | 10; 138; 124 | 4; 14 | 14
Check: 14 × 31: 4, 10, 120, 300, 434, +14, 448
448 = (31 × 14) + 14

7. 14)527 — 14)527; 420 | 30; 107; 98 | 7; 9 | 37
Check: 37 × 14: 28, 120, 70, 300, 518, +9, 527
527 = (14 × 37) + 9

8. 15)681 — 15)681; 600 | 40; 81; 75 | 5; 6 | 45
Check: 45 × 15: 25, 200, 50, 400, 675, +6, 681
681 = (15 × 45) + 6

G-234

Name

Compute. Check your results. *Partial quotients may vary.*

1. $15\overline{)517}$ Check: $34 \times 15 = 20 + 150 + 40 + 300 = 510$; $510 + 7 = 517$

   $15\overline{)517}$: 450 | 30; 67; 60 | 4; 7 | 34

   $517 = (15 \times 34) + 7$

2. $41\overline{)826}$ Check: $41 \times 20 = 20 + 800 = 820$; $820 + 6 = 826$

   $41\overline{)826}$: 820 | 20; 6 | 20

   $826 = (41 \times 20) + 6$

3. $33\overline{)642}$ Check: $19 \times 33 = 27 + 30 + 270 + 300 = 627$; $627 + 15 = 642$

   $33\overline{)642}$: 330 | 10; 312; 297 | 9; 15 | 19

   $642 = (33 \times 19) + 15$

4. $40\overline{)723}$ Check: $18 \times 40 = 320 + 400 = 720$; $720 + 3 = 723$

   $40\overline{)723}$: 400 | 10; 323; 320 | 8; 3 | 18

   $723 = (40 \times 18) + 3$

5. $50\overline{)3192}$ Check: $63 \times 50 = 150 + 3000 = 3150$; $3150 + 42 = 3192$

   $50\overline{)3192}$: 3000 | 60; 192; 150 | 3; 42 | 63

   $3192 = (50 \times 63) + 42$

6. $62\overline{)1293}$ Check: $20 \times 62 = 40 + 1200 = 1240$; $1240 + 53 = 1293$

   $62\overline{)1293}$: 1240 | 20; 53 | 20

   $1293 = (62 \times 20) + 53$

7. $43\overline{)2708}$ Check: $62 \times 43 = 6 + 180 + 80 + 2400 = 2666$; $2666 + 42 = 2708$

   $43\overline{)2708}$: 2580 | 60; 128; 86 | 2; 42 | 62

   $2708 = (43 \times 62) + 42$

8. $67\overline{)5102}$ Check: $76 \times 67 = 42 + 490 + 360 + 4200 = 5092$; $5092 + 10 = 5102$

   $67\overline{)5102}$: 4690 | 70; 412; 402 | 6; 10 | 76

   $5102 = (67 \times 76) + 10$

reference page

G-235

● Pages 236 and 237 provide a review of computing a mixed fraction for a fraction and of computing a fraction for a mixed fraction. Discuss the example at the top of page 236. First review the idea that $547 \div 36$ and $\frac{547}{36}$ are the same number. Then discuss computing a mixed fraction for $\frac{547}{36}$. One mixed fraction for $\frac{547}{36}$ is $15\frac{7}{36}$. Also discuss computing a fraction for $15\frac{7}{36}$. The pupils should recall that $15\frac{7}{36}$ means $15 + \frac{7}{36}$. They should also see that 15 is $\frac{36 \times 15}{36}$ and that $15 + \frac{7}{36}$ is $\frac{36 \times 15}{36} + \frac{7}{36}$, or $\frac{547}{36}$.

Assign the exercises on pages 236 and 237 for independent practice. Ask the pupils to complete the first two exercises on page 236. Tell them they need not write every step, in the other exercises, unless they wish to.

● Pages 238 and 239 provide the children with more practice in computing a mixed fraction for a fraction and a fraction for a mixed fraction.

Referring to the mixed fractions machine at the top of page 238, help the pupils see that putting a fraction into this machine results in a mixed fraction for the same number. Tell the pupils to imagine they are putting each fraction in exercises 1 through 8 into the mixed fractions machine. In each instance they are to tell what comes out of the machine. Some pupils will use the division algorism to compute the mixed fractions; others may do the exercises without it.

Next refer to the fractions machine in the middle of page 238. The children should observe that putting a mixed fraction into this machine results in a fraction for the same number. Then assign exercises 9 through 16 for independent work. The pupils should write only the steps necessary for their individual computation.

Refer to the illustration and paragraph at the top of page 239. Have individuals explain how to use the machines for the fractions in exercises 1 through 9. Then tell the pupils to complete exercises 1 through 9 independently. Follow the same procedure with the machines and exercises on the bottom half of the page. When the pupils have completed the exercises, discuss the questions at the bottom of the page, helping the pupils see that the result of the two computations is the original number. The fractions machine undoes the work of the mixed fractions machine, and the mixed fractions machine undoes the work of the fractions machine.

● To introduce page 240, ask the class to discuss the question. The pupils should explain that $5 \times 2\frac{1}{5}$ equals 11 because 5×2 is 10, $5 \times \frac{1}{5}$ is 1, and $10 + 1$ is 11. Let some pupils work exercises 1 through 12 at the chalkboard, and discuss their results.

Fractional numbers are quotients of whole numbers. *See pupil page suggestions.*

If $a \times \square = b$, then $\square = \frac{b}{a}$; $a \times \frac{b}{a} = b$.

What is a fraction for $547 \div 36$?

$$547 \div 36 = \frac{547}{36}$$

What is a mixed fraction for $547 \div 36$?

$36\overline{)547}$: 360 | 10; 187; 180 | 5; 7; 7 | $\frac{7}{36}$; 0 | $15 + \frac{7}{36}$

$$547 \div 36 = 15\frac{7}{36}$$

What is a fraction for $15\frac{7}{36}$?

$$15 + \frac{7}{36} = \frac{36 \times 15}{36} + \frac{7}{36} = \frac{540 + 7}{36}$$

$$15\frac{7}{36} = \frac{547}{36}$$

$36 \times 15 = 30 + 150 + 360 = 540$

Write a fraction for the mixed fraction.

1. $12\frac{2}{3}$ $\quad 12 + \frac{2}{3} = \frac{12 \times 3}{3} + \frac{2}{3} = \frac{38}{3}$

2. $10\frac{4}{5}$ $\quad 10 + \frac{4}{5} = \frac{10 \times 5}{5} + \frac{4}{5} = \frac{54}{5}$

3. $6\frac{11}{12}$ $\quad \frac{83}{12}$

4. $20\frac{3}{4}$ $\quad \frac{83}{4}$

5. $9\frac{3}{8}$ $\quad \frac{75}{8}$

6. $11\frac{1}{6}$ $\quad \frac{67}{6}$

reference page

G-236

Name

Write a fraction for the mixed fraction.

1. $23\frac{4}{5} = \frac{119}{5}$

2. $31\frac{1}{2} = \frac{63}{2}$

3. $18\frac{2}{7} = \frac{128}{7}$

4. $52\frac{2}{3} = \frac{158}{3}$

Write a mixed fraction for the fraction.

5. $\frac{47}{2} = 23\frac{1}{2}$

6. $\frac{47}{3} = 15\frac{2}{3}$

7. $\frac{65}{2} = 32\frac{1}{2}$

8. $\frac{65}{3} = 21\frac{2}{3}$

9. $\frac{168}{13} = 12\frac{12}{13}$

10. $\frac{428}{16} = 26\frac{12}{16}$

11. $\frac{143}{12} = 11\frac{11}{12}$

12. $\frac{840}{29} = 28\frac{28}{29}$

13. $\frac{514}{25} = 20\frac{14}{25}$

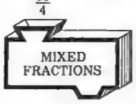
14. $\frac{93}{17} = 5\frac{8}{17}$

15. $\frac{1217}{35} = 34\frac{27}{35}$

16. $\frac{4982}{111} = 44\frac{98}{111}$

Name

$\frac{31}{4}$



$7\frac{3}{4}$

$\frac{31}{4}$

The machines are arranged so the mixed fractions from the mixed fractions machine will spill into the fractions machine.

Compute the mixed fraction.
Then compute the fraction.

1. $\frac{8}{7}$ $1\frac{1}{7}$, $\frac{8}{7}$ 2. $\frac{10}{7}$ $1\frac{3}{7}$, $\frac{10}{7}$ 3. $\frac{10}{2}$ $5\frac{0}{2}$, $\frac{10}{2}$

4. $\frac{31}{5}$ $6\frac{1}{5}$, $\frac{31}{5}$ 5. $\frac{45}{9}$ $5\frac{0}{9}$, $\frac{45}{9}$ 6. $\frac{27}{8}$ $3\frac{3}{8}$, $\frac{27}{8}$

7. $\frac{46}{6}$ $7\frac{4}{6}$, $\frac{46}{6}$ 8. $\frac{75}{10}$ $7\frac{5}{10}$, $\frac{75}{10}$ 9. $\frac{93}{12}$ $7\frac{9}{12}$, $\frac{93}{12}$

What is the result of the two computations? Why? The final result is the same as the original fraction; one machine undoes what the other machine does.

Now the machines are arranged so the fractions from the fractions machine will spill into the mixed fractions machine.

Compute the fraction.
Then compute the mixed fraction.

$2\frac{1}{3}$

$\frac{7}{3}$

$2\frac{1}{3}$

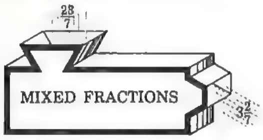
10. $3\frac{3}{4}$ $\frac{15}{4}$, $3\frac{3}{4}$ 11. $5\frac{2}{3}$ $\frac{17}{3}$, $5\frac{2}{3}$ 12. $9\frac{4}{7}$ $\frac{67}{7}$, $9\frac{4}{7}$

13. $12\frac{1}{2}$ $\frac{25}{2}$, $12\frac{1}{2}$ 14. $6\frac{3}{10}$ $\frac{63}{10}$, $6\frac{3}{10}$ 15. $8\frac{5}{9}$ $\frac{77}{9}$, $8\frac{5}{9}$

16. $4\frac{6}{7}$ $\frac{34}{7}$, $4\frac{6}{7}$ 17. $7\frac{0}{9}$ $\frac{63}{9}$, $7\frac{0}{9}$ 18. $27\frac{3}{11}$ $\frac{300}{11}$, $27\frac{3}{11}$

What is the result of the two computations? Why? The final result is the same as the original mixed fraction; one machine undoes what the other machine does.

$\frac{28}{7}$



If a fraction is put into the mixed fractions machine, a mixed fraction comes out.

What mixed fraction will come out of the machine?

1. $\frac{8}{3} = 2\frac{2}{3}$

2. $\frac{8}{4} = 2\frac{0}{4}$

3. $\frac{32}{7} = 4\frac{4}{7}$

4. $\frac{32}{9} = 3\frac{5}{9}$

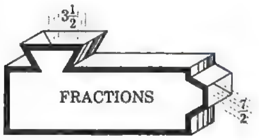
5. $\frac{32}{8} = 4\frac{0}{8}$

6. $\frac{18}{4} = 4\frac{2}{4}$

7. $\frac{18}{1} = 18\frac{0}{1}$

8. $\frac{26}{12} = 2\frac{2}{12}$

$3\frac{1}{2}$



If a mixed fraction is put into the fractions machine, a fraction comes out.

What fraction will come out of the machine?

9. $2\frac{1}{4} = \frac{9}{4}$

10. $4\frac{3}{5} = \frac{23}{5}$

11. $6\frac{5}{7} = \frac{47}{7}$

12. $3\frac{0}{3} = \frac{9}{3}$

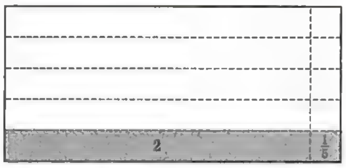
13. $8\frac{1}{2} = \frac{17}{2}$

14. $7\frac{5}{10} = \frac{75}{10}$

15. $5\frac{3}{8} = \frac{43}{8}$

16. $14\frac{2}{9} = \frac{128}{9}$

See pupil page suggestions.



$5 \times 2\frac{1}{5} = 11$ Why?

The number for the whole region is $5 \times 2\frac{1}{5}$. $5 \times 2 = 10$ and $5 \times \frac{1}{5} = 1$; $10 + 1 = 11$.

Compute.

1. $5 \times 3\frac{1}{5} =$ 16 2. $3 \times 4\frac{1}{3} =$ 13 3. $5 \times 6\frac{1}{5} =$ 31 4. $3 \times 5\frac{2}{3} =$ 17

5. $4 \times 6\frac{1}{4} =$ 25 6. $6 \times 4\frac{2}{6} =$ 26 7. $4 \times 7\frac{3}{4} =$ 31 8. $5 \times 5\frac{3}{5} =$ 28

9. $8 \times 2\frac{3}{8} =$ 19 10. $8 \times 4\frac{5}{8} =$ 37 11. $7 \times 6\frac{2}{7} =$ 44 12. $2 \times 10\frac{1}{2} =$ 21

● On page 241, the multiplication algorism is used to compute $47 \times 39\frac{13}{47}$. Ask pupils to explain the computation of each partial product. You may wish to provide another example, such as $49\frac{2}{15} \times 15$ or $67\frac{1}{38} \times 38$. Then, with the pupils, work each exercise on the page.

● Pages 242 through 246 contain exercises that involve computing mixed fractions and checking division by multiplication.

With the class, work the example at the top of page 242. In discussing the check, review the fact that no computation is required for a pupil to know that $28 \times \frac{4}{28}$ is 4. $\frac{4}{28}$ is 4 divided by 28 and is therefore the number that gives 4 when multiplied by 28. Help the pupils understand that the sum of the partial products in the check is the original dividend, thus indicating that the answer $32\frac{4}{28}$ is correct.

Use exercises 1 through 9 on page 242 as a class activity. Then assign exercises 1 through 6 on page 243 for independent work. For each exercise have the pupils compute the mixed fraction, write an equation that relates the fraction and the mixed fraction, rewrite the division equation as a multiplication equation, and check by multiplying.

● Work the example at the top of page 244 with the class. Then have them complete and discuss exercises 1 through 5 as a class activity. The exercises on pages 245 and 246 may be assigned for independent work.

● Pages 247 through 250 give the children practice in using the division algorism to compute mixed fractions. Encourage them to use multiples of 10, 100, and 1000 as partial quotients. Assign as many of the exercises at one time as is appropriate for the class.

Name

How can we use the multiplication algorism to compute $47 \times 39\frac{13}{47}$?

$$\begin{array}{rl} 39\frac{13}{47} & \\ \times\ 47 & \\ \hline 13 & (47 \times \frac{13}{47}) \\ 63 & (7 \times \underline{\ 9\ }) \\ 210 & (\underline{\ 7\ } \times 3 \text{ tens}) \\ 360 & (4 \text{ tens} \times \underline{\ 9\ }) \\ 1200 & (\underline{4 \text{ tens}} \times \underline{3 \text{ tens}}) \\ \hline 1846 & \end{array}$$

Compute. *

1.
$$\begin{array}{rl} 18\frac{5}{14} & \\ \times\ 14 & \\ \hline 5 & (14 \times \frac{5}{14}) \\ 32 & (8 \times 4) \\ 40 & (10 \times 4) \\ 80 & (10 \times 8) \\ 100 & (10 \times 10) \\ \hline 257 & \end{array}$$

2.
$$\begin{array}{rl} 31\frac{5}{61} & \\ \times\ 61 & \\ \hline 5 & (61 \times \frac{5}{61}) \\ 1 & (1 \times 1) \\ 30 & (1 \times 30) \\ 60 & (60 \times 1) \\ 1800 & (60 \times 30) \\ \hline 1896 & \end{array}$$

3.
$$\begin{array}{rl} 181\frac{11}{32} & \\ \times\ 32 & \\ \hline 11 & (32 \times \frac{11}{32}) \\ 2 & (2 \times 1) \\ 160 & (2 \times 80) \\ 200 & (2 \times 100) \\ 30 & (30 \times 1) \\ 2400 & (30 \times 80) \\ 3000 & (30 \times 100) \\ \hline 5803 & \end{array}$$

4.
$$\begin{array}{rl} 247\frac{25}{99} & \\ \times\ 99 & \\ \hline 25 & (99 \times \frac{25}{99}) \\ 63 & (9 \times 7) \\ 360 & (9 \times 40) \\ 1800 & (9 \times 200) \\ 630 & (90 \times 7) \\ 3600 & (90 \times 40) \\ 18000 & (90 \times 200) \\ \hline 24{,}478 & \end{array}$$

reference page

* Order of partial products may vary.

We divide to compute a mixed fraction for a fraction.

What is a mixed fraction for $\frac{900}{28}$?

$$\begin{array}{r|r|l} 28 & 900 & \\ & 840 & 30 \\ \hline & 60 & \\ & 56 & 2 \\ \hline & 4 & \\ & 4 & \frac{4}{28} \\ \hline & 0 & 32\frac{4}{28} \end{array}$$

$900 \div 28 = 32\frac{4}{28}$

Check: Does $900 = 28 \times 32\frac{4}{28}$?

$$\begin{array}{rl} 32\frac{4}{28} & \\ \times\ 28 & \\ \hline 4 & (28 \times \frac{4}{28}) \\ 16 & \\ 240 & (28 \times 32) \\ 40 & \\ 600 & \\ \hline 900 & (28 \times 32\frac{4}{28}) \end{array}$$

Compute.

1. $2 \times \frac{3}{2} = \underline{\ 3\ }$
2. $8 \times \frac{7}{8} = \underline{\ 7\ }$
3. $15 \times \frac{13}{15} = \underline{\ 13\ }$
4. $68 \times \frac{31}{68} = \underline{\ 31\ }$
5. $125 \times \frac{7}{125} = \underline{\ 7\ }$
6. $\frac{3}{4} \times 4 = \underline{\ 3\ }$
7. $392 \times \frac{127}{392} = \underline{\ 127\ }$
8. $\frac{93}{152} \times 152 = \underline{\ 93\ }$
9. $1428 \times \frac{243}{1428} = \underline{\ 243\ }$

reference page

Name

Compute a mixed fraction. Check. *Partial products may vary.*

1. $\frac{799}{6}$ $133\frac{1}{6}$ Check:
$$\begin{array}{r} 133\frac{1}{6} \\ \times 6 \\ \hline 1 \\ 18 \\ 180 \\ 600 \\ \hline 799 \end{array}$$

2. $\frac{958}{4}$ $239\frac{2}{4}$ Check:
$$\begin{array}{r} 239\frac{2}{4} \\ \times 4 \\ \hline 2 \\ 36 \\ 120 \\ 800 \\ \hline 958 \end{array}$$

3. $\frac{1078}{3}$ $359\frac{1}{3}$ Check:
$$\begin{array}{r} 359\frac{1}{3} \\ \times 3 \\ \hline 1 \\ 27 \\ 150 \\ 900 \\ \hline 1078 \end{array}$$

4. $272 \div 37$ $7\frac{13}{37}$ Check:
$$\begin{array}{r} 7\frac{13}{37} \\ \times 37 \\ \hline 13 \\ 49 \\ 210 \\ \hline 272 \end{array}$$

5. $341 \div 17$ $20\frac{1}{17}$ Check:
$$\begin{array}{r} 20\frac{1}{17} \\ \times 17 \\ \hline 1 \\ 140 \\ 200 \\ \hline 341 \end{array}$$

6. $875 \div 19$ $46\frac{1}{19}$ Check:
$$\begin{array}{r} 46\frac{1}{19} \\ \times 19 \\ \hline 1 \\ 54 \\ 360 \\ 60 \\ 400 \\ \hline 875 \end{array}$$

G-243

Name

Compute a mixed fraction. Check. *Partial products may vary.*

1. $\frac{543}{96} = 5\frac{63}{96}$ Check:
$$\begin{array}{r} 5\frac{63}{96} \\ \times 96 \\ \hline 63 \\ 30 \\ 450 \\ \hline 543 \end{array}$$

2. $\frac{315}{33} = 9\frac{18}{33}$ Check:
$$\begin{array}{r} 9\frac{18}{33} \\ \times 33 \\ \hline 18 \\ 27 \\ 270 \\ \hline 315 \end{array}$$

3. $\frac{827}{27} = 30\frac{17}{27}$ Check:
$$\begin{array}{r} 30\frac{17}{27} \\ \times 27 \\ \hline 17 \\ 210 \\ 600 \\ \hline 827 \end{array}$$

4. $\frac{555}{45} = 12\frac{15}{45}$ Check:
$$\begin{array}{r} 12\frac{15}{45} \\ \times 45 \\ \hline 15 \\ 10 \\ 50 \\ 80 \\ 400 \\ \hline 555 \end{array}$$

5. $\frac{9321}{31} = 300\frac{21}{31}$ Check:
$$\begin{array}{r} 300\frac{21}{31} \\ \times 31 \\ \hline 21 \\ 300 \\ 9000 \\ \hline 9321 \end{array}$$

6. $\frac{8765}{25} = 350\frac{15}{25}$ Check:
$$\begin{array}{r} 350\frac{15}{25} \\ \times 25 \\ \hline 15 \\ 250 \\ 1500 \\ 1000 \\ 6000 \\ \hline 8765 \end{array}$$

G-245

Compute a mixed fraction for $\frac{826}{35}$.

$$\begin{array}{r|r|l} 35 & 826 & \\ & 700 & 20 \\ \hline & 126 & \\ & 105 & 3 \\ \hline & 21 & \\ & 21 & \frac{21}{35} \\ \hline & 0 & 23\frac{21}{35} \end{array}$$

$\frac{826}{35} = 23\frac{21}{35}$

Check:
$$\begin{array}{rl} 23\frac{21}{35} & \\ \times 35 & \\ \hline 21 & (35 \times \frac{21}{35}) \\ 15 & \\ 100 & (35 \times 23) \\ 90 & \\ 600 & \\ \hline 826 & (35 \times 23\frac{21}{35}) \end{array}$$

$826 = 35 \times 23\frac{21}{35}$

Compute a mixed fraction. Check. *Partial products may vary.*

1. $\frac{693}{33}$ $21\frac{0}{33}$ Check:
$$\begin{array}{r} 21\frac{0}{33} \\ \times 33 \\ \hline 3 \\ 60 \\ 30 \\ 600 \\ \hline 693 \end{array}$$

2. $\frac{495}{29}$ $17\frac{2}{29}$ Check:
$$\begin{array}{r} 17\frac{2}{29} \\ \times 29 \\ \hline 2 \\ 63 \\ 90 \\ 140 \\ 200 \\ \hline 495 \end{array}$$

3. $\frac{756}{19}$ $39\frac{15}{19}$ Check:
$$\begin{array}{r} 39\frac{15}{19} \\ \times 19 \\ \hline 15 \\ 81 \\ 270 \\ 90 \\ 300 \\ \hline 756 \end{array}$$

4. $\frac{459}{37}$ $12\frac{15}{37}$ Check:
$$\begin{array}{r} 12\frac{15}{37} \\ \times 37 \\ \hline 15 \\ 14 \\ 70 \\ 60 \\ 300 \\ \hline 459 \end{array}$$

5. Mark fed his dogs the same amount of food each day. If he used 15 pounds of food in 7 days, how much did he use each day? *$\frac{15}{7}$ or $2\frac{1}{7}$ pounds*

G-244

Compute a mixed fraction. Check.

1. $\frac{7864}{37} = 212\frac{20}{37}$

2. $\frac{2173}{41} = 53\frac{0}{41}$

3. $\frac{6653}{300} = 22\frac{53}{300}$

4. $\frac{8749}{29} = 301\frac{20}{29}$

5. $\frac{8054}{210} = 38\frac{74}{210}$

6. Karen baked a batch of cookies every day for 5 days. She used the same amount of flour each day. If she used 12 cups of flour in 5 days, how much did she use each day? *$\frac{12}{5}$ or $2\frac{2}{5}$ cups*

G-246

Name

Compute each mixed fraction. *Partial quotients may vary.*

1. $\frac{9764}{25} = 390\frac{14}{25}$

25 ⟌ 9764	
7500	300
2264	
2250	90
14	390

2. $\frac{9764}{35} = 278\frac{34}{35}$

35 ⟌ 9764	
7000	200
2764	
2450	70
314	
280	8
34	278

3. $\frac{9764}{45} = 216\frac{44}{45}$

45 ⟌ 9764	
9000	200
764	
450	10
314	
270	6
44	216

4. $\frac{9764}{55} = 177\frac{29}{55}$

55 ⟌ 9764	
5500	100
4264	
3850	70
414	
385	7
29	177

5. In 6 days Steven walked 32 miles delivering papers. He walked the same distance each day. How far did he walk each day to deliver papers? $\frac{32}{6}$ or $5\frac{2}{6}$ miles

Name

Compute each mixed fraction. *Partial quotients may vary.*

1. 8 ⟌ 488 $\frac{488}{8} = 61\frac{0}{8}$

8 ⟌ 488	
480	60
8	
8	1
0	1

2. 16 ⟌ 488 $\frac{488}{16} = 30\frac{8}{16}$

16 ⟌ 488	
320	20
168	
160	10
8	30

3. 80 ⟌ 4880 $\frac{4880}{80} = 61\frac{0}{80}$

80 ⟌ 4880	
4800	60
80	
80	1
0	61

4. 800 ⟌ 48800 $\frac{48800}{800} = 61\frac{0}{800}$

800 ⟌ 48800	
48000	60
800	
800	1
0	61

5. 160 ⟌ 4880 $\frac{4880}{160} = 30\frac{80}{160}$

160 ⟌ 4880	
4800	30
80	30

6. 1600 ⟌ 48800 $\frac{48800}{1600} = 30\frac{800}{1600}$

1600 ⟌ 48800	
48000	30
800	30

Compute each mixed fraction. *Partial quotients may vary.*

1. 65 ⟌ 9764 $\frac{9764}{65} = 150\frac{14}{65}$

65 ⟌ 9764	
6500	100
3264	
3250	50
14	150

2. 75 ⟌ 9764 $\frac{9764}{75} = 130\frac{14}{75}$

75 ⟌ 9764	
7500	100
2264	
2250	30
14	130

3. 85 ⟌ 9764 $\frac{9764}{85} = 114\frac{74}{85}$

85 ⟌ 9764	
8500	100
1264	10
850	
414	
340	4
74	114

4. 95 ⟌ 9764 $\frac{9764}{95} = 102\frac{74}{95}$

95 ⟌ 9764	
9500	100
264	
190	2
74	102

5. 105 ⟌ 9764 $\frac{9764}{105} = 92\frac{104}{105}$

105 ⟌ 9764	
9450	90
314	
210	2
104	92

6. 115 ⟌ 9764 $\frac{9764}{115} = 84\frac{104}{115}$

115 ⟌ 9764	
9200	80
564	
460	4
104	84

Compute each mixed fraction. *Partial quotients may vary.*

1. 32 ⟌ 488 $\frac{488}{32} = 15\frac{8}{32}$

32 ⟌ 488	
320	10
168	
160	5
8	15

2. 320 ⟌ 4880 $\frac{4880}{320} = 15\frac{80}{320}$

320 ⟌ 4880	
3200	10
1680	
1600	5
80	15

3. 44 ⟌ 2401 $\frac{2401}{44} = 54\frac{25}{44}$

44 ⟌ 2401	
2200	50
201	
176	4
25	54

4. 45 ⟌ 2401 $\frac{2401}{45} = 53\frac{16}{45}$

45 ⟌ 2401	
2250	50
151	
135	3
16	53

5. 46 ⟌ 2401 $\frac{2401}{46} = 52\frac{9}{46}$

46 ⟌ 2401	
2300	50
101	
92	2
9	52

6. 47 ⟌ 2401 $\frac{2401}{47} = 51\frac{4}{47}$

47 ⟌ 2401	
2350	50
51	
47	1
4	51

Supplemental Experiences

■ Copy the following tables on the chalkboard:

A

1 × 36	36
2 × 36	
4 × 36	
8 × 36	

B

1 × 36	
2 × 36	
3 × 36	
4 × 36	
5 × 36	
6 × 36	
7 × 36	
8 × 36	
9 × 36	

Have a pupil complete table A using addition. For example, 2 × 36 is 36 + 36; 4 × 36 is twice 2 × 36, or 72 + 72; 8 × 36 is twice 4 × 36, or 144 + 144. Have another child check the final multiple by computing 8 × 36. Then ask the other pupils for suggestions on how to complete table B using the information in table A. One pupil may suggest that 9 × 36 is (8 × 36) + (1 × 36), or 288 + 36. Try any suggestions the pupils make.

After table B has been completed, have the class work independently on the following division exercises:

$36\overline{)856}$ $36\overline{)983}$ $36\overline{)12467}$ $36\overline{)9863}$

$36\overline{)1205}$ $36\overline{)10001}$ $36\overline{)839}$ $36\overline{)45063}$

Follow the procedure just discussed to develop tables and exercises in which 57 is the divisor.

■ Write the following rows of exercises on the chalkboard:

$26\overline{)728}$ 0 | 28 $28\overline{)1093}$ 1 | 39 $39\overline{)9873}$ 6 | 253

$9\overline{)1843}$ 7 | 204 $204\overline{)6353}$ 29 | 31 $31\overline{)8179}$ 26 | 263

$83\overline{)976}$ 63 | 11 $11\overline{)7696}$ 7 | 699 $699\overline{)83,313}$ 132 | 119

$443\overline{)789,000}$ 17 | 1781 $1781\overline{)93,456}$ 844 | 52 $52\overline{)2783}$ 27 | 53

$8\overline{)6983}$ 7 | 872 $872\overline{)65,401}$ 1 | 75 $75\overline{)525}$ 0 | 7

Suggest to the class that in each row they use the greatest partial quotient from the first exercise as the divisor in the second exercise, and the greatest partial quotient from the second exercise as the divisor in the third exercise.

■ On the chalkboard write part of the check for a division exercise.

$$\begin{array}{r} 4\ 3 \\ \times \square\square \\ \hline 1\ 8\ 0\ 6 \\ + \square\square \\ \hline 1\ 8\ 2\ 1 \end{array}$$

Have two pupils write the divisor and dividend in the division algorism. Then have the division algorism completed and the missing information written in the steps of the check.

$$\begin{array}{r|l} 43\overline{)1821} & \\ 1720 & 40 \\ \hline 101 & \\ 86 & 2 \\ \hline 15 & 42 \end{array} \qquad \begin{array}{r} 43 \\ \times 42 \\ \hline 1806 \\ + 15 \\ \hline 1821 \end{array}$$

KEY IDEA

Division undoes multiplication; multiplication undoes division.

Scope

To practice using multiplication as a check for division and division as a check for multiplication.

Fundamentals

The algorism for multiplication is based on a knowledge of the basic multiplication combinations, numeration, and the distributive property. Because of the algorism's structure, there are shortcuts that can be emphasized to increase efficiency.

Pupils should develop the ability to see that the computation of such products as 9 × 80 involves the combination 9 × 8 tens = 72 tens and the fact that 72 tens is 720. The children are aware that the number and order of partial products used in the multiplication algorism does not affect the final answer. They learn that two or more partial products may be consolidated into one.

Since division undoes multiplication—

$$(6 \times 3) \div 3 \text{ is } 6$$

—and multiplication undoes division—

$$(6 \div 3) \times 3 \text{ is } 6$$

—division may be used to check multiplication, and multiplication may be used to check division.

Readiness for Understanding

Knowledge of multiplication.

Developmental Experiences

felt-tip pen
tagboard cards
box

► Make a set of cards showing exercises like these, one exercise to a card.

$$\begin{array}{r} 30 \\ \times\ 9 \\ \hline \end{array} \quad \begin{array}{r} 600 \\ \times\ \ 7 \\ \hline \end{array} \quad \begin{array}{r} 20 \\ \times 90 \\ \hline \end{array} \quad \begin{array}{r} 300 \\ \times\ 60 \\ \hline \end{array} \quad \begin{array}{r} 700 \\ \times 300 \\ \hline \end{array} \quad \begin{array}{r} 9 \\ \times 4 \\ \hline \end{array}$$

Place the cards in a box. Have a pupil select one of the cards, read the exercise, and give the standard numeral for the product. Tell him he may write the exercise on the chalkboard and compute the product, or he may do the computation mentally.

As the pupils work with the various multiples of 10, 100, and 1000, help them see that when each factor involves tens, the product involves hundreds: tens × tens = ten tens, or hundreds. Have this information written on a chart.

Give each pupil an opportunity to compute a product. The activity should result in the following information:

ones × ones = ones
ones × tens = tens
ones × hundreds = hundreds
tens × tens = ten tens, or hundreds
tens × hundreds = ten hundreds, or thousands
hundreds × hundreds = hundred hundreds, or ten thousands

This chart should be posted so that the pupils can refer to it when necessary.

► Write 76 × 43 in vertical form on the chalkboard. To the right of the exercise show the expanded numeral for both factors. Call on a pupil to compute the ones and record his result. Have two other pupils compute the tens, a fourth pupil compute the hundreds, and another pupil write the computed sum of these partial products.

$$\begin{array}{rcrcr} 76 & = & 70 + 6 & = & 7 \text{ tens} + 6 \\ \underline{\times\ 43} & = & \underline{40 + 3} & = & \underline{4 \text{ tens} + 3} \\ & & & & 18 \\ & & & & 210 \\ & & & & 240 \\ & & & & \underline{2800} \\ & & & & 3268 \end{array}$$

Review with the class the fact that in computing the product it is possible to record fewer partial products than was just done. Next to the computation still on the chalkboard, write again the exercise 76 × 43 in vertical form. Again have someone compute the product of the ones and record the result. Then tell the class to compute the product 3 × 7 tens and the product 4 tens × 6. Have them add the two products, and call on a volunteer to write the sum (45 tens) beneath the 18.

$$\begin{array}{rcrcr} 76 & = & 70 + 6 & = & 7 \text{ tens} + 6 \\ \underline{\times\ 43} & = & \underline{40 + 3} & = & \underline{4 \text{ tens} + 3} \\ & & & & 18 \\ & & & & 210 \\ & & & & 240 \\ & & & & \underline{2800} \\ & & & & 3268 \end{array} \qquad \begin{array}{r} 76 \\ \underline{\times\ 43} \\ 18 \\ 450 \\ \underline{2800} \\ 3268 \end{array}$$

Ask someone else to compute the hundreds and record his result. Then call on a volunteer to compute and record the sum of the partial products. Relate the two partial products, 210 and 240, in the expanded algorism to the sum of these partial products in the shortened algorism.

Now two multiplication algorisms are shown on the chalkboard. Point out that the two algorisms are alike in the information they give: 76 × 43 = 3268. The computed product is not affected by using the shorter method of recording partial products.

Have the class compute such products as 37 × 54, 86 × 94, 45 × 72, and 63 × 29. Ask the children to use three partial products in each instance.

Pages 251 through 253

● Pages 251 through 253 provide practice in handling partial products in whole-number multiplication. They also provide review of a shortcut, combining a pair of partial products in the multiplication of two-digit whole numbers.

With the class, work the example at the top of page 251. Be sure the pupils recall that division undoes multiplication, just as multiplication undoes division. They should observe that the computed product 1504 is divided by one of the factors, in this instance 32. The fact that the computed quotient (47) is the other factor indicates that the product has been computed correctly; 1504 ÷ 32 = 47 implies 32 × 47 = 1504. Ask the pupils whether computing the quotient 1504 ÷ 47 would also serve as a check on the multiplication (yes). Have a child compute this quotient at the chalkboard. Then assign the exercises on pages 251 and 252 for independent work. Assign no more than 8 exercises at one time, however. Be sure the children understand that they can divide the computed product by either of the factors when checking their work.

Discuss the example at the top of page 253 with the class. Call for volunteers to explain how each partial product in the shortcut is derived. Help them conclude that the second and third steps of the expanded algorisms are combined into one step in the shortcut.

Have the pupils complete exercises 1 through 6 orally. Encourage them to explain any shortcuts they use in computing the sums of the products.

Have the pupils work exercises 7 and 8 as a group. Then assign the remaining exercises for independent work. Encourage everyone to use the shortcut in at least four of the exercises. The pupils should be aware that it is natural to be somewhat "wobbly" when first attempting something new and different. They may find, however, that if they practice using the shortcut, they will make fewer and fewer errors. Do not be too concerned at the number of errors made by pupils on their first attempts at using the shortcut. Remind the pupils that they may divide the computed product by either of the given factors when checking their work.

Name

We can divide to check multiplication.

```
   32        32)1504
 × 47           1280  40
   14            224
  210            224   7
   80              0  47
 1200
 1504
```

Compute. Check your work. *Order of partial products may vary. Partial quotients may vary.*

```
1.   58      58)1218
   × 21         1160  20
      8           58
     50           58   1
    160            0  21
   1000
   1218

2.   97      97)3298
   × 34         2910  30
     28          388
    360          388   4
    210            0  34
   2700
   3298

3.  135       7)945
   ×  7         700  100
     35         245
    210         210   30
    700          35
    945          35    5
                  0  135

4. 4576       9)41184
   ×  9         36000  4000
     54          5184
    630          4500   500
   4500           684
  36000           630    70
  41,184           54
                   54     6
                    0  4576

5.   84      84)6132
   × 73         5880  70
     12          252
    240          252   3
    280            0  73
   5600
   6132

6.  302     302)16912
   × 56        15100  50
     12         1812
   1800         1812   6
    100            0  56
  15000
  16,912

7.   16      64)1024
   × 64          640  10
     24          384
     40          384   6
    360            0  16
    600
   1024

8.   47      47)2209
   × 47         1880  40
     49          329
    280          329   7
    280            0  47
   1600
   2209
```

reference page

G-251

Name

Here is a shortcut in multiplication.

Expanded Algorism		Shortened Algorism	
74		74	
× 32		× 32	
8	(ones × ones)	8	ones
140	(ones × tens)	260	tens
120	(tens × ones)		
2100	(tens × tens)	2100	ten tens
2368		2368	

Which steps are combined? *See pupil page suggestions.*

Compute the sum of the products.

1. $(2 \times 7 \text{ tens}) + (4 \times 3 \text{ tens}) =$ *26 tens or 260*
2. $(5 \times 4 \text{ tens}) + (2 \times 3 \text{ tens}) =$ *26 tens or 260*
3. $(6 \times 10) + (3 \times 20) =$ *120*
4. $(2 \times 40) + (3 \times 30) =$ *170*
5. $(7 \times 80) + (20 \times 2) =$ *600*
6. $(6 \times 70) + (50 \times 9) =$ *870*

Compute.

```
7.   65        8.   94        9.   53       10.   34
   × 23           × 71           × 43           × 56
     15              4              9             24
    180 }           90 }          150 }          180 }
    100 } 280      280 } 370      120 } 270      200 } 380
   1200           6300           2000           1500
   1495           6674           2279           1904

11.  67       12.   87       13.   28       14.   49
   × 69           × 47           × 86           × 95
     63             49             48             45
    540 }          560 }          120 }          200 }
    420 } 960      280 } 840      640 } 760      810 } 1010
   3600           3200           1600           3600
   4623           4089           2408           4655
```

reference page

G-253

Compute. Check your work. *Order of partial products may vary. Partial quotients may vary.*

```
1.   78      87)6786
   × 87         6090  70
     56          696
    490          696   8
    640            0  78
   5600
   6786

2.  320     320)15680
   × 49        12800  40
      0         2880
    180         2880   9
   2700            0  49
    800
  12000
  15,680

3.  876     876)7884
   ×  9        7884  9
     54           0  9
    630
   7200
   7884

4.  408     408)123216
  × 302        122400  300
     16           816
    800           816    2
   2400             0  302
 120000
 123,216

5.  873     873)52380
   × 60        52380  60
    180            0  60
   4200
  48000
  52,380

6.  751     800)600800
  × 800        560000  700
    800         40800
  40000         40000   50
 560000           800
 600,800          800    1
                    0  751

7.  903     903)189630
  × 210        180600  200
     30          9030
   9000          9030   10
    600             0  210
 180000
 189,630

8.  273     273)24297
   × 89        21840  80
     27         2457
    630         2457   9
   1800            0  89
    240
   5600
  16000
  24,297
```

9. Mary used 31 balls of yarn every week for 9 weeks. How much yarn did she use during that time? *279 balls of yarn*

G-252

Developmental Experiences

▶ Write the product 87 × 34 in vertical form on the chalkboard. Have a pupil compute the product using the expanded algorism. Next to this write the exercise 87 × 34 again. Tell the children you want them to help compute this product using another shortcut. Have someone compute the product of the ones (28). Explain to the class that you are going to write the 8 ones on the chalkboard and they are to remember the 2 tens.

```
   87 = 80 + 7 = 8 tens + 7           87
 × 34 = 30 + 4 = 3 tens + 4         × 34
                       2[8]            8
                        320
                        210
                       2400
                       2958
```

Next have someone compute 4 × 8 tens (32 tens) and add the 2 tens remembered (34 tens). Have someone else compute 3 tens × 7 (21 tens). Then ask, "How many tens are there all together?" (34 tens + 21 tens,

or 55 tens) Explain that you are going to write on the chalkboard only 5 tens of the 55 tens and that they are to remember the other 50 tens, or 5 hundreds. Let a child show how the 5 tens are written in the expanded algorism.

$$\begin{array}{r} 87 = 80 + 7 = 8 \text{ tens} + 7 \\ \underline{\times 34 = 30 + 4 = 3 \text{ tens} + 4} \\ 28 \\ 320 \\ 210 \\ 2400 \\ \hline 2958 \end{array} \qquad \begin{array}{r} 87 \\ \underline{\times 34} \\ 58 \end{array}$$

As the final step, have someone compute the product of the tens (3 tens × 8 tens, or 24 hundreds) and add the remembered 5 hundreds (29 hundreds). Write 29 hundreds on the board. Then have a pupil show the 29 hundreds in the expanded algorism. The pupils should observe that both algorisms result in the same product.

$$\begin{array}{r} 87 = 80 + 7 = 8 \text{ tens} + 7 \\ \underline{\times 34 = 30 + 4 = 3 \text{ tens} + 4} \\ 28 \\ 320 \\ 210 \\ 2400 \\ \hline 2958 \end{array} \qquad \begin{array}{r} 87 \\ \underline{\times 34} \\ 2958 \end{array}$$

Continue this way to help the pupils compute such products as the following, using the short algorism.

64 × 23
78 × 55
27 × 46
93 × 28
79 × 56

Then adapt the shortcut to computation of such products as 276 × 58, 435 × 91, and 743 × 24.

Pages 254 and 255

● Pages 254 and 255 provide practice in using division to check multiplication exercises and in using multiplication to check division exercises. Work the example at the top of page 254 with the class. The exercises on pages 254 and 255 should be assigned for independent work. Include both multiplication and division exercises in each assignment, but assign no more than five exercises at one time. Be sure the pupils understand that although there are two possible checks for each multiplication exercise, one check is sufficient. The children should be encouraged to use the shortcut but they should be free to use any method in computing the products.

To check multiplication, we can divide by either factor.

$$\begin{array}{r} 107 \\ \underline{\times 46} \\ 42 \\ 600 \\ 280 \\ \underline{4000} \\ 4922 \end{array} \qquad \begin{array}{r|l} 46 \;\; 4922 & \\ 4600 & 100 \\ 322 & \\ 322 & 7 \\ \hline 0 & 107 \end{array} \qquad \begin{array}{r|l} 107 \;\; 4922 & \\ 4280 & 40 \\ 642 & \\ 642 & 6 \\ \hline 0 & 46 \end{array}$$

Compute and check. *Order of partial products may vary. Partial quotients may vary.*

1. $\begin{array}{r} 1478 \\ \underline{\times \;\; 8} \\ 64 \\ 560 \\ 3200 \\ \underline{8000} \\ 11{,}824 \end{array} \quad \begin{array}{r|l} 1478 \;\; 11824 & \\ 11824 & 8 \\ \hline 0 & 8 \end{array}$

2. $\begin{array}{r} 345 \\ \underline{\times 407} \\ 35 \\ 280 \\ 2100 \\ 2000 \\ 16000 \\ \underline{120000} \\ 140{,}415 \end{array} \quad \begin{array}{r|l} 407 \;\; 140415 & \\ 122100 & 300 \\ 18315 & \\ 16280 & 40 \\ 2035 & \\ 2035 & 5 \\ \hline 0 & 345 \end{array}$

3. $\begin{array}{r} 264 \\ \underline{\times 58} \\ 32 \\ 480 \\ 1600 \\ 200 \\ 3000 \\ \underline{10000} \\ 15{,}312 \end{array} \quad \begin{array}{r|l} 264 \;\; 15312 & \\ 13200 & 50 \\ 2112 & \\ 2112 & 8 \\ \hline 0 & 58 \end{array}$

4. $\begin{array}{r} 5816 \\ \underline{\times \;\; 49} \\ 54 \\ 90 \\ 7200 \\ 45000 \\ 240 \\ 400 \\ 32000 \\ \underline{200000} \\ 284{,}984 \end{array} \quad \begin{array}{r|l} 5816 \;\; 284984 & \\ 232640 & 40 \\ 52344 & \\ 52344 & 9 \\ \hline 0 & 49 \end{array}$

5. $\begin{array}{r} 425 \\ \underline{\times 136} \\ 30 \\ 120 \\ 2400 \\ 150 \\ 600 \\ 12000 \\ 500 \\ 2000 \\ \underline{40000} \\ 57{,}800 \end{array} \quad \begin{array}{r|l} 425 \;\; 57800 & \\ 42500 & 100 \\ 15300 & \\ 12750 & 30 \\ 2550 & \\ 2550 & 6 \\ \hline 0 & 136 \end{array}$

6. $\begin{array}{r} 608 \\ \underline{\times 204} \\ 32 \\ 2400 \\ 1600 \\ \underline{120000} \\ 124{,}032 \end{array} \quad \begin{array}{r|l} 204 \;\; 124032 & \\ 122400 & 600 \\ 1632 & \\ 1632 & 8 \\ \hline 0 & 608 \end{array}$

reference page

G-254

Name

Compute a mixed fraction and check.

1. $\frac{15567}{247}$ $63\frac{6}{247}$ 2. $\frac{6521}{304}$ $21\frac{137}{304}$ 3. $\frac{8490}{251}$ $33\frac{207}{251}$ 4. $\frac{3932}{87}$ $45\frac{17}{87}$

5. $\frac{9291}{2000}$ $4\frac{1291}{2000}$ 6. $\frac{78560}{402}$ $195\frac{170}{402}$ 7. $\frac{38304}{504}$ $76\frac{0}{504}$ 8. $\frac{56550}{628}$ $90\frac{30}{628}$

9. $\frac{4444}{333}$ $13\frac{115}{333}$ 10. $\frac{843}{261}$ $3\frac{60}{261}$ 11. $\frac{2785}{34}$ $81\frac{31}{34}$ 12. $\frac{62150}{888}$ $69\frac{878}{888}$

G-255

Developmental Experiences

► Write on the chalkboard the exercise 94 × 68 in vertical form using expanded notation. Then have a pupil compute the product using the expanded algorism. Next to this write the exercise again, this time using standard numerals. Tell the class that they are going to examine another shortcut in computing products. Explain that in this shortcut 94 is multiplied first by 8 and then by 60. These two partial products then are added.

Have someone describe how he would compute the product of 94 and 8. Have the pupil write this partial product on the chalkboard. Let a second pupil describe each of the steps involved in finding this first partial product.

$$\begin{array}{r} 94 = 90 + 4 = 9 \text{ tens} + 4 \\ \underline{\times 68} = \underline{60 + 8} = \underline{6 \text{ tens} + 8} \\ 32 \\ 720 \\ 240 \\ \underline{5400} \\ 6392 \end{array} \qquad \begin{array}{r} 94 \\ \underline{\times 68} \\ 752 \end{array}$$

Ask another pupil to describe how to compute the product of 94 and 60. He should write the second partial product below the first, aligning the ones, tens, and hundreds.

$$\begin{array}{r} 94 = 90 + 4 = 9 \text{ tens} + 4 \\ \underline{\times 68} = \underline{60 + 8} = \underline{6 \text{ tens} + 8} \\ 32 \\ 720 \\ 240 \\ \underline{5400} \\ 6392 \end{array} \qquad \begin{array}{r} 94 \\ \underline{\times 68} \\ 752 \\ 5640 \end{array}$$

Then have a third pupil compute the sum of these two partial products.

$$\begin{array}{r} 94 \\ \underline{\times 68} \\ 752 \\ \underline{5640} \\ 6392 \end{array}$$

Continue this way to have the pupils describe computing such products as 35 × 19, 428 × 62, 306 × 72, and 294 × 583 using this shortcut.

Pages 256 through 266

● Pages 256 through 258 provide practice in computing products. The exercises on page 256 involve two-digit factors, and those on pages 257 and 258 involve three-digit factors. In presenting pages 256 and 257, discuss the example at the top of the page with the class, and then have the pupils compute orally the sums of products given in the first set of exercises. Assign the remaining exercises on pages 256 and 257 and the exercises on page 258 for independent work. Tell the pupils to use the shortcut in at least one exercise in each assignment.

Here is another shortcut in multiplication.

$$\begin{array}{rl} 36 & \\ \underline{\times 54} & \\ 24 & (4 \times 6) \\ 120 & (4 \times 30) \\ 300 & (50 \times 6) \\ \underline{1500} & (50 \times 30) \\ 1944 & \end{array} \qquad \begin{array}{rl} 36 & \\ \underline{\times 54} & \\ 144 & (4 \times 36) \\ \underline{1800} & (50 \times 36) \\ 1944 & \end{array}$$

Which steps are combined? *The first and second steps, and the third and fourth steps are combined.*

Compute.

1. (2 × 4) + (2 × 20) = *48*
2. (7 × 6) + (7 × 30) = *252*
3. (5 × 70) + (5 × 8) = *390*
4. (3 × 5) + (3 × 60) = *195*
5. (20 × 4) + (20 × 20) = *480*
6. (80 × 5) + (80 × 60) = *5200*
7. (90 × 70) + (90 × 8) = *7020*
8. (40 × 6) + (40 × 30) = *1440*

9. $\begin{array}{r} 24 \\ \underline{\times 22} \\ 48 \\ \underline{480} \\ 528 \end{array}$ 10. $\begin{array}{r} 65 \\ \underline{\times 83} \\ 195 \\ \underline{5200} \\ 5395 \end{array}$ 11. $\begin{array}{r} 36 \\ \underline{\times 47} \\ 252 \\ \underline{1440} \\ 1692 \end{array}$ 12. $\begin{array}{r} 78 \\ \underline{\times 95} \\ 390 \\ \underline{7020} \\ 7410 \end{array}$

13. $\begin{array}{r} 51 \\ \underline{\times 34} \\ 204 \\ \underline{1530} \\ 1734 \end{array}$ 14. $\begin{array}{r} 66 \\ \underline{\times 66} \\ 396 \\ \underline{3960} \\ 4356 \end{array}$ 15. $\begin{array}{r} 84 \\ \underline{\times 96} \\ 504 \\ \underline{7560} \\ 8064 \end{array}$ 16. $\begin{array}{r} 37 \\ \underline{\times 87} \\ 259 \\ \underline{2960} \\ 3219 \end{array}$

reference page

G-256

Name

Which steps are combined in this shortcut in multiplication?

$$\begin{array}{r} 428 \\ \underline{\times 637} \\ 56 \\ 140 \\ 2800 \\ 240 \\ 600 \\ 12000 \\ 4800 \\ 12000 \\ \underline{240000} \\ 272636 \end{array} \qquad \begin{array}{rl} 428 & \\ \underline{\times 637} & \\ 2996 & (7 \times 428) \\ 12840 & (30 \times 428) \\ \underline{256800} & (600 \times 428) \\ 272636 & \end{array}$$

The first, second, and third, the fourth, fifth, and sixth, and the seventh, eighth, and ninth steps are combined.

Compute.

1. (2 × 3) + (2 × 10) + (2 × 200) = *426*
2. (30 × 3) + (30 × 10) + (30 × 200) = *6390*
3. (4 × 1) + (4 × 30) + (4 × 200) = *924*
4. (100 × 1) + (100 × 30) + (100 × 200) = *23,100*
5. (20 × 1) + (20 × 30) + (20 × 200) = *4620*
6. (500 × 500) + (500 × 60) + (500 × 4) = *282,000*
7. (400 × 3) + (400 × 10) + (400 × 200) = *85,200*
8. (90 × 700) + (90 × 30) + (90 × 8) = *66,420*
9. (90 × 8) + (90 × 30) + (90 × 700) = *66,420*
10. (800 × 600) + (800 × 10) + (800 × 7) = *493,600*

Compute and check. Use a shortcut if you wish. *Order of partial products may vary. Partial quotients may vary.*

11. $\begin{array}{r} 231 \\ \underline{\times 124} \\ 924 \\ 4620 \\ \underline{23100} \\ 28,644 \end{array}$ 12. $\begin{array}{r} 213 \\ \underline{\times 432} \\ 426 \\ 6390 \\ \underline{85200} \\ 92,016 \end{array}$ 13. $\begin{array}{r} 564 \\ \underline{\times 583} \\ 1692 \\ 45120 \\ \underline{282000} \\ 328,812 \end{array}$

$$\begin{array}{r|l} 231 \,\big)\, 28644 & \\ \underline{23100} & 100 \\ 5544 & \\ \underline{4620} & 20 \\ 924 & \\ \underline{924} & 4 \\ 0 & 124 \end{array} \qquad \begin{array}{r|l} 432 \,\big)\, 92016 & \\ \underline{86400} & 200 \\ 5616 & \\ \underline{4320} & 10 \\ 1296 & \\ \underline{1296} & 3 \\ 0 & 213 \end{array} \qquad \begin{array}{r|l} 564 \,\big)\, 328812 & \\ \underline{282000} & 500 \\ 46812 & \\ \underline{45120} & 80 \\ 1692 & \\ \underline{1692} & 3 \\ 0 & 583 \end{array}$$

reference page

G-257

Compute and check. *Order of partial products may vary. Partial quotients may vary.*

1. 73 × 19 = 657 + 730 = 1387

Check: 73) 1387: 730 (10), 657 (9), remainder 0; quotient 19

2. 37 × 96 = 222 + 3330 = 3552

Check: 96) 3552: 2880 (30), 672 (7), remainder 0; quotient 37

3. 983 × 272 = 1966 + 68810 + 196600 = 267,376

Check: 983) 267 376: 196 600 (200), 70 776; 68 810 (70), 1966; 1966 (2), remainder 0; quotient 272

4. 738 × 391 = 738 + 66420 + 221400 = 288,558

Check: 391) 288558: 273700 (700), 14858; 11730 (30), 3128; 3128 (8), remainder 0; quotient 738

5. 617 × 819 = 5553 + 6170 + 493600 = 505,323

Check: 819) 505323: 491400 (600), 13923; 8190 (10), 5733; 5733 (7), remainder 0; quotient 617

6. 999 × 678 = 7992 + 69930 + 599400 = 677,322

Check: 678) 677322: 610200 (900), 67122; 61020 (90), 6102; 6102 (9), remainder 0; quotient 999

reference page

Name

What is a mixed fraction for $\frac{7432}{20}$?

20) 7432: 6000 (300), 1432; 1400 (70), 32; 20 (1), 12; 12 ($\frac{12}{20}$), remainder 0; quotient $371\frac{12}{20}$

$7432 \div 20 = 371\frac{12}{20}$

Does $20 \times 371\frac{12}{20}$ equal 7432?

$371\frac{12}{20}$
$\times$ 20

12 ← $(20 \times \frac{12}{20})$
20 ← (20×1)
1400 ← (20×70)
6000 ← (20×300)
7432

Compute a mixed fraction and check. *Order of partial products may vary. Partial quotients may vary.*

1. $\frac{777}{22} = 35\frac{7}{22}$

2. $\frac{5959}{34} = 175\frac{9}{34}$

3. $\frac{5555}{66} = 84\frac{11}{66}$

4. $\frac{6363}{61} = 104\frac{19}{61}$

Compute a mixed fraction and check. *Order of partial products may vary. Partial quotients may vary.*

1. $\frac{7575}{65} = 116\frac{35}{65}$

2. $\frac{5959}{75} = 79\frac{34}{75}$

3. $\frac{98980}{314} = 315\frac{70}{314}$

4. $\frac{1500}{200} = 7\frac{100}{200}$

5. $\frac{6008}{123} = 48\frac{104}{123}$

6. $\frac{6000}{450} = 13\frac{150}{450}$

7. $\frac{7468}{291} = 25\frac{193}{291}$

8. $\frac{66778}{450} = 148\frac{178}{450}$

Pages 259 through 263 provide practice in computing mixed fractions. Discuss with the class the example at the top of pages 259 and 261. Then have the pupils complete the exercises on pages 259 through 263 independently. Assign four exercises at a time. The children should be allowed to do the multiplication by writing all the partial products if they wish, but those who are able to use shortcuts should be encouraged to do so.

Do not equate errors in computation with lack of understanding. The length of an assignment may contribute to careless errors. Keep this in mind when analyzing a pupil's performance and try to point out the pupil's successes rather than his failures. This, along with short varied practice exercises, will do more to encourage accuracy than pages of drill material.

Name

What is a mixed fraction for $\frac{38569}{132}$?

132	38569	
	13200	100
	25369	
	13200	100
	12169	
	5280	40
	6889	
	5280	40
	1609	
	1320	10
	289	
	264	2
	25	292

$38569 \div 132 = 292\frac{25}{132}$

Does $132 \times 292\frac{25}{132}$ equal 38569?

$$\begin{array}{r} 292\frac{25}{132} \\ \times\ 132 \\ \hline 25 \\ 4 \\ 180 \\ 400 \\ 60 \\ 2700 \\ 6000 \\ 29200 \\ \hline 38569 \end{array}$$

Compute a mixed fraction. Check. Order of partial products may vary. Partial quotients may vary.

1. $\frac{289}{189}$ $1\frac{100}{189}$

2. $\frac{3625}{700}$ $5\frac{125}{700}$

G-261

Compute a mixed fraction. Check. Order of partial products may vary. Partial quotients may vary.

1. $\frac{2375}{311}$ $7\frac{198}{311}$

2. $\frac{6920}{210}$ $32\frac{200}{210}$

3. $\frac{3501}{931}$ $3\frac{708}{931}$

4. $\frac{8943}{427}$ $20\frac{403}{427}$

5. $\frac{15929}{522}$ $30\frac{269}{522}$

6. $\frac{18000}{127}$ $141\frac{93}{127}$

G-262

Name

Compute a mixed fraction. Check. Order of partial products may vary. Partial quotients may vary.

1. $\frac{257}{35}$ $7\frac{12}{35}$

2. $\frac{462}{18}$ $25\frac{12}{18}$

3. $\frac{53794}{228}$ $235\frac{214}{228}$

4. $\frac{67129}{6503}$ $10\frac{2099}{6503}$

5. $\frac{83000}{3291}$ $25\frac{725}{3291}$

6. $\frac{23478}{5189}$ $4\frac{2722}{5189}$

G-263

● Pages 264 and 265 provide practice with the division algorism.

Complete these pages as a class activity. Write exercise 1 from page 264 on the chalkboard.

12)	b	
	1	4488

Have a pupil show the given information in an equation, using b to represent the dividend.

$$b = (12 \times 4488) + 1$$

Ask another pupil to compute the dividend (53,857). Have the computed dividend written in place of the b on the chalkboard. Then let other pupils complete the steps of the indicated division. As they do this, ask what partial quotients could be used. Help the class see that 4000 + 400 + 80 + 8, the expanded form of 4488, gives a clue as to what partial quotients to use.

12)	53857	
	48000	4000
	5857	
	4800	400
	1057	
	960	80
	97	
	96	8
	1	4488

Follow the same procedure in having the pupils complete exercises 2 through 4.

Next read the first two story exercises on page 265 with the class. Let two pupils work at the chalkboard to compute the lengths of the orbits in hundreds of miles.

Titov			Cooper		
17	4287		22	5564	
	3400	200		4400	200
	887			1164	
	850	50		1100	50
	37			64	
	34	2		44	2
	3	252		20	252

The children should see that in exercise 1 an orbit is about 252 hundred miles long. This distance can also be expressed as 25 thousand 2 hundred miles. The pupils should note in exercise 2 that the remainder 20 is almost 22, so each of Cooper's orbits is about 253 hundred miles long. Have them compare the lengths of the two orbits (both orbits were longer than 25 thousand miles but shorter than 26 thousand miles). Have a pupil tell the approximate difference in the lengths of the two orbits (the difference is about 100 miles). Follow the same procedure in having the pupils complete exercises 3, 4, and 5 together.

● Use page 266 to give the children an opportunity to further investigate the definition of fractional numbers. Pupils may work together in groups of 2 or 3 to provide an interchange of ideas. When the class has finished, discuss the results. Let some pupils display their solutions.

Name

Answer the questions. Show your work. *Partial quotients and partial products may vary.*

1. Gherman Titov orbited the earth 17 times in 1961 in his ship, Vostok 2. It traveled about 4287 hundred miles while in orbit. About how many hundreds of miles did it travel in each orbit?

   4287 = (17 × 252) + 3
   It traveled about 252 hundred miles each orbit.

2. In 1963, L. Gordon Cooper orbited the earth 22 times in his ship, Faith 7. It traveled about 5564 hundred miles while in orbit. About how many hundreds of miles did it travel in each orbit?

   5564 = (22 × 252) + 20.
   It traveled about 252 hundred miles each orbit.

3. It took 91 minutes for Faith 7 to travel 25,291 miles around the earth. About how many miles did Faith 7 travel in one minute?

   25,291 = (91 × 277) + 84
   It traveled about 277 miles in one minute.

4. An early Mercury capsule reached a height of 55 miles. In March, 1965, Virgil Grissom and John Young orbited the earth 3 times in a Gemini spacecraft. During their first orbit, they reached a height of 140 miles. During their second orbit, they reached a height of 105 miles. How much higher than the Mercury capsule did the Gemini craft go in its first orbit? *140 − 55 = 85*

   In its second orbit? *105 − 55 = 50*

   The Gemini went 85 miles higher in the first orbit; it went 50 miles higher in the second orbit.

5. In November, 1967, the United States launched the first Saturn V moon rocket. Saturn V shot an unmanned Apollo spacecraft 11,386 miles into space. Then Saturn V returned the Apollo spacecraft to the earth at a speed of 25,000 miles per hour. This is about how many miles per minute? *25,000 = (60 × 416) + 40*

   This is about 416 miles per minute.

What number is b? Copy and complete. *Order of partial products may vary*
Partial quotients may vary

1. $12\overline{)b}$; remainder 1, quotient 4488

   b = (12 × 4488) + 1
   = 53,857

12	53857	
	48000	4000
	5857	
	4800	400
	1057	
	960	80
	97	
	76	8
	1	4488

4488 × 12 = 8976 + 44880 = 53856; + 1 = 53,857

2. $321\overline{)b}$; remainder 60, quotient 214

   b = (321 × 214) + 60
   = 68,754

321	68754	
	64200	200
	4554	
	3210	10
	1344	
	1284	4
	60	214

214 × 321 = 214 + 4280 + 64200 = 68694; + 60 = 68,754

3. $4876\overline{)b}$; remainder 1140, quotient 19

   b = (4876 × 19) + 1140
   = 93,784

4876	93784	
	48760	10
	45024	
	43884	9
	1140	19

4876 × 19 = 43884 + 48760 = 92644; + 1140 = 93,784

4. $5035\overline{)b}$; remainder 1737, quotient 83

   b = (5035 × 83) + 1737
   = 419,642

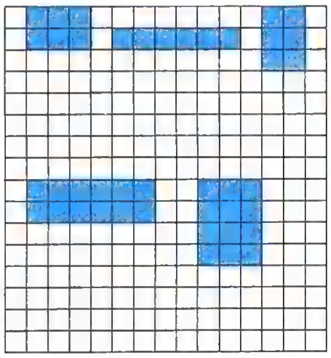
5035	419642	
	402800	80
	16842	
	15105	3
	1737	83

5035 × 83 = 15105 + 402800 = 417905; + 1737 = 419,642

Draw and shade a rectangle with an area of—

1. 6 ☐. (Can you do this 3 ways?)

2. 12 ☐. (more than one way)

1. 

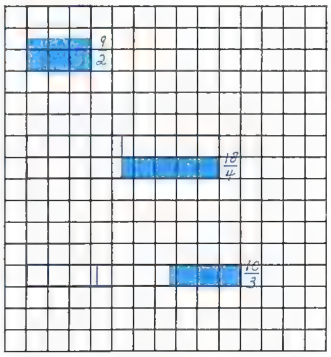
*2.

Draw and shade any figure with an area of—

3. $\frac{9}{2}$ ☐. $\left(\frac{9 \times ☐}{2}\right)$

4. $\frac{18}{4}$ ☐.

5. $\frac{10}{3}$ ☐.

*3. 

*4

*5

* *answers may vary.*

Supplemental Experiences

■ On 3-inch by 12-inch tagboard cards, show sums like those illustrated below. Put the cards in a pack face down on a table next to the bulletin board. Across the bulletin board pin two sets of the digits from 0 through 9.

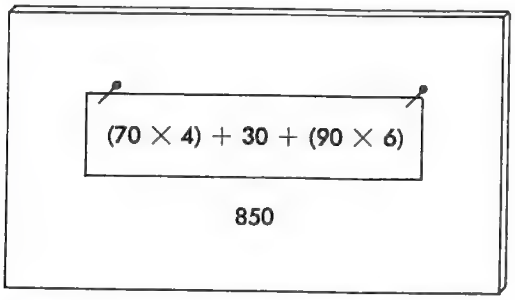
$\boxed{(30 \times 7) + (50 \times 3) + 80}$

$\boxed{(20 \times 90) + (60 \times 40) + 60}$

$\boxed{(70 \times 4) + 30 + (90 \times 6)}$

$\boxed{(4 \times 80) + 60 + (9 \times 50)}$

Have the pupils take turns choosing a card from the pack and pinning it to the bulletin board. Give each child adequate time to compute the sum mentally and place the appropriate digits under his card. Then have him describe his computational steps to the class. For example, with the sum (70 × 4) + 30 + (90 × 6), a pupil may say that 7 tens × 4 is 28 tens, 28 tens + 3 tens is 31 tens, and 31 tens + 54 tens (9 tens × 6) is 85 tens, or 850.



If a child makes an error, help him correct it.

■ On each of ten index cards, write a number less than 100. On ten other cards write numbers greater than 100. Place the cards in two stacks, and have a pupil draw a card from one of the stacks. Ask him to use the number just drawn to make a sentence about an item in a supermarket.

Have a second child draw a card from the other stack and use the number in a sentence that continues the story. Then have a third pupil ask a question that can be answered by computing with the two numbers selected. A fourth pupil should then write an equation to show the number relationship.

For example:

$\boxed{16}$	A box contains 16 pineapples.
$\boxed{398}$	The box of pineapples costs \$3.98.
Question:	How much does one pineapple cost?
Equation:	$398 = 16q + r$

OR

$\boxed{16}$	There are 16 marshmallows in a bag.
$\boxed{398}$	There are 398 bags of marshmallows in the warehouse.
Question:	How many marshmallows are in the warehouse?
Equation:	$16 \times 398 = m$

Other story situations may require addition or subtraction.

■ On the chalkboard, have four pupils compute the product 73 × 56 in different ways. Their results may appear as follows:

$$\begin{array}{r} 73 \\ \underline{\times\, 56} \\ 18 \\ 420 \\ 150 \\ \underline{3500} \\ 4088 \end{array} \qquad \begin{array}{r} 73 \\ \underline{\times\, 56} \\ 18 \\ 570 \\ \underline{3500} \\ 4088 \end{array}$$

$$\begin{array}{r} 73 \\ \underline{\times\, 56} \\ 18 \\ 150 \\ 420 \\ \underline{3500} \\ 4088 \end{array} \qquad \begin{array}{r} 73 \\ \underline{\times\, 56} \\ 438 \\ \underline{3650} \\ 4088 \end{array}$$

Continue this way with the products 73 × 62, 48 × 75, 69 × 21, 436 × 58, and 387 × 92. If the ability of the class permits, this activity can be adapted to such products as 143 × 657, 569 × 208, and 742 × 837.

■ The following exercises may be used if more division practice is desired. Dictate only 2 or 3 exercises per assignment.

1. $98\overline{)\,5078}$	2. $73\overline{)\,4900}$
3. $88\overline{)\,1200}$	4. $213\overline{)\,3506}$
5. $472\overline{)\,5809}$	6. $116\overline{)\,2237}$
7. $880\overline{)\,9800}$	8. $220\overline{)\,4805}$
9. $110\overline{)\,7986}$	10. $725\overline{)\,4999}$
11. $832\overline{)\,5601}$	12. $458\overline{)\,1309}$
13. $700\overline{)\,78001}$	14. $640\overline{)\,89921}$
15. $254\overline{)\,69900}$	16. $852\overline{)\,18523}$
17. $611\overline{)\,27111}$	18. $213\overline{)\,18413}$
19. $547\overline{)\,19386}$	20. $365\overline{)\,31666}$
21. $298\overline{)\,17451}$	22. $999\overline{)\,32322}$
23. $888\overline{)\,54445}$	24. $808\overline{)\,56560}$

■ The following is a quiz you may wish to use.

SUGGESTED QUIZ

Compute the mixed fraction.

1. $\frac{10}{3}$ $3\frac{1}{3}$ 2. $\frac{35}{8}$ $4\frac{3}{8}$ 3. $\frac{25}{7}$ $3\frac{4}{7}$

Compute the fraction.

4. $3\frac{2}{5}$ $\frac{17}{5}$ 5. $7\frac{3}{4}$ $\frac{31}{4}$ 6. $9\frac{1}{2}$ $\frac{19}{2}$

Compute the mixed fraction, and check your work.

7. $\frac{958}{6} = 159\frac{4}{6}$

6) 958	
600	100
358	
300	50
58	
54	9
4	159

$$\begin{array}{r} 159 \\ \times\ 6 \\ \hline 54 \\ 300 \\ 600 \\ \hline 954 \\ +\ 4 \\ \hline 958 \end{array}$$

8. $875 \div 17 = 51\frac{8}{17}$

17) 875	
850	50
25	
17	1
8	51

$$\begin{array}{r} 51 \\ \times\ 17 \\ \hline 7 \\ 350 \\ 10 \\ 500 \\ \hline 867 \\ +\ 8 \\ \hline 875 \end{array}$$

Compute. Check by dividing.

9. $$\begin{array}{r} 59 \\ \times\ 84 \\ \hline 4956 \end{array}$$

59) 4956	
0	84

10. $$\begin{array}{r} 978 \\ \times\ 36 \\ \hline 35208 \end{array}$$

36) 35208	
0	978

The steps used in computation may vary.

UNIT 18
BAR GRAPHS
Pages 267 Through 276

OBJECTIVE
To introduce bar graphs.

The pupil learns that the bar graph is a way to display data. He practices reading and making bar graphs. The concepts of range and arithmetic mean are introduced.

See Key Topics in Mathematics for the Intermediate Teacher: Statistics.

KEY IDEA
A graph is a picture of measurements.

CONCEPTS
mean
partial sum
range

KEY IDEA
A graph is a picture of measurements.

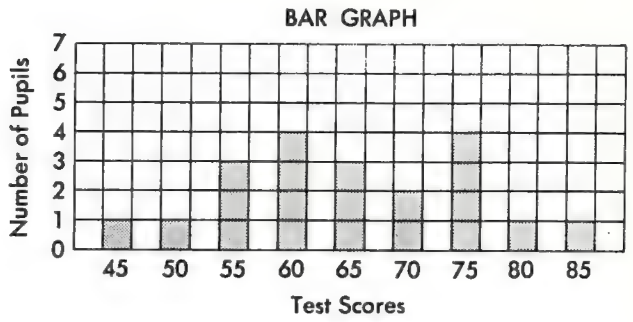
Scope
To interpret graphs.
To compute statistics from a graph.

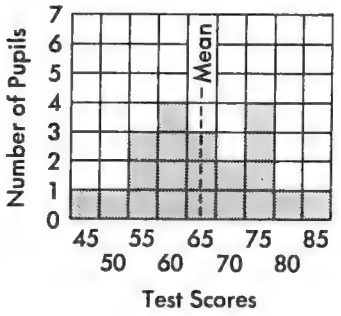
Fundamentals
A bar graph is a picture of a set of measurements. The bar graph organizes data so that they can be easily investigated. For example, consider these data, which might be the scores achieved by your class on some test: 60, 75, 80, 45, 60, 85, 55, 75, 70, 65, 60, 50, 65, 70, 65, 75, 55, 60, 55, 75.

To organize these data meaningfully, we tabulate the distribution of the scores (the number of pupils that achieved each score).

TEST SCORE	NUMBER OF PUPILS	PARTIAL SUM
85	1	85
80	1	80
75	4	300
70	2	140
65	3	195
60	4	240
55	3	165
50	1	50
45	1	45
	20	1300

A bar graph, however, can convey the same information better.





To acquire more information about the set of scores, we introduce two statistics:

1) the *arithmetic mean*, or simply, *mean*
2) the *range*

The mean is the sum of the scores divided by the total number of scores: 1300 ÷ 20 = 65. The mean of this set of scores is 65. The arithmetic mean is often called the average. The range is the difference between the highest and lowest scores; the range of this set is 85 − 45, or 40.

Readiness for Understanding
Knowledge of exact quotient.

Developmental Experiences
for each child
$\frac{1}{2}$-inch squared paper

► Use a story exercise to introduce the table as a way of organizing data. Tell the class this story about Mr. Bell and his pupils:

> Mr. Bell had his class do push-ups. He wrote down the number of push-ups that each pupil did.

Let the class suggest thirty-five names to represent the pupils in Mr. Bell's class. List these names on the chalkboard, and after each name write the number of push-ups done by that pupil. Use these numbers:

0, 3, 5, 4, 1, 0, 1, 1, 0, 9, 0, 0, 4, 0, 0, 1, 0, 1, 1, 4, 0, 4, 3, 1, 0, 1, 1, 0, 4, 3, 1, 0, 4, 1, 0. Then continue telling the story.

Mr. Bell decided to use a table to show the number of push-ups his class had done, so he made a chart that looked like this:

Number of push-ups	Number of students
0	//
1	/
2	
3	/
4	/
5	/
6	
7	
8	
9	
10	
11	

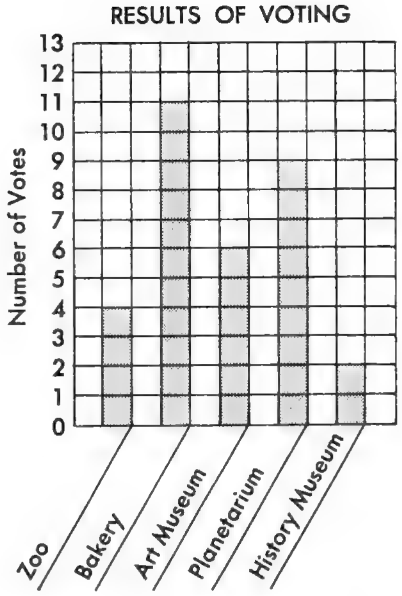
Copy Mr. Bell's table on the chalkboard, and complete it with the help of the class. When the table is complete, ask a pupil to tell the least number of push-ups anyone did and the greatest number (0 and 9).

▶ Use a story exercise to introduce bar graphs as a way of organizing data. Tell the following story to your pupils:

Mrs. Mayer's class planned a field trip for the last day of school. The class voted on a place to visit. There were 4 children who voted to go to the zoo, 11 who voted to go to the bakery, 6 who voted to go to the art museum, 9 who voted to go to the planetarium, and 2 who voted to go to the history museum.

One way to present this information is to record the facts in the form of a bar graph. Draw a grid on the chalkboard, number the left-hand side from 0 through 13, and write the trip choices below the grid.

Discuss the meaning of these numbers and places. Tell the class that labeling a graph is important; it should tell what the graph is about. Ask the pupils to suggest an appropriate title for Mrs. Mayer's graph. It could be called RESULTS OF VOTING. Write the suggested title above the graph. Ask how many children voted to go to the zoo (4). Color a bar on the graph to show this information. Ask the pupils to color in the remaining bars, thus showing all the information in the story.



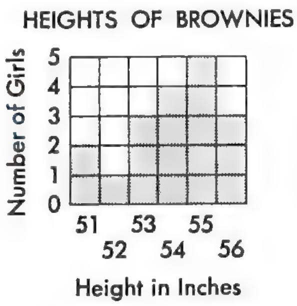
Discuss the information shown on the graph by asking questions such as these:

Where will the children go on their field trip?

Which place received the second greatest number of votes?

Discuss what elements are necessary for a graph to be a helpful aid. The graph should have a title to identify it, the divisions of the graph should be numbered and labeled, and each bar should be labeled.

▶ Copy the following graph on the chalkboard:



Then discuss the situation depicted by the graph. There are 18 girls in a Brownie troop. Their heights range from 51 to 56 inches. The first column of the graph indicates that there are 2 girls who are 51 inches tall, and so on. Ask the class questions like these:

How many Brownies are 54 inches tall?
How many are 55 inches tall?
How tall are the Brownies in this troop?

The last question will produce many different answers, each one depending on the pupil's point of view. Explain that there is a way to find a number that is an average of the heights of these Brownies. Let the class help you compute the sum of the heights of all the Brownies.

$$51 + 51 + 52 + 53 + 53 + 53 + 54 + 54 + 54 + 54 + 55 + 55 + 55 + 55 + 55 + 56 + 56 + 56 = 972$$

Ask a child to divide this sum by the total number of Brownies.

$$972 \div 18 = 54$$

Tell the children that the resulting number (54) is called the *arithmetic mean*, or just the *mean*. The mean height of the Brownies is 54 inches.

Have pupils point out that the shortest Brownie is 51 inches tall and the tallest is 56. Tell the class that the range of heights is the difference, 56 inches $-$ 51 inches, or 5 inches.

▶ Write this exercise on the chalkboard:

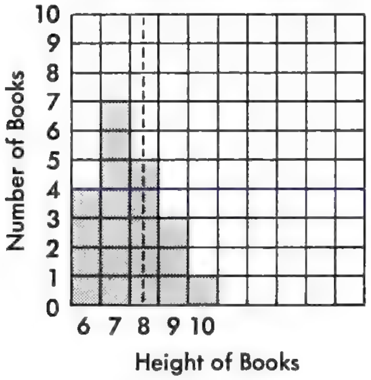
John has 20 books on a bookshelf. The heights of the books are recorded below.

HEIGHT OF BOOKS	NUMBER OF BOOKS
6 inches	4
7 inches	7
8 inches	5
9 inches	3
10 inches	1

Give each child a sheet of paper marked off in $\frac{1}{2}$-inch squares, and have each pupil draw a bar graph showing the data given on the chalkboard.

Tell the children to guess what the mean height is and to draw a line on their bar graphs showing this mean height.

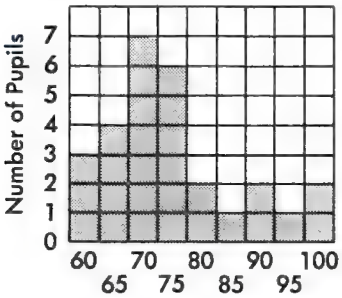
HEIGHTS OF BOOKS



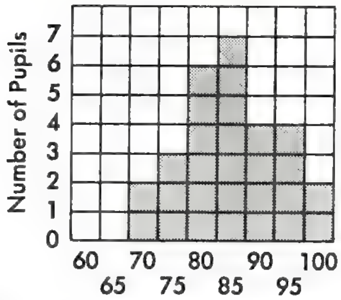
Then help the pupils to compute the mean height of the books ($7\frac{1}{2}$ inches). Then compute with the class the range of the heights of the books (4 inches). Encourage them to compare the computed mean with their estimated means and to discuss their comparisons.

▶ Draw these bar graphs on the chalkboard:

SCORES OF PUPILS IN A MENTAL ARITHMETIC TEST



SCORES OF SAME PUPILS IN A SPELLING TEST



Ask the pupils to look at the graphs and to say what they think the mean score is in each one. The pupils should eventually see a clue for estimating the mean. First find a place on the graph where most of the scores seem to be. For example, for the spelling test, some child may point out that most of the scores are between 80 and 95. The mean score for this test, therefore, should be between 80 and 95. For the arithmetic test, some child might say that he thinks the mean is between 65 and 80 because most of the scores are between 65 and 80.

Have the pupils check the guesses they suggest by computing the mean scores (the mean score is 85 for the spelling test and 75 for the arithmetic test).

▶ On the chalkboard copy the following table, including each number from 41 through 59 in the height column:

HEIGHT IN INCHES	NUMBER OF CHILDREN
41	___
42	___
43	___
.	.
.	.
.	.
57	___
58	___
59	___

Let the pupils measure each other's height (to the nearest inch) and record each height by placing a check mark in the appropriate blank in the second column. If a pupil happens to be shorter than 41 inches or taller than 59 inches, add these numbers to the table.

When the pupils have all been measured, ask one pupil to go to the chalkboard, count the check marks for each height, and write the number beside the marks. Tell the pupils to draw a bar graph that shows this information. Ask them to compute the mean height and the range of the heights. The mean height will probably not be a whole number.

Pages 267 through 276

- Page 267 introduces the bar graph.

Use the page for class discussion. Select pupils to read and answer the questions concerning the bar graph. Discuss their answers.

- After discussing page 267, the pupils should be able to complete page 268 independently. When the class has finished, discuss their answers and any problems they may have had in answering the questions.

- Discuss the results of the voting and complete the table on page 269 as a class activity. Let the pupils complete the bar graph independently. When they are finished, let them compare their results.

- Page 270 provides practice in completing a bar graph. Have the pupils complete the page independently. Then discuss the results. Ask questions that will lead to a comparison of the enrollment in various grades.

- Discuss the bar graph at the top of page 271 with the pupils. Explain to the pupils that to find the mean height of the Cub Scouts, we first add the heights and then divide the sum by the total number of Cub Scouts.

Ask the class to compute the sum and discuss their answers.

Have the children find out how many Cub Scouts there are. Allow independent investigation, then discussion. Then allow the pupils to discuss the remainder of the page.

- Page 272 provides practice in computing the mean. Work exercises 1 and 2 as a class activity. Assign the remaining exercises for independent work followed by class discussion.

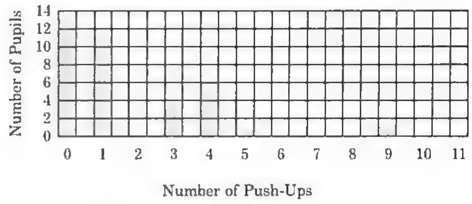
Name

UNIT 18 BAR GRAPHS

For Class Discussion

On September 15, Mr. Bell's class began push-up exercises. They made a bar graph to show how many push-ups the pupils did. The bar graph looked like this.

PUSH-UP EXERCISES–SEPTEMBER 15



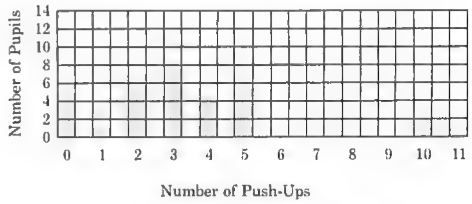
1. What was the greatest number of push-ups done by any pupil? 9
2. What was the least number of push-ups done by any pupil? 0
3. How many pupils did exactly 3 push-ups? 3
   Exactly 4 push-ups? 6 Exactly 9 push-ups? 1 Exactly 1 push-up? 11
   Exactly 7 push-ups? 0 Exactly 0 push-ups? 13 Exactly 5 push-ups? 1
4. What number of push-ups was done by the greatest number of pupils? 0
5. How many pupils did more than 5 push-ups? 1
6. How many pupils did at least 1 push-up? 22 More than 1 push-up? 11
7. How many pupils did at least 3 push-ups? 11 More than 3 push-ups? 8

reference page

G-267

On December 15, Mr. Bell's class again recorded how many push-ups they did. They made another bar graph to show their results.

PUSH-UP EXERCISES–DECEMBER 15



1. What was the greatest number of push-ups any pupil did on December 15? 11 On September 15? 9
2. What was the least number of push-ups any pupil did on December 15? 0 On September 15? 0
3. On December 15, how many pupils did exactly 2 push-ups? 7
   Exactly 3 push-ups? 8 Exactly 5 push-ups? 7
   Exactly 4 push-ups? 4 Exactly 6 push-ups? 0
4. How many pupils did not do any push-ups on September 15? 13
   On December 15? 5
5. How many pupils did more than 2 push-ups on September 15? 11
   On December 15? 22
6. How many pupils did at least 1 push-up on September 15? 22
   On December 15? 30
7. How many pupils did more than 3 push-ups on September 15? 8
   On December 15? 14

G-268

Name

The children in Mrs. Horton's class wanted to write to a fourth-grade class in some other country. Each child wrote his choice on a slip of paper.

When the slips were collected, they looked like this:

Holland	France	Ireland	England	Holland
Ireland	Holland	Denmark	Holland	England
Holland	Ireland	Denmark	Holland	Japan
England	Holland	Holland	Denmark	France
Ireland	Holland	Ireland	Holland	Ireland
Denmark	Ireland	Ireland	England	Japan

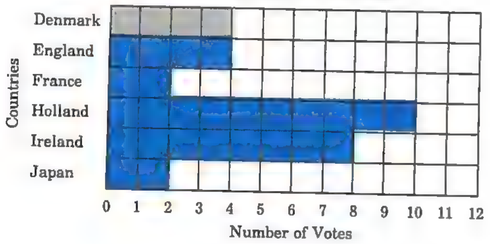
The children counted the votes and then began a table. Complete the table.

Votes for Countries	
Denmark	4
England	4
France	2
Holland	10
Ireland	8
Japan	2

Which country had the most votes? Holland

Next, the children made a bar graph showing the votes. Complete the shading.

VOTES FOR COUNTRIES



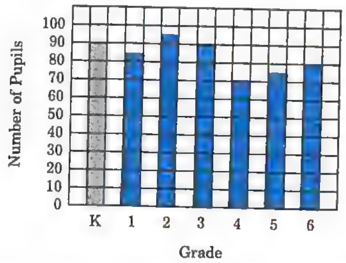
reference page

The number of pupils in each grade at Bay School is listed below.

Grade	Number of Pupils
Kindergarten	90
First	85
Second	95
Third	90
Fourth	70
Fifth	75
Sixth	80

Complete the graph. Then answer each question.

NUMBER OF PUPILS AT BAY SCHOOL



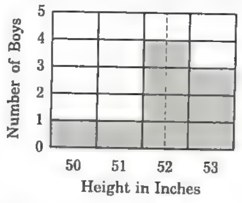
1. Which grade has the most pupils? Second grade
2. Which grade has the fewest pupils? Fourth grade
3. Which grades have 90 or more pupils? Kindergarten, second, and third grades
4. Which grades have fewer than 90 pupils? First, fourth, fifth, and sixth grades
5. How many pupils go to Bay School? 585

Name

For Class Discussion

The boys in Jim's Cub Scout den decided to make a graph of their heights. This is the graph they made.

HEIGHTS OF CUB SCOUTS IN JIM'S DEN



Jim calculated the average, or *mean*, height of the Scouts. First he added all the heights.

50 + 51 + 52 + 52 + 52 + 53 + 53 + 53 = 468

Then he divided the sum by the total number of Cub Scouts.

468 ÷ 9 = 52

His result is the *mean height* of the Scouts.

1. What is the mean height of the Cub Scouts in Jim's den? 52 inches
2. Does the dotted line on the graph show the mean? Yes
3. What is the height of the tallest Cub Scout? 53 inches
4. What is the height of the shortest Cub Scout? 50 inches
5. How does the mean height compare with the least height? 2 inches more than least height
   The greatest height? 1 inch less than greatest height

reference page

Answer each question.

1. Jack earned $26.95 by working 7 afternoons at the drugstore. He earned the same amount each afternoon. How much did he earn each afternoon? $3.85

2. On Monday the average temperature was 45 degrees. On Tuesday it was 35 degrees. Compute the average temperature for Monday and Tuesday. 40°

3. Last week Mrs. King's fourth-grade class made a chart of the noon temperatures. Compute the average noon temperature. 27°

Monday	38 degrees
Tuesday	26 degrees
Wednesday	18 degrees
Thursday	22 degrees
Friday	31 degrees

4. Six children made the following scores. Find the mean score.

a. Bob: | 10 | 30 | 20 |
Number of scores 3
Total 60 Mean 20

b. Jim: | 60 | 40 | 60 | 30 | 80 | 90 |
Number of scores 6
Total 360 Mean 60

c. Tom: | 30 | 40 | 40 | 50 |
Number of scores 4
Total 160 Mean 40

d. May: | 95 | 43 | 0 | 75 | 82 |
Number of scores 5
Total 295 Mean 59

e. Sue: | 56 | 88 | 34 | 42 |
Number of scores 4
Total 220 Mean 55

f. Kay: | 91 | 87 | 83 | 99 | 93 | 89 | 81 | 97 |
Number of scores 8
Total 720 Mean 90

● Pages 273 and 274 provide further practice in making and interpreting bar graphs. Assign the pages as independent work. Discuss the pupils' results when the pages have been completed.

● Page 275 requires the children to make and interpret a bar graph in order to answer specific questions. Assign the exercises for independent work. Then have the pupils discuss their answers.

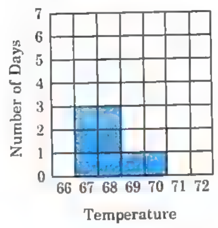
● Assign exercises 1 through 7 on page 276 for independent work. After studying the bar graph on this page and the one made for page 275, the pupils should realize that the mean score is greater for the second test but that the range of scores is less. Their conclusion should be that fewer pupils got low scores on the second test.

Name

Use the information given to make a bar graph. Each bar of the graph should show the number of days a temperature was recorded.

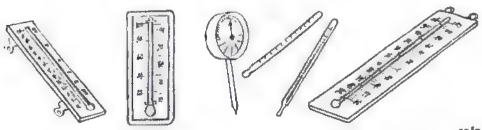
Noon Temperatures
68°
68°
67°
67°
68°
67°
70°
69°
Total 544°

NOON TEMPERATURES



Answer each question.

1. What is the mean noon temperature? 68°
2. What is the greatest noon temperature? 70°
3. What is the least noon temperature? 67°
4. To find the range of noon temperatures, subtract the least temperature from the greatest. The difference is the *range*. What is the range of noon temperatures? 3°



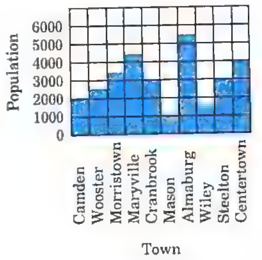
reference page

G-273

Use the information given to make a graph of the population of small towns near Midville.

Town	Population
Camden	2000
Wooster	2500
Morristown	3500
Maryville	4500
Cranbrook	3000
Mason	1000
Almaburg	5500
Wiley	1500
Steelton	3000
Centertown	4000

POPULATION OF THE SMALL TOWNS NEAR MIDVILLE



Answer each question.

1. Which town has the most people? Almaberg
2. Which town has the fewest people? Mason
3. What is the range of the populations of the towns? 4500
4. What is the total population of the towns? 30,500
5. What is the mean population of the towns? 3050
6. In which towns is the population above the mean? Morristown, Maryville, Almaburg, and Centertown
7. In which towns is the population below the mean? Camden, Wooster, Cranbrook, Mason, Wiley, and Steelton

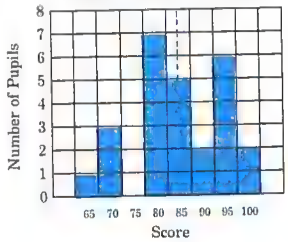
G-274

Name

On Monday, Mrs. Reed gave a spelling test. The pupils' scores are listed. Use the information given to make a bar graph.

	Scores		
90	65	95	70
70	80	70	85
80	100	95	80
80	80	85	95
80	85	95	95
100	90	80	85
95	85		

FIRST SPELLING SCORES



1. What is the highest score? 100
2. What is the lowest score? 65
3. What is the range of scores? 35
4. Complete the table.

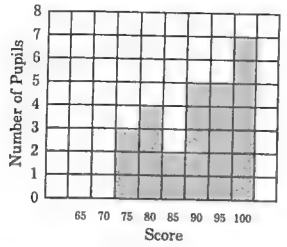
Score	Number of Pupils	Partial Sum
65	1	65
70	3	210
75	0	0
80	7	560
85	5	425
90	2	180
95	6	570
100	2	200
	Total 26	Total 2210

5. The mean score = 2210 ÷ 26 = 85.
6. Show the mean by making a dotted line on the graph.

G-275

On Friday, Mrs. Reed gave a second spelling test.
The pupils made a graph of their scores.

SECOND SPELLING TEST SCORES



1. How many pupils had scores of 80? 4
2. How many pupils had scores of 90? 5
3. Without computing, do you think the mean is higher or lower for this spelling test than for the first? Higher
4. What is the mean of the scores on the second test? 90
5. What is the highest score on the second test? 100
6. What is the lowest score on the second test? 75
7. What is the range of the scores on the second test? 25

G-276

Supplemental Experience

■ The pupils may enjoy making a bar graph of the enrollment in their school. This can be completed as a class or team activity.

Discuss how the left-hand side of the graph can be numbered. The class may decide to use any of these methods:

Find which grade has the greatest enrollment, and divide this number by the number of squares in one column of the grid. Suppose there are 68 pupils in the fifth grade and 10 squares to a column. The quotient of $68 \div 10$ is $6\frac{8}{10}$. Since the quotient is almost 7, let each square represent 7 children.

Choose a convenient number larger than the quotient, and let one square represent it. For example, let one square represent 8 children.

It might be easier to work with multiples of 5, 10, or 20. For example, let each square represent 10 children.

Review the importance of correct labeling. The numbering should start with zero. As the pupils make the graph, they will see that they must estimate the heights of the tallest bars in the process of labeling the graph.

When the pupils have completed the bar graph, ask questions about the grade with the greatest enrollment, the grade with the least enrollment, the total enrollment, and comparison of the enrollment in different grades.

UNIT 19
STORY EXERCISES
Pages 277 Through 288

OBJECTIVE

To explore the number structure of story exercises.

The pupil examines story exercises to increase his ability to abstract a mathematical sentence from a situation expressed in words. He learns that the number structure of a story exercise may be represented by more than one equation and that the same equation may be identified with many different story exercises.

See Key Topics in Mathematics for the Intermediate Teacher: Problem Solving.

KEY IDEA

Equations show number relationships.

KEY IDEA

Equations show number relationships.

Scope

To practice reading word exercises that involve numbers.

Fundamentals

This unit is primarily concerned with improving the pupil's ability to express in equations the number relationships described in story situations. As the pupil expresses these number relationships, he is demonstrating an understanding of the number structure of the story. Most exercises will not require computation. The goal is to have the pupil demonstrate an understanding of the number structure of the story by writing an appropriate equation.

The story exercises have been constructed so that two numbers are specifically stated and a third number is known when the exercise has been read and understood. For example,

> Vose School has 5 fourth grades with the same number of pupils in each class. There are 140 fourth graders. How many pupils are in each class?

Without computation, we can state that there are $140 \div 5$ pupils in each class. Any one of eight equations expresses the number structure of the above exercise.

$140 \div 5 = p$	$5p = 140$
$p = 140 \div 5$	$140 = 5p$
$140 \div p = 5$	$p \times 5 = 140$
$5 = 140 \div p$	$140 = p \times 5$

Consider another example.

> Tom is 58 inches tall. His father is 72 inches tall. How much taller than Tom is his father?

The question asked can be answered immediately if the exercise is understood. Tom's father is $72 - 58$ inches taller than Tom. Each of these equations represents one way to express the difference between Tom's height and his father's height.

$72 - 58 = t$	$t + 58 = 72$
$t = 72 - 58$	$72 = t + 58$
$72 - t = 58$	$58 + t = 72$
$58 = 72 - t$	$72 = 58 + t$

Readiness for Understanding

Ability to read comprehensively.
Knowledge of placeholder.

Developmental Experiences

▶ On the chalkboard, write several phrases taken from story exercises.

> 16 inches taller than Peter
> three times as many
> 7 fewer boomerangs than Jack has

Tell the pupils that you want them to translate each phrase into mathematical terms. Explain that they are to express the number described in each phrase as a product, a quotient, a sum, or a difference. Review the fact that a box or letter may be used to represent a number that is not specified. The children may give answers like these:

16 inches taller than Peter	$16 + p$
three times as many	$3 \times \square$
7 fewer boomerangs than Jack has	$j - 7$

Tell the children that sometimes all of the information they need is found in a phrase but sometimes the rest of the story may change their ideas about the number. Write this phrase and let pupils decide how they would express the number.

> 14 girls and 7 boys living on Oak Street

Some pupils may decide on $14 + 7$ while others would like more information. Add a few more words.

There are 14 girls and 7 boys living on Oak Street.

Write any suggested numbers on the chalkboard. Then add other phrases to the story. Let the children suggest their ideas about the number expressed each time.

At the end of the phrase, begin a new sentence:

How many . . .

Let the pupils discuss this. Then complete the sentence:

. . . times as many girls as boys live on Oak Street?

Use the following example in the same way.

Mr. Jones had a board 6 feet long that he sawed into 2 pieces. (Write *at the beginning*.)

One piece is 4 feet long. (Write *at the end*.)

How long is the other piece? (Write *at the end*.)

The children should understand that it is important to have all of the relevant information before making a decision about the number.

Some pupils may enjoy creating stories in a similar manner.

► Write a story exercise on the chalkboard and read it with the class. Tell the pupils that you are going to write an equation that shows the number relationship in the story.

There were 35 fleas on Miss Jackson's dog, Wolfgang. She gave him a bath and some of the fleas hopped away. Now there are 31 fleas on Wolfgang. How many fleas hopped away?

$$35 - c = 31$$

Point out the fact that in writing the equation you have eliminated all information that does not affect the number relationship in the story. Ask the pupils to name some of the things that do not affect the answer to the question. They may suggest the dog's name and his owner's name, the fact that the dog had a bath, and the fact that fleas hop.

Write another story exercise on the chalkboard. This time, have the pupils suggest an equation that shows the number relationship.

During January, Pat made \$3.75 babysitting for the Marshalls. If she was paid \$.75 each time, how many times did she babysit during January?

Then ask the pupils to identify the information that does not affect the number relationship of the story (Pat's name, the fact that Pat was paid for babysitting, and the fact that she babysat in January).

Continue in this manner to use other story exercises. In each instance, have the pupils write an equation that expresses the number structure of the story exercise and identify the information in the story that is irrelevant to the number structure. Some suggested stories are:

John and Bob played a dart game. John scored 26 points and Bob scored 34 points. What was the difference in their scores?

Mrs. Bartelsen brought back 10 yards of silk from her trip to Thailand. She gave 3 yards to her niece Andrea. How much of the silk did she have left?

To reach his favorite fishing spot, Jack rowed a boat on Lake Perry for 12 minutes. Bill rowed the boat back. Since he was rowing against the current it took him 2 times as long. How long did it take Bill to row back?

A team of 9 boys gathered wood for a campfire. They collected 63 pieces of wood. If each boy collected the same number of pieces, how many did each collect?

Thirteen smoots from Kansas and 17 smoots from Algeria went gockling. How many smoots went gockling?

Story exercises like the last one will help the pupils realize that the particular people, places, objects, and so on do not affect the number relationship of a story.

Do not ask the pupils to answer the story question at this time. This activity will help pupils develop their ability to find essential information and write equations that give the number relationship in the story.

► Write the equation $8 + c = 12$ on the chalkboard. Tell the children that they are to make up a story exercise for which $8 + c = 12$ shows the number relationship. If the pupils have trouble making up an appropriate story exercise, give several examples like the following:

Cinderella looked at the clock. It showed 8 o'clock. Her coach will turn into a pumpkin at 12 o'clock. How long would it be before her carriage turned into a pumpkin?

Jacob had 12 glibons of squares, but he lost some. Now he has 8 glibons of squares. How many did he lose?

Help the pupils to see that, although the stories are different, the number relationships are the same. Have two or three pupils make up other stories for which $8 + c = 12$ shows the number relationship. Then write the equation $24 \div 8 = n$ on the chalkboard and have four or five pupils tell stories for which this is an appropriate equation.

Continue in this way to write equations and have the pupils tell 4 or 5 different stories for each one.

Pages 277 through 288

● Pages 277 and 278 provide practice in writing equations to express the numerical ideas in story exercises that involve addition or subtraction.

With the class, read the story exercise at the top of page 277. Then have the pupils tell which equations show the number relationship expressed in the story. The pupils should observe that all the given equations show the sum-addend relationship of the numbers in the story. Next, ask how each equation expresses the situation of the story. One child may suggest that all of the addition equations tell about the 24 panels. Another may suggest that the equations $24 - 9 = 15$ and $15 = 24 - 9$ describe the number of panels Mr. French still has. A third pupil may state that the equations $24 - 15 = 9$ and $9 = 24 - 15$ describe the number of panels that were used. Finally, have the pupils identify the information that does not affect the number relationship of the story.

Exercises 1 through 6 can be assigned for independent work. Review the fact that a box or a letter may be used to represent the number that is not specified and be sure the pupils understand that they need to write only one addition equation for each story. After the class has completed this assignment, ask a child to write his equation for exercise 1 on the chalkboard and to identify the sum and addends in his equation. If other pupils have selected different addition equations for exercise 1, have them listed on the chalkboard. Follow a similar procedure with exercises 2 through 6.

Help the pupils see that, although the stories differ, the equations for exercises 1 through 3 are the same and those for exercises 4 through 6 are the same.

Now have the pupils write one subtraction equation for each story exercise. Help them understand that, because of the relation of subtraction to addition, the subtraction equation can be derived from the addition equation. For example, given the addition equation $27 + 19 = c$ for exercise 1, one of the addends (27 or 19) can be subtracted from the sum (c) to give the other addend. After the pupils have completed this assignment, call for a volunteer to write a subtraction equation for exercise 1 on the chalkboard. Tell him to identify the sum and addends in his equation. Then have pupils who have selected different subtraction equations write their equations on the chalkboard. Continue in this way with the subtraction equations for exercises 2 through 6.

● Follow a procedure similar to that suggested for page 277 in having the pupils complete page 278. They will observe that exercises 1 through 3 have the same number structure, exercises 4 through 6 have the same number structure, and exercises 7 through 9 have the same number structure.

Name

UNIT 19 STORY EXERCISES

See pupil page suggestions.

Mr. French bought 24 wood panels. He put 9 panels on one wall. He has 15 panels left for another wall.

Which equations show the number relationship in the story?

$24 = 9 + 15$	$9 + 15 = 24$	$24 - 9 = 15$	$15 = 24 - 9$
$24 = 15 + 9$	$15 + 9 = 24$	$24 - 15 = 9$	$9 = 24 - 15$

Each equation shows this number relationship.

Write one addition equation for each story. *Equations may vary.*

1. Mr. French used 27 long nails and 19 short nails. How many nails did he use? *$27 + 19 = a$*
2. There are 46 doors in Maplewood School. All but 19 doors need repair. How many doors need repair? *$19 + a = 46$*
3. Mr. Jacobs has 27 cupboards. He wants 46 cupboards. How many cupboards must he make? *$46 = 27 + a$*
4. Mr. Post had a board 110 inches long. He cut it into 2 pieces. One piece is 39 inches long. How long is the other piece? *$39 + a = 110$*
5. Mr. Selker cut a board into 2 pieces. One piece is 71 inches long. The other is 39 inches long. How long was the board before he cut it? *$a = 39 + 71$*
6. Mr. Paul sawed a piece of wood 71 inches long from a 110-inch board. How long is the other piece? *$71 + a = 110$*

Write one subtraction equation for each story. *Equations may vary.*

1. *$a - 27 = 19$*
2. *$46 - 19 = a$*
3. *$46 - 27 = a$*
4. *$110 - 39 = a$*
5. *$a - 71 = 39$*
6. *$110 - 71 = a$*

reference page

H-277

See pupil page suggestions.

Sam weighs 78 pounds. Jeff weighs 18 pounds more than Sam. Jeff weighs 96 pounds.

Which equations show the number relationship in the story?

$96 - 78 = 18$	$18 = 96 - 78$	$78 + 18 = 96$	$96 = 78 + 18$
$96 - 18 = 78$	$78 = 96 - 18$	$18 + 78 = 96$	$96 = 18 + 78$

Each equation shows this number relationship.

Write one subtraction equation for each story. *Equations may vary.*

1. Betty had a 24-inch piece of ribbon. After she used some, she had 6 inches left. How much did she use? *$24 - 6 = b$*
2. After Sara used 18 inches of ribbon, she had 6 inches left. How much ribbon did Sara start with? *$6 = b - 18$*
3. Paula used 18 inches of ribbon from a 24-inch piece of ribbon. How much did she have left? *$24 - 18 = b$*
4. Peter's father is 16 inches taller than Peter. Peter is 56 inches tall. How tall is his father? *$b - 16 = 56$*
5. Tom is 56 inches tall. His father is 72 inches tall. How much taller than Tom is his father? *$72 - 56 = b$*
6. Paul's father is 72 inches tall. He is 16 inches taller than Paul. How tall is Paul? *$72 - 16 = b$*
7. Fred carried 16 quarts of water in 2 pails. There were 6 quarts in one pail. How much was in the other pail? *$b = 16 - 6$*
8. Sue carried 16 quarts of water. If she carried 10 quarts in one pail, how much did she carry in the other pail? *$16 - 10 = b$*
9. Ann carried 10 quarts of water in one pail and 6 quarts in another. How much did she carry in the 2 pails? *$10 = b - 6$*

Write one addition equation for each story. *Equations may vary.*

1. *$24 = b + 6$*
2. *$18 + 6 = b$*
3. *$18 + b = 24$*
4. *$16 + 56 = b$*
5. *$72 = 56 + b$*
6. *$b + 16 = 72$*
7. *$16 = 6 + b$*
8. *$10 + b = 16$*
9. *$10 + 6 = b$*

H-278

● Pages 279 and 280 provide practice in giving multiplication and division equations for story exercises.

With the class, read the story at the top of page 279. Guide the pupils to observe that each of the given equations shows the number relationship expressed in this story. After you have read the stories with the class and the pupils understand the procedure to be followed, assign exercises 1 through 9 for independent work. When the pupils have completed this assignment, have them identify the details in exercise 1 that do not affect the equations. Then ask for a volunteer to write the four multiplication equations for exercise 1 on the chalkboard. Have the class identify the product and factors in each equation. Tell the pupils that if they have written any one of the four equations for exercise 1, they have completed exercise 1 correctly. Follow the same procedure with each of the remaining exercises. Then ask the pupils which stories resulted in the same equations (exercises 1 through 3, 4 through 6, and 7 through 9).

Next have the children write one division equation for each exercise. When they have completed this assignment, follow the procedure used in checking the multiplication equations.

The children should complete page 280 in the same way as page 279 except that they are to write the division equations first. Since writing correct equations is the goal of this activity, no attempt should be made to have the pupils solve the equations at this time.

● Pages 281 and 282 give the pupils practice in writing equations for story exercises.

With the class, read the story at the top of page 281. The pupils will observe that the given multiplication equations apply to the story but the given addition equations do not. Then have nine pupils read aloud the stories in the exercises. Explain to the children that they are to write one addition or one multiplication equation for each story. Then have them complete exercises 1 through 9 on their own. After they have completed this assignment, have them write one subtraction or one division equation for each story. Then, on the chalkboard, have some pupils write the equations they have selected for specific exercises.

● Page 282 should be completed in the same way as page 281 except that the pupils are to write first one subtraction or division equation and then one addition or multiplication equation for each story exercise. Have pupils identify the information that does not affect the equations.

● Use page 283 to allow the pupils to test their ability to write equations for story exercises. Read the stories with the class and have the pupils complete exercises 1 through 12 on their own. Be sure they understand that they need to write only one equation for each story. After the assignment has been completed, discuss those exercises that have caused difficulty. The pupils may observe that some of the story exercises have the same number structure. (Exercises 1, 5, and 9; 2, 4, and 8; 3 and 10; 6 and 11; and 7 and 12 have the same structure.)

Name

See pupil page suggestions.

A farmer packed 36 boxes of apples, each weighing 45 pounds. The 36 boxes of apples weighed 1620 pounds.

Which equations show the number relationship in the story?

$45 \times 36 = 1620$	$1620 = 45 \times 36$	$1620 \div 36 = 45$	$45 = 1620 \div 36$
$36 \times 45 = 1620$	$1620 = 36 \times 45$	$1620 \div 45 = 36$	$36 = 1620 \div 45$

Each equation shows this number relationship.

Write one multiplication equation for each story. Equations may vary.

1. Kay bought 5 boxes of cards with 24 cards in each box. How many cards did she buy? $24 \times 5 = a$
2. George has 120 cards. There are 24 cards in each box. How many boxes of cards does he have? $a \times 24 = 120$
3. Tim has 5 boxes of cards with the same number in each box. He has 120 cards. How many are in a box? $5 \times a = 120$
4. Frank made 3 toys a day until he had 54 toys. How many days did it take him? $3 \times a = 54$
5. Sam made 54 toys. He made the same number each day for 18 days. How many toys did he make each day? $54 = a \times 18$
6. Jane made 3 toys a day for 18 days. How many toys did she make? $3 \times 18 = a$
7. Bill bought 6 games that cost 59¢ apiece. How much did he pay for the games? $\$.59 \times 6 = a$
8. Diane paid \$3.54 for 6 games. If all of the games cost the same amount, what was the cost of one game? $6 \times a = \$3.54$
9. Janet spent \$3.54 for some games that cost 59¢ apiece. How many games did she buy? $\$.59 \times a = \$3.54$

Write one division equation for each story. Equations may vary.

1. $a \div 5 = 24$
2. $24 = 120 \div a$
3. $120 \div 5 = a$
4. $54 \div 3 = a$
5. $a = 54 \div 18$
6. $a \div 3 = 18$
7. $\$.59 = a \div 6$
8. $\$3.54 \div a = 6$
9. $\$3.54 \div \$.59 = a$

reference page

H-279

See pupil page suggestions.

Gayle had \$2.50 to spend on candy for a party. She bought 50 candy bars that cost 5¢ each.

Which equations show the number relationship in the story?

$250 \div 50 = 5$	$5 = 250 \div 50$	$50 \times 5 = 250$	$250 = 50 \times 5$
$250 \div 5 = 50$	$50 = 250 \div 5$	$5 \times 50 = 250$	$250 = 5 \times 50$

Each equation shows this number relationship.

Write one division equation for each story. Equations may vary.

1. Bob has 12 9-inch pieces of track for his model train. How long a track can he make with the 12 pieces? $12 = a \div 9$
2. John has enough 9-inch pieces of track to make a track 108 inches long. How many pieces does he have? $108 \div 9 = a$
3. Each piece of Jim's track has the same length. How long is each piece if 12 pieces make a 108-inch track? $108 \div 12 = a$
4. There are 24 pens in each box. How many pens are in 13 boxes? $13 = a \div 24$
5. The same number of pens are in each of 13 boxes. If there are 312 pens, how many pens are in each box? $312 \div 13 = a$
6. If 24 pens are placed in each box, how many boxes can be filled with 312 pens? $312 \div 24 = a$
7. It takes 12 hours for a machine in the post office to sort 65,040 letters. How many letters does it sort in one hour? $65{,}040 \div 12 = a$
8. A machine sorts 5420 letters in an hour. How many letters will it sort in 12 hours? $a \div 12 = 5420$
9. A machine has 65,040 letters to sort. If it sorts 5420 an hour, how long will it take to sort all the letters? $65{,}040 \div 5420 = a$

Write one multiplication equation for each story. Equations may vary.

1. $9 \times 12 = a$
2. $9 \times a = 108$
3. $12 \times a = 108$
4. $a = 24 \times 13$
5. $13 \times a = 312$
6. $312 = 24 \times a$
7. $a \times 12 = 65{,}040$
8. $a = 5420 \times 12$
9. $5420 \times a = 65{,}040$

H-280

Name

See pupil page suggestions.

Mary baked 3 pans of cookies with 24 cookies on each pan. How many cookies did Mary bake?

Which equations show the number relationship in the story?

$3 + 24 = c$	$c = 3 + 24$	$3 \times 24 = c$	$c = 3 \times 24$
$24 + 3 = c$	$c = 24 + 3$	$24 \times 3 = c$	$c = 24 \times 3$

The multiplication equations show this number relationship.

Write one addition equation or one multiplication equation for each story. **Equations may vary.*

1. Brian collects \$19.50 from all the customers on his paper route. He collects 50¢ from each. How many customers does he have? *$1950 = 50 \times a$*
2. Scott collects \$19.50 from his 39 customers. Each pays the same amount. How much does he collect from each customer? *$39 \times a = 1950$*
3. Ray collects 50¢ from each of the 39 customers on his paper route. How much money does he collect? *$a = 50 \times 39$*
4. There are 100 tablets in 1 box and 144 tablets in another box. How many tablets are in the two boxes? *$100 + 144 = a$*
5. There are 244 tablets in 2 boxes. If there are 100 tablets in one box, how many tablets are in the other? *$100 + a = 244$*
6. There are 244 tablets in 2 boxes. There are 144 tablets in one box. How many tablets are in the second box? *$244 = a + 144$*
7. Ridge School has 140 fourth graders in 5 classes, with the same number in each class. How many pupils are in each class? *$5 \times a = 140$*
8. West School has 140 fourth graders. There are 28 pupils in each class. How many fourth-grade classes are in West School? *$28 \times a = 140$*
9. Lee School has 5 fourth-grade classes with 28 pupils in each class. How many fourth graders are in Lee School? *$5 \times 28 = a$*

Write one subtraction equation or one division equation for each story. *Equations may vary.*

1. *$a = 1950 \div 50$*
2. *$1950 \div 39 = a$*
3. *$a \div 39 = 50$*
4. *$a - 144 = 100$*
5. *$a = 244 - 100$*
6. *$244 - 144 = a$*
7. *$a = 140 \div 50$*
8. *$140 \div 28 = a$*
9. *$a \div 5 = 28$*

H-281

** Symbols for dollars and cents need not be used.*

See pupil page suggestions.

Mike and Ed planned a puppet show. Mike sold 8 tickets and Ed sold 14 tickets. How many tickets did they sell?

Which equations show the number relationship in the story?

$t - 14 = 8$	$8 = t - 14$	$t \div 14 = 8$	$8 = t \div 14$
$t - 8 = 14$	$14 = t - 8$	$t \div 8 = 14$	$14 = t \div 8$

The subtraction equations show this number relationship.

Write one subtraction equation or one division equation for each story. *Equations may vary.*

1. Rick has 4 boxes of pins with 300 in each box. How many pins does Rick have? *$b \div 4 = 300$*
2. Dan has 1200 pins in boxes. There are 300 in each box. How many boxes of pins does Dan have? *$1200 \div 300 = b$*
3. Martha has 1200 pins in 4 boxes. If each box has the same number of pins in it, how many are in one box? *$b = 1200 \div 4$*
4. Jay has 35 books on a shelf. Keith borrowed 17 of the books. Then how many books were on the shelf? *$35 - 17 = b$*
5. Walt had 18 books left after David borrowed 17. How many books did Walt have before any were borrowed? *$18 = b - 17$*
6. Bonnie had 35 books. After Val borrowed some, Bonnie had 18 books. How many books did Val borrow? *$35 - 18 = b$*
7. Fred ate 7 of the 56 cookies his mother baked. How many cookies were left? *$56 - 7 = b$*
8. Pat's mother baked 56 cookies. After Pat ate some there were 49 left. How many cookies did Pat eat? *$56 - b = 49$*
9. After Lewis ate 7 of the cookies his mother had baked, 49 were left. How many cookies had she baked? *$49 = b - 7$*

Write one addition equation or one multiplication equation for each story. *Equations may vary.*

1. *$300 \times 4 = b$*
2. *$b \times 300 = 1200$*
3. *$1200 = 4 \times b$*
4. *$17 \times b = 35$*
5. *$b = 17 + 8$*
6. *$b + 18 = 35$*
7. *$7 + b = 56$*
8. *$b + 49 = 56$*
9. *$b = 7 + 49$*

H-282

Name

Write an equation for each story. *Equations may vary. Symbols for dollars and cents need not be used.*

1. Jack sold 12 of his 40 chickens. How many chickens did he have after the sale? *$12 + a = 40$*
2. Ann spent \$4.14 for 6 dolls. If all the dolls sold for the same price, how much did one doll cost? *$414 \div 6 = a$*
3. It takes 6 hours for a newspaper press to print 24,216 papers. How many papers can the press print in one hour? *$6 \times a = 24{,}216$*
4. Herb bought 6 games at 69¢ apiece. How much did he pay for all the games? *$a = 69 \times 6$*
5. Marie has 28 books left after giving 12 away. How many books did Marie have before she gave any away? *$a - 12 = 28$*
6. Cindy spelled 17 out of 20 words correctly. How many words did she misspell? *$17 + a = 20$*
7. Don caught 12 fish. He caught 3 times as many as Ken. How many fish did Ken catch? *$12 \div 3 = a$*
8. Karen paid \$4.14 for notebooks costing 69¢ apiece. How many notebooks did she buy? *$414 \div 69 = a$*
9. Dean had 40 marbles. Some dropped through a hole in his marble bag, leaving only 28. How many marbles dropped through the hole? *$40 - 28 = a$*
10. A press printed 4036 newspapers an hour. How long will it take to print 24,216 papers? *$4036 \times a = 24{,}216$*
11. Polly misspelled 3 words and spelled 17 words correctly. How many words did she try to spell? *$a = 17 + 3$*
12. Ted caught 4 frogs. Tony caught 3 times as many as Ted. How many frogs did Tony catch? *$4 \times 3 = a$*

H-283

● Page 284 introduces what is meant by solving an equation. As the page is discussed, the children should understand that solving an equation may involve computation. The unspecified number, shown as a letter or □, is the result of the computation.

Have the class discuss the examples at the top of the page. Then continue with the following examples.

$$c - 10 = 14$$
$$c = 14 - 10$$
$$8 \div 2 = c$$
$$c \div 3 = 4$$

Ask some pupils to tell how each equation can be solved.

Let several pupils work at the chalkboard while the rest work on paper to solve the equations in exercises 1 through 3. Discuss any problems that arise. Then assign exercises 4 through 12 for independent work.

● Pages 285 through 288 provide practice in writing and solving equations for stories. Have the pupils write an equation for each story exercise and write a sentence to answer each story question. As you discuss exercise 4, on page 287, point out that some of the numbers given in a story exercise may have no bearing on the question.

How can an equation be solved?

$27 + 19 = c$

Compute the sum.

$$\begin{array}{r} 27 \\ +\ 19 \\ \hline 16 \\ 30 \\ \hline 46 \end{array} \quad c = 46$$

$27 + m = 46$

Think of the relation of subtraction to addition.

$46 - 27 = m$

$$\begin{array}{r} \overset{3}{4}6 \\ -\ 27 \\ \hline 19 \end{array} \quad m = 19$$

$59 \times 6 = h$

Compute the product.

$$\begin{array}{r} 59 \\ \times\ 6 \\ \hline 54 \\ 300 \\ \hline 354 \end{array} \quad h = 354$$

$12b = 108$

Think of the relation of multiplication to division.

$108 \div 12 = b$

$$\begin{array}{r|l} 12\overline{)108} & \\ 72 & 6 \\ \hline 36 & \\ 36 & 3 \\ \hline 0 & 9 \end{array} \quad b = 9$$

Solve each equation.

1. $39 = 110 - j$ $j =$ 71
2. $s - 71 = 29$ $s =$ 100
3. $24 - r = 6$ $r =$ 18
4. $f = 56 + 16$ $f =$ 72
5. $6 = b - 10$ $b =$ 16
6. $5 \times 24 = c$ $c =$ 120
7. $120 = 24p$ $p =$ 5
8. $54 \div 18 = b$ $b =$ 3
9. $t \div 24 = 13$ $t =$ 312
10. □ + 6 = 59 □ = 53
11. 17 = □ ÷ 7 □ = 119
12. 18 = 72 ÷ □ □ = 4

reference page

H-284

Name

Write one equation for each story. Solve the equation. Answer each question.
Equations may vary. Symbols for dollars and cents need not be used.

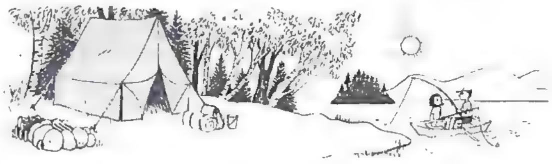
1. Jean spent 36¢ for 4 pencils. How much did she spend for each pencil? $36 = 4a$ $a = 9$
She spent 9 cents for each pencil.

2. Marc is 47 inches tall. Dennis is 53 inches tall. How much taller than Marc is Dennis? $53 - 47 = b$ $b = 6$
Dennis is 6 inches taller.

3. Judy saved her entire allowance for 4 weeks. By saving the same amount each week, she saved $1.40. How much is Judy's allowance each week? $c = 1.40 \div 4$ $c = .35$
Her allowance each week is $.35.

4. Greg and Marsha caught 36 fish. Greg caught 19 fish. How many did Marsha catch? $36 = 19 + d$ $d = 17$
Marsha caught 17 fish.

5. A camping trip costs $196 for 7 days. What is the cost for 1 day? $196 \div 7 = b$ $b = 28$
The cost for one day is $28.



H-285

Write one equation for each story. Solve the equation. Answer each question.
Equations may vary.

1. The load limit on a bridge is 4000 pounds. Mr. Swenson's truck weighs 3700 pounds. How much less than the limit is this? $3700 + d = 4000$ $d = 300$
This is 300 pounds less than the limit.

2. Mrs. Brown baked 9 pizzas and cut each one into 12 pieces. How many pieces of pizza did she cut? $e = 12 \times 9$ $e = 108$
She cut 108 pieces of pizza.

3. On opening day East School registered all of its pupils. After one week, 19 children left the school. Then there were 842 pupils in East School. How many pupils registered on opening day? $f - 19 = 842$ $f = 861$
861 registered on opening day.

4. Jill has 1023 shells. She has three times as many shells as Kent. How many shells does Kent have? $1023 \div 3 = g$ $g = 341$
Kent has 341 shells.

5. Mrs. Jones spent $7.43. She received $2.57 change. How much money did she give the clerk for her purchase? $7.43 + 2.57 = m$ $m = 10.00$
Mrs. Jones gave the clerk $10.00.

H-286

Name

Write one equation for each story. Solve the equation. Answer each question.
Equations may vary. Symbols for dollars and cents need not be used.

1. Carol packed 135 cookies in 9 boxes. She put the same number in each box. How many cookies did Carol put in each box? $135 \div 9 = l$ $l = 15$
She put 15 cookies in each box.

2. Dick earned $41 in January and $33 in February. How much did he earn in the two months? $41 + 33 = j$ $j = 74$
Dick earned $74.

3. Ellen shared her collection of 56 leaves equally with 3 other girls. How many leaves does each girl have? $56 = 4 \times m$ $m = 14$
Each girl has 14 leaves.

4. In one month River Camp had 287 campers the first week, 432 the second week, 305 the third week, and 249 the fourth week. How many more campers were there the first week than the fourth week? $287 - 249 = n$ $n = 38$
There were 38 more campers the first week.

5. Dottie bought 6 pencils for 23¢. What is the cost of each pencil? $23 \div 6 = p$ $p = 3\frac{5}{6}$
Each pencil costs $3\frac{5}{6}$ cents.

H-287

Write one equation for each story. Solve the equation. Answer each question.
Equations may vary. Symbols for dollars and cents need not be used.

1. The Adolph family traveled 144 miles to get from their home to Riverside. Then they traveled the same distance to get home. How far did they travel on the whole trip? $g = 2 \times 144$ $g = 288$
They traveled 288 miles on the whole trip.

2. By noon the outside temperature had risen 14 degrees to 67 degrees. How cold had it been before the temperature began to rise? $67 = r + 14$ $r = 53$
It had been 53°.

3. Nan can drive her car 231 miles on 11 gallons of gasoline. How many miles can Nan's car go on 1 gallon of gasoline? $s = 231 \div 11$ $s = 21$
Nan's car can go 21 miles on a gallon of gasoline.

4. Don owes his father 15¢. Don's allowance is 75¢. How much money will Don get this time if he asks his father to keep the amount that he owes him? $75 - 15 = l$ $l = 60$
Don will get $.60.

5. Two famous national parks are Grand Canyon Park, which covers 673,203 acres, and Yellowstone Park, which covers 2,213,207 acres. How many acres larger than Grand Canyon Park is Yellowstone Park? $2{,}213{,}207 - 673{,}203 = u$
$u = 1{,}540{,}004$ Yellowstone Park is 1,540,004 acres larger.

H-288

Supplemental Experiences

■ Write lists of equations and story exercises on the chalkboard, or duplicate the lists so that each pupil has a copy.

Have the students match each story exercise in column A with an equation from column B that shows the number relationship in the story.

A

1. Carol asked the candy store clerk for a dollar's worth of 5-cent jawbreakers. How many did the clerk give Carol?
2. The fourth grade wanted to buy a kangaroo rat that cost $4.75 and a miniature orange tree that cost $6.50. How much money did they need?
3. Mrs. Wetherton offered to pay Gayle $5.00 for giving a birthday party for her twin daughters. She gave Gayle $2.50 to spend on favors for the party. How many 10-cent favors could Gayle buy with the money?
4. Gene bought 12 bags of jumping beans at $.43 a bag. How much did Gene spend on jumping beans?
5. Alice earned $12.15 walking dogs during the summer. She bought a wool skirt for $8.47. How much money did she have left?
6. Each blop swallowed 56 gomas. How many gomas did 7 blops swallow?

B

a. $b \div 10 = 250$
b. $43 \div n = 12$
c. $d - 847 = 1215$
d. $100 - 5 = n$
e. $g \div 7 = 56$
f. $5p = 100$
g. $847 + s = 1215$
h. $56 \div 7 = g$
i. $650 - 475 = m$
j. $10w = 250$
k. $5 \times 100 = p$
l. $w = 475 + 650$
m. $12 \times 43 = p$

■ Write the following chart and story exercises on the chalkboard. Tell the pupils to use the information given on the chart to answer each story question.

School Supplies

crayons	16¢	notebook paper	15¢	pens	39¢
erasers	8¢	paste	10¢	rulers	14¢
notebooks	69¢	pencils	4¢	tablets	12¢

1. Sue and Dick Stevens each need to buy a tablet, a pencil, an eraser, and a box of crayons. What is the total cost of their supplies?
2. Peter Moore needs a notebook and a pen. Nancy Moore needs a ruler and a notebook. Marie Moore needs a notebook, a ruler, and a pen. How much money do the Moore children need to buy these supplies?
3. Carl, Ellen, Teresa, and Jeremy Adams each need a tablet, a ruler, and paste. Ellen also needs a notebook, a pen, and 2 packages of notebook paper.

What is the total cost of supplies for the Adams family?

4. Sarah needs a box of crayons, an eraser, a notebook, a package of notebook paper, paste, 2 pencils, a pen, a ruler, and a tablet. How much will her supplies cost?
5. The Caldwell family has six children. Five of them need one tablet and one eraser each. Four of them need one pen, one ruler, and one box of crayons each. All of them need 2 pencils each. Two of them need one notebook and 2 packages of notebook paper each. How much change will the family get from a $10 bill after buying these supplies?

■ On the chalkboard, write a story in which an insufficient amount of information is given to answer the story question.

Mark had 302 football cards. He has more than Mike. How many football cards do the two boys have together?

Explain to the class that it is possible to write an equation showing the number relationship of this story although the question cannot be answered. Have the class help select a different placeholder for each number not specifically named, such as *a* for the number of cards Mike has and *b* for the number of cards both boys have together. Have a pupil write an equation for the story. He may write any of the following:

$302 + a = b$	$b - 302 = a$
$a + 302 = b$	$b - a = 302$
$b = 302 + a$	$a = b - 302$
$b = a + 302$	$302 = b - a$

Adapt the described procedure to the following story exercises.

Archie bought 3 hotdogs and one 10-cent root beer. How much did Archie spend?

Brian has a dozen nickels. Together Kerry and Sheila have 32 nickels. How many nickels does Kerry have?

Marilyn spent 25¢ for admission to the swimming pool, $1.25 for a new bathing cap, and 15¢ to rent a locker. How much money does she have left?

Sam wants to go to a professional football game. He has $.75. How much more money does he need to buy a ticket to the game?

Suzanne earned money by running a play group during the summer. If 14 children attended, how much did she charge for each child?

Mrs. Francis received 12 letters from Tanzania. She distributed the stamps equally among her children. How many stamps did each child receive?

UNIT 20
OPEN SENTENCES
Pages 289 Through 300

OBJECTIVE

To explore open sentences.

The pupil learns that some sentences are true and some are not. He learns that replacing the variable in an open sentence may result in either a true sentence or a false sentence. He investigates several equations that have an unlimited number of solutions.

See Key Topics in Mathematics for the Intermediate Teacher: Mathematical Sentences.

KEY IDEA

Some sentences are true; some are not.

CONCEPTS

true
open sentence
solution

KEY IDEA

Some sentences are true; some are not.

Scope

To distinguish between true and false sentences.
To introduce the pupil to open sentences.

Fundamentals

Mathematical sentences may take the form of equations or inequalities.

$2 + 3 = 5$
$3 < 5$
$5c = 45$
$7 + \square = 10$

The pupil has had experience using a placeholder to name a sum, a product, a missing factor, or a missing addend.

$2 + a = 5$	(missing addend)
$9 \times 7 = \square$	(product)
$3a = 18$	(missing factor)
$17 + 21 = b$	(sum)

In these cases, although the standard numeral is missing, each statement is thought of as a true statement and the number is actually known. For example, if $3a = 18$, then $a = 18 \div 3$.

Now the pupil is introduced to the idea that mathematical sentences may be true or false.

True sentences: $3 + 2 = 5$
$3 < 8 + 6$
$4 + 1 > 2 + 1$

False sentences: $14 - 3 = 9$
$6 \times 3 > 21$
$10 < 100 \div 20$

He is then introduced to the idea of an open sentence.

Open sentences: $3 + x = 5$
$2y > 15$
$y + 9 < 18$
$\square \times \square = 9$

These are introduced as sentences which may be used to *generate* other sentences, which are either true or false.

$3 + x = 5$		$\square \times \square = 9$	
$3 + 1 = 5$	(F)	$1 \times 1 = 9$	(F)
$3 + 2 = 5$	(T)	$2 \times 2 = 9$	(F)
$3 + 3 = 5$	(F)	$3 \times 3 = 9$	(T)

In the sentence $3 + x = 5$, x is a variable. This means that x does not represent a specific number, but represents any number of a specified set, the replacement set. If the replacement set is the Set of Whole Numbers, x may be 0, or 1, or 2, or any other whole number. In this case, $3 + x = 5$ generates the sentences $3 + 1 = 5$, $3 + 2 = 5$, $3 + 3 = 5$, and so on. Of these, only $3 + 2 = 5$ is a true sentence. The number 2 is the solution to the equation $3 + x = 5$.

When the replacement set is the Set of Whole Numbers, the sentence $2y > 15$ represents the sentences $2 \times 0 > 15$, $2 \times 1 > 15$, $2 \times 2 > 15$, $2 \times 3 > 15$, and so on, because y may be any whole number. The sentence $2y > 15$ is true for all whole numbers greater than 7, that is, for 8, 9, 10, 11, and so on. Thus the inequality $2y > 15$ has many solutions. The solution set is $\{8, 9, 10, \ldots\}$. Note that if the replacement set is the Set of Fractional Numbers, the solution set would be different. It would consist of all fractional numbers greater than $7\frac{1}{2}$.

When the replacement set is the Set of Whole Numbers, the inequality $y + 9 < 8$ has no solution since the sentences $0 + 9 < 8$, $1 + 9 < 8$, $2 + 9 < 8$, $3 + 9 < 8$, and so on, are all false; the solution set in the Set of Whole Numbers for $y + 9 < 8$ is the empty set $\{\ \}$.

An open sentence may have any number of solutions. This is illustrated in the following table. The replacement set is the Set of Whole Numbers.

Open Sentence	Solution Set	Number of Solutions
$4 + x = 3$	$\{\ \}$	0
$5 - y = 2$	$\{3\}$	1
$6 + y < 8$	$\{0, 1\}$	2
$10 > z + 7$	$\{0, 1, 2\}$	3
$5 < x + 2$	$\{4, 5, 6, \ldots\}$	Unlimited
$5 + x = 5 + x$	$\{0, 1, 2, \ldots\}$	Unlimited

The new words used in this unit are of little importance; the main objective is to introduce the pupil to open sentences and let him determine when a sentence is true. Note that the pupils will use the Set of Whole Numbers as the replacement set throughout the unit.

Readiness for Understanding

Knowledge of equations.

Developmental Experiences

8 tagboard cards (3″ × 9″)
8 tagboard cards (4″ × 10″)
felt-tip pen
pocket chart

► Write these sentences on the chalkboard.

$$17 = 9 + 8$$
$$3 + 4 < 3 \times 4$$
$$8 \times 2 > 8 \div 2$$

Ask the pupils to tell what each sentence means. The first sentence is an equation; it tells us that seventeen is nine plus eight. The second sentence is an inequality; it tells us that three plus four is less than three times four. The third sentence is also an inequality; it tells us that eight times two is greater than eight divided by two.

Point out that an equation such as $3 + 4 = 7$ says "the number $3 + 4$ is the number 7." It is comparable to a sentence such as "Columbus is the capital of Ohio." In this sentence, Columbus and capital of Ohio are the same location. In an inequality, two different numbers are being compared. They are similar to sentences such as "John is heavier than Henry" and "Betty is shorter than Dotty."

Have the pupils give other examples of equations and inequalities.

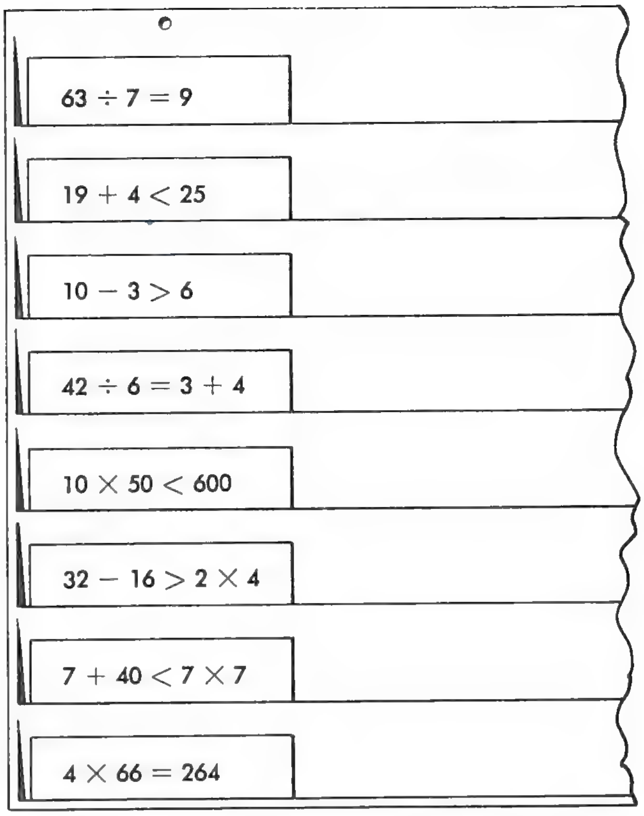
► Write the following word sentences on the board or on a transparency for the overhead projector.

FORTY-TWO IS GREATER THAN SIX TIMES FIVE.	$42 > 6 \times 5$
THIRTY-ONE MINUS FIVE IS LESS THAN FORTY MINUS THIRTEEN.	$31 - 5 < 40 - 13$
FORTY-TWO DIVIDED BY SEVEN IS LESS THAN FIFTY-SIX DIVIDED BY EIGHT.	$42 \div 7 < 56 \div 8$
THE SUM OF THREE AND SEVEN IS LESS THAN THE PRODUCT OF THREE AND SEVEN.	$3 + 7 < 3 \times 7$
FOURTEEN PLUS NINE EQUALS NINETEEN PLUS FOUR.	$14 + 9 = 19 + 4$

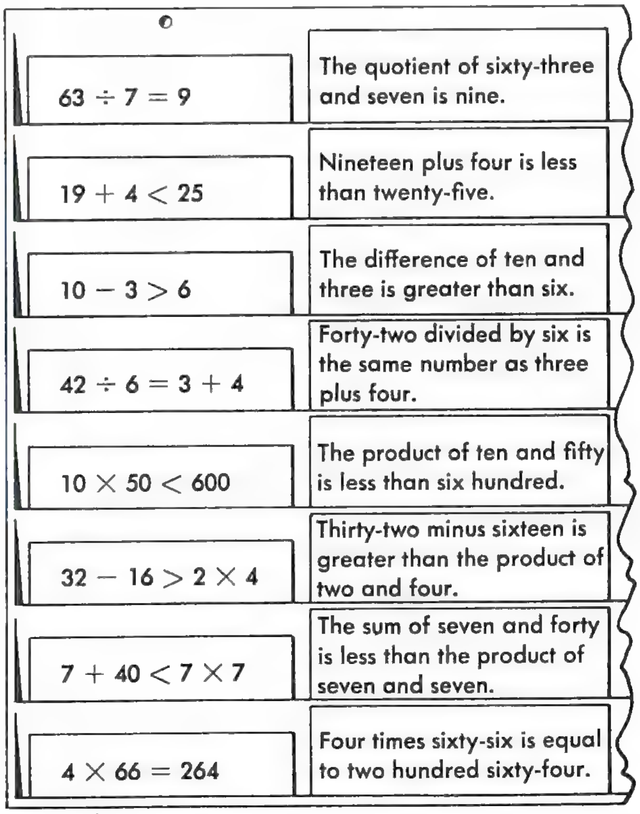
Have the pupils write the equation or inequality for each sentence.

► On each of eight 3 by 9 inch tagboard cards, write a different equation or inequality. Place the cards in a pocket chart.

On eight 4 by 10 inch tagboard cards, write the same equations and inequalities, using words. Place these cards on a table. Point to each card in the pocket chart on which the *less than* symbol (<) is used. Tell the pupils that this symbol means "is less than." Do the same thing with the *greater than* symbol (>). Then call



some pupils forward in turn to match each word sentence with the sentence that uses mathematical symbols. Have each pupil read aloud the word sentence as he places it in the pocket chart.



Page 289

● Discuss the top of page 289 with the class. Have the pupils identify each sample sentence as an equation or an inequality. Then have the exercises completed at the chalkboard, and let the pupils discuss them.

Name

UNIT 20 OPEN SENTENCES

For Class Discussion *See pupil page suggestions.*

In mathematics we can write sentences with words or with mathematical symbols:

Ten equals seven plus three.	$10 = 7 + 3$
Eight minus four is greater than two.	$8 - 4 > 2$
Five times nine is less than six times eight.	$5 \times 9 < 6 \times 8$

The symbol for *greater than* is >.
The symbol for *less than* is <.

Sixteen is less than three times six.	$16 < 3 \times 6$
Two plus four is equal to five plus one.	$2 + 4 = 5 + 1$
The difference of nineteen and eight is greater than the product of five and two.	$19 - 8 > 5 \times 2$

Write each sentence, using mathematical symbols.

1. Seven minus five equals two. $7-5=2$
2. The sum of three and seven is less than twenty-one. $3+7<21$
3. The sum of two and two is equal to the product of two and two. $2+2=2\times 2$
4. Seventy-two divided by nine is greater than seven. $72\div 9>7$
5. Thirty-six is the same number as twenty plus sixteen. $36=20+16$

Which of the above statements are true?
Which are false? *All are true.*

reference page

H-289

Developmental Experiences

▶ Write these sentences on the chalkboard.

England is one of the 50 states in the United States.
Mars is a planet.
A yard is longer than 12 inches.
An elephant is smaller than a mouse.

Ask the pupils to tell whether each sentence is true or false. Then ask them to give several other examples of true sentences and of false sentences.

Write the following equations and inequalities on the chalkboard and have the pupils tell whether they are true or false. As each false sentence is identified by a pupil, have him explain why he thinks it is false.

$17 - 9 = 6$	$72 + 96 = 96 + 72$
$3 \times 7 > 8$	$9 \times 30 < 2 \times 70$
$40 \div 8 > 40 \div 5$	$16 \times 0 < 16 - 0$

Key:

F because $17 - 9 = 8$	T because $168 = 168$.
T because $21 > 8$.	F because $270 > 140$.
F because $5 < 8$.	T because $0 < 16$.

Ask for other examples of true and false sentences.

Pages 290 and 291

● On page 290, the pupils are asked to identify sentences as being either true or false. Clarify the procedure for each set of exercises. Then assign page 290 for independent work. After the page has been completed, discuss the six sentences in exercises 9 and 10. Ask the pupils to tell what they notice about each set of three sentences. In their own words they may point out that the numbers involved are the same, but the relations are different. In each set only one sentence is true. Summarize by discussing the key statement at the bottom of the page and relating it to each set of exercises.

See pupil page suggestions.

Tell whether each of these equations is true or false, and why.

1. $18 \times 10 = 1800$ *False because $18 \times 10 = 180$*
2. $72 \div 9 = 64 \div 8$ *True because $72 \div 9$ is 8 and $64 \div 8$ is also 8*
3. $324 + 676 = 990$ *False because $324 + 676 = 1000$*
4. $543 + 729 = 1272$ *True because $543 + 729 = 1272$*

Tell whether each of these inequalities is true or false, and why.

5. $10 \times 8 > 5 \times 16$ *False because $10 \times 8 = 80$ and $5 \times 16 = 80$*
6. $6 + 18 > 18 - 6$ *True because 24 is greater than 12*
7. $63 \div 9 < 63 \div 7$ *True because 7 is less than 9*
8. $72 + 27 > 83 + 38$ *False because 99 is not greater than 121*

Tell whether each sentence is true or false.

9.	a. $3 \times 4 > 12$ *False*	10. a. $18 \div 2 > 16 \div 2$ *True*
	b. $3 \times 4 < 12$ *False*	b. $18 \div 2 < 16 \div 2$ *False*
	c. $3 \times 4 = 12$ *True*	c. $18 \div 2 = 16 \div 2$ *False*

For any whole numbers a and b, one and only one of these statements is true:

$a > b$ $a = b$ $a < b$

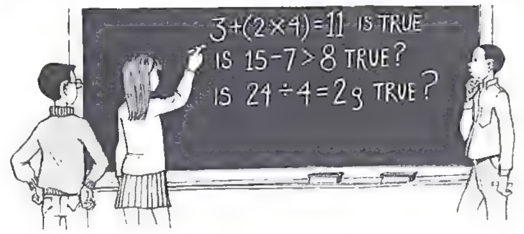
H-290

● Discuss the sentences in the illustration at the top of page 291. Ask the pupils to tell whether they are true, false, or neither. In the third sentence, $24 \div 4 = 2g$, the multiplication notation $2g$ is used. Point out that 2×6 is sometimes read "2 sixes." Ask what the pupils think $2g$ means (two g's). This is a short way of writing $2 \times g$. Point out that the sentence $24 \div 4 = 2g$ is neither true nor false. Explain to the class that when letter placeholders are used, we do not use the multiplication sign, $\times$, because it might be confused with the placeholder, x. Present several other examples and let the pupils tell their meaning.

$4t \qquad 9r \qquad 7z \qquad 5y$

Have the pupils complete the rest of the page as an independent activity. Then discuss the page with the class.

Name ______________________________



Tell whether each sentence is true, false, or neither. Write T, F, or N.
See pupil page suggestions.

1. A puppy has four eyes. F
2. ☐ can swim. N
3. A mouse is an insect. F
4. An ostrich has two legs. T
5. There are fifty states in the United States. T
6. There are ☐ pupils in the fourth-grade class. N
7. $n + 14 < 65$ N
8. $63 \div 7 < 9$ F
9. $64 \div 8 = 4m$ N
10. $7 \times 4 > (3 \times 6) + 4$ T
11. $6 + 8 + 10 < 6g$ N
12. $5a = 6 \times (1 + 4)$ N
13. $27 - 18 = 28 - 17$ F
14. $0 \times (3 + 8) = 10 + 1$ F
15. $(6 \times 3) \times 7 = 6 \times (3 \times 7)$ T
16. $5 \times 17 > n$ N

reference page

H-291

Developmental Experiences

▶ The concept of true and false sentences can be used to provide a review of the basic number combinations. List several equations and inequalities on the chalkboard. Instruct the pupils to find the false sentences in the list.

Each time a false sentence is identified, have it listed separately. Ask the pupils to tell why each sentence is false. For example, the pupils may say that the equation $15 - 7 = 9$ is false for any of the following reasons:

"Because 15 − 7 equals 8."
"Because 15 − 6 equals 9."
"Because 16 − 7 equals 9."
"Because 15 − 7 is less than 9."

Pages 292 and 293

● Discuss the top part of page 292 with the class. Ask the pupils to read the four sentences that mention the Mississippi, Nile, Cuyahoga, and Potomac rivers. Let the pupils discuss the bottom of the page briefly. They should see that, because $3 + \square = 5 \times 10$ generates these sentences, 10, 47, 30, 40, 60, and 100 have replaced ☐. Ask if any of the sentences are true.

For Class Discussion *See pupil page suggestions.*

Some sentences can be used to "generate" other sentences.
What do we mean by generate? Watch!

Generating Sentence: The ☐ River is in the United States.

Nile
↓
The ☐ River is in the United States.
Mississippi
↓
The ☐ River is in the United States.
Cuyahoga
↓
The ☐ River is in the United States.
Potomac
↓
The ☐ River is in the United States.

Is the Mississippi River in the United States? *Yes*
Is the Nile River in the United States? *No*

The generating sentence gave us three true sentences and one false sentence.
Give another sentence that it generates. *The Amazon River is in the United States. Sentences may vary and may be true or false.*

1. $3 + \square = 5 \times 10$ is the generating sentence for the following sentences. How do you know? *For answer, see pupil page suggestions.*

$3 + 10 = 5 \times 10 \qquad 3 + 40 = 5 \times 10$
$3 + 47 = 5 \times 10 \qquad 3 + 60 = 5 \times 10$
$3 + 30 = 5 \times 10 \qquad 3 + 100 = 5 \times 10$

2. What is the generating sentence for these sentences?

$5 \times 10 > 60$
$5 \times 11 > 60$
$5 \times 12 > 60$
$5 \times 13 > 60$

reference page

H-292

● Briefly review the concept of true and false sentences. On the chalkboard, draw the chart that appears on page 293, and discuss sentences 1, 2, and 16 with the class.

Some pupils may need to do some written computation in order to determine the truth or falsity of the sentences on this page. When sentences such as those in exercises 4, 8, and 12 are discussed, there may be disagreement as to whether they are true or false. Do not tell the class which is right. Ask questions that help the pupils draw their own conclusions about the nature of a generating sentence.

These pupil responses indicate a fairly good grasp of the idea of *generating sentence.*

"Sometimes it's true and sometimes it's false."
"It all depends."
"You have to put a number in before you can tell."
"It's both."
"It's neither."

Continue to use the sentences on page 293 for class discussion. Discuss the fact that both equations and inequalities are mathematical sentences.

Name

Complete the chart. *See pupil page suggestions.*
Which sentences are true? Which are false? Which are generating sentences?

	True	False	Generating
1. Lake Erie is one of the Great Lakes.	✓		
2. $\square \times \square = 16$			✓
3. One foot is longer than one mile.		✓	
4. The River is in the United States.			✓
5. Rhode Island is smaller than Texas.	✓		
6. Twelve divided by three is less than twelve divided by four.		✓	
7. The sum of eighteen and thirty-six is greater than fifty.	✓		
8. The difference between seventeen and seven is $\square$.			✓
9. $8 \times 8 = 65$		✓	
10. $4 \times 7 = 7 \times 4$	✓		
11. $63 \div 7 > 7$	✓		
12. $2 + \bigcirc = 56$			✓
13. $156 - 32 = \square$			✓
14. $3 + 18 > n$			✓
15. $6 \times 4 < 8 \times 3$		✓	
16. $3 + (4 \times 7) = 49$		✓	
17. $\Diamond + 19 = 36$			✓
18. $327 + t = 692$			✓

H-293

Developmental Experiences

► These exercises may be used for practice. Have the pupils tell whether the sentences are true or false.

$5 \times 9 < 4 \times 10$	(False)	$63 \div 9 < 9 \times 7$	(True)
$6 \div 3 < 2$	(False)	$9 \times 9 = 81$	(True)
$8 \times 9 = 9 \times 8$	(True)	$48 = 6 \times 7$	(False)
$6 \times 40 > 7 \times 30$	(True)	$36 \div 4 > 12 \times 3$	(False)
$6 \times 40 < 7 \times 30$	(False)	$36 \div 4 < 12 \times 3$	(True)
$6 \times 40 = 7 \times 30$	(False)	$36 \div 4 = 12 \times 3$	(False)

Pages 294 through 296

● Use page 294 to introduce the idea that in mathematics, generating sentences are open sentences.

Tell the pupils to compare the open sentences with the sentences on the right. Note that computation cannot be used to determine whether an open sentence is true or false. Have the pupils determine which sentences are true and which are false.

Assign the exercises for independent work; follow this with a brief discussion.

See pupil page suggestions.

In mathematics, sentences used to generate other sentences are open sentences.

Open sentences	Not open
$3 + \square = 5 \times 10$	$3 + 47 = 5 \times 10$ $3 + 60 = 5 \times 10$
$\square + 10 = 10 + \square$	$0 + 10 = 10 + 0$ $1 + 10 = 10 + 1$
$\triangle - 5 > 2 \times 10$	$30 - 5 > 2 \times 10$ $20 - 5 > 2 \times 10$ $21 - 5 > 2 \times 10$ $25 - 5 > 2 \times 10$
$\square \times \square = 9$	$2 \times 2 = 9$ $3 \times 3 = 9$
$6 = \square \times \square$	$6 = 2 \times 2$ $6 = 3 \times 3$

Which of the following sentences are true (T), false (F), or open (O) sentences?

1. $5 > 10 - 6$ T
2. $6 + 20 = 2 \times 13$ T
3. $5 + 20 < 2 \times 13$ T
4. $5 + \square < 2 \times 13$ O
5. $6 + \square = 50 + 10$ O
6. $3 \times 17 = 50 + \square$ O
7. $18 \times 18 = 9 \times 36$ T
8. $60 = \square \times 7$ O

reference page

H-294

● Before starting page 295, write this sentence on the chalkboard.

$$5 \times (t + 2) = 35$$

Ask the pupils if the sentence is true, false, or open (open). Let the pupils suggest numbers to be used as a replacement for t and rewrite the equation on the chalkboard, using their replacements. Tell the pupils to solve their equations and to tell whether they are true or false. If 3 and 5 are used as replacements, the resulting sentences may look like this:

$5 \times (3 + 2) = 35$	$5 \times (5 + 2) = 35$
$5 \times 5 = 35$	$5 \times 7 = 35$
$25 = 35$ False	$35 = 35$ True

Using 3 makes the sentence false, but using 5 gives us a true sentence. At this time, do not stress finding a replacement that will give a true sentence.

Work the top of the page as a class activity and clarify any misunderstandings. Establish the procedure for the bottom of the page when working through the example. Answer any questions the pupils have; then allow different pupils to work the exercises at the chalkboard. Remind the pupils that they may use any numbers they wish to replace the variable, and that it is not necessary to get a true sentence for each exercise. When discussing each exercise, have the pupils explain why their equations for an exercise are true or false.

Name

For Class Discussion *See pupil page suggestions.*

Is this sentence true, false, or open? $n \times (4 + 2) = 30$
Open

If 3 replaces n, is the sentence true? $3 \times (4 + 2) = 30$
no
$3 \times 6 = 30$
$18 = 30$

If 6 replaces n, is the sentence true? $6 \times (4 + 2) = 30$
no
$6 \times 6 = 30$
36 $= 30$

If 5 replaces n, is the sentence true? $5 \times (4 + 2) = 30$
yes
$5 \times$ *6* $= 30$
36 $= 30$

Write three true or false sentences for each open sentence.
Write T or F to tell if each sentence is true or false. *Replacements will vary.*

Example: $7 + d = 19$
F $7 + 9 = 19$
F $7 + 13 = 19$
T $7 + 12 = 19$

1. $r \times (4 + 5) = 18$
T $2 \times (4+5) = 18$
F $3 \times (4+5) = 18$
F $4 \times (4+5) = 18$

2. $d = 5 + 67$
F $70 = 5 + 67$
T $72 = 5 + 67$
F $74 = 5 + 67$

3. $(k + 3) + 8 > 0$
T $(1+3)+8 > 0$
T $(6+3)+8 > 0$
T $(8+3)+8 > 0$

4. $36 = 9b$
F $36 = 9 \times 3$
F $36 = 9 \times 9$
T $36 = 9 \times 4$

5. $\square \div 8 < \square \div 7$
T $(1 \div 8) < (1 \div 7)$
T $(5 \div 8) < (5 \div 7)$
T $(7 \div 8) < (7 \div 7)$

6. $(9 \times \square) \div 9 = \square$
T $(9 \times 3) \div 9 = 3$
T $(9 \times 5) \div 9 = 5$
T $(9 \times 7) \div 9 = 7$

7. $\square \times 2 \times \square = 2 \times \square \times \square$
T $3 \times 2 \times 3 = 2 \times 3 \times 3$
T $4 \times 2 \times 4 = 2 \times 4 \times 4$
T $5 \times 2 \times 5 = 2 \times 5 \times 5$

reference page

H-295

● Before beginning page 296, write the following open sentence (inequality) on the chalkboard.

$$3g < 17$$

Using different replacements for g, have several pupils write the sentence. Ask other pupils to tell whether the resulting sentences are true or false.

$3 \times 7 < 17$	$3 \times 2 < 17$
$21 < 17$ False	$6 < 17$ True

The pupils will observe that there are many solutions for g. The idea that an equation or inequality may have more than one solution should be examined in class discussion. Use the inequalities in the following chart. The discussion should lead to the conclusion that an equation or inequality may have no solution, a unique solution (exactly one solution), two solutions, or any number of solutions.

0 solutions	$g < g$
1 solution	$2g < 2$
2 solutions	$2g < 4$
Many solutions	$2g < 99$
Unlimited set	$g < 2g$

The pupils should be cautioned not to stop after finding one solution for an open sentence; they should try other numbers to determine whether or not there are other solutions.

Present an equation such as $g - 0 = g$. Using different replacements for g, have several pupils rewrite the sentence. Remind them that both g's in the equation must be replaced by the same number. Have the pupils tell whether the resulting sentences are true or false.

$7 - 0 = 7$	$26 - 0 = 26$
$7 = 7$ True	$26 = 26$ True
$0 - 0 = 0$	$532 - 0 = 532$
$0 = 0$ True	$532 = 532$ True

The pupils may observe that each whole-number replacement will make this sentence true because any number minus zero is that number.

Assign the page for independent work. Review the statements at the bottom of the page. Ask for examples to be placed on the chalkboard.

See pupil page suggestions.

Open sentences are written with variables.

$6 \times \square > 5 \times 6$ $\quad$ $16 - k = 16 \div 2$

variables

$27 \div (2 + 1) < q + 4$ $\quad$ $7 + \diamond = \diamond + 7$

By replacing the variables with a number we obtain a true sentence or a false sentence.

The variables have been replaced with whole numbers. Tell whether each sentence is true (T) or false (F).

1. *open* $6 \times \square > 5 \times 6$
   a. $6 \times 4 > 5 \times 6$ F
   b. $6 \times 5 > 5 \times 6$ F
   c. $6 \times 6 > 5 \times 6$ T
   d. $6 \times 7 > 5 \times 6$ T

2. *open* $16 - k = 16 \div 2$
   a. $16 - 6 = 16 \div 2$ F
   b. $16 - 8 = 16 \div 2$ T
   c. $16 - 10 = 16 \div 2$ F
   d. $16 - 12 = 16 \div 2$ F

3. *open* $27 \div (2 + 1) < q + 4$
   a. $27 \div (2 + 1) < 29 + 4$ T
   b. $27 \div (2 + 1) < 27 + 4$ T
   c. $27 \div (2 + 1) < 5 + 4$ F
   d. $27 \div (2 + 1) < 6 + 4$ T

4. *open* $7 + \diamond = \diamond + 7$
   a. $7 + 4 = 4 + 7$ T
   b. $7 + 8 = 8 + 7$ T
   c. $7 + 16 = 16 + 7$ T
   d. $7 + 57 = 57 + 7$ T

Some replacements for variables give true sentences.
Some replacements give false sentences.

reference page

H-296

Developmental Experiences

▶ Write this sentence on the chalkboard.

$$3y + 4 = 37$$

Ask for different replacements for y. Have the pupils write the resulting sentences and determine whether each sentence is true or false.

$(3 \times 2) + 4 = 37$	$(3 \times 11) + 4 = 37$
$6 + 4 = 37$	$33 + 4 = 37$
$10 = 37$ False	$37 = 37$ True

Explain that each replacement for y that gives a true sentence is a solution for the equation. Thus, 11 is a solution for the equation above. Write $y = 11$ on the chalkboard. Tell the pupils that this particular equation has only one solution, but that many open sentences have more than one solution. Some open sentences have no solution.

Present other open equations to the class. Encourage the pupils to use a trial-and-success method to find solutions. Do not present any formal rules for solving equations. After experimentation with a number of different replacements in open sentences, the pupils will discover for themselves more refined ways to find solutions. At this level, we should encourage them to experiment and discover their own methods.

After the pupils have explored, help them discover how they can use each trial to improve the next trial. For example, if a pupil suggests 10 as a replacement for m in the open equation $7m + 9 = 93$, have him determine whether the resulting sentence is true or false (false). Ask, "Is 93 greater than, or less than, 79? Do you think your new replacement should be greater than, or less than, 10?" (greater than 10) Suppose the pupil's next guess is 15. Have the pupils determine whether the resulting sentence is true or false (false). Ask, "Do you think the solution will be greater than, or less than, 15?" (less than 15) "Between what two numbers will the solution lie?" (10 and 15) Continue in this way to help the pupil narrow his range of guesses, but do not tell him what numbers to try. Questions such as "How can you make a better trial?" and "How can you achieve success?" should be used frequently. Encourage the pupils to discuss their reasoning as they work.

Pages 297 and 298

- Use page 297 as a class discussion page following the approach in the previous activity. The pupils might enjoy acting out the discussion.

- Page 298 is to be done as a class activity. The pupils may want to try more replacements, using greater numbers, before saying Bill is correct. If so, let the pupils experiment with a few.

Name

For Class Discussion

Cathy asked Jim to help her find a solution to this open sentence:

$$3n + 7 = 25$$

She explained, "We have to find a replacement for n that will give us a true sentence." Jim said, "Let's try 4." He wrote:

$(3 \times 4) + 7 = 25$
$12 + 7 = 25$
$19 = 25$ False

"Four is not a solution," said Cathy, "because it gives us a false sentence. Nineteen is less than 25, so we need a number greater than 4." She wrote:

$(3 \times 7) + 7 = 25$
$21 + 7 = 25$
$28 = 25$ False

"Another false sentence!" exclaimed Jim. "Twenty-eight is greater than 25, so 7 is too great. The number we need to make a true sentence is between 4 and 7."

"That means n must be 5 or 6," said Cathy. "We know it is closer to 7 than to 4, because 25 is closer to 28 than to 19; n must be 6."

Is Cathy correct? Cathy wrote:

$(3 \times 6) + 7 = 25$
$18 + 7 = 25$
$25 = 25$ Therefore: $n = 6$.

"You were right," said Jim, "6 gives a true sentence, so it is a solution for the open sentence. But maybe 5 will work, too."

Is 5 a solution for the open sentence? *no*

reference page

H-297

For Class Discussion

$$7 + p = p + 5 + 2$$

Susan and Bill tried to find solutions for this open sentence:

$$7 + p = p + 5 + 2$$

"Let's start with 1 as a replacement for p," Susan suggested. She wrote:

$7 + 1 = 1 + 5 + 2$
$8 = 8$

"Good," said Bill, "you found the solution on the first try. Let's do the next exercise."

"Wait," said Susan, "there may be some more solutions. Some equations do have more than one solution. I'll try 0, 2, and 3. You try 4, 5, and 6."

What results did Bill and Susan get? *0, 2, 3, 4, 5, and 6 are also solutions.*

"You were right," said Bill, "there are many solutions for this open sentence. I think any whole number is a solution."

Was Bill correct? *yes, any whole number is a solution for $7+p=p+5+2$.*

Susan said, "My brother showed me a way to list the numbers in our solution set." She wrote:

$\{0, 1, 2, \ldots\}$

"The dots mean that the numbers go on and on."

Bill asked, "What if there is only one number in the solution set?"

Susan said, "In an open sentence such as $7 + q = 15$, 8 is the only solution. We can list it this way." She wrote:

$\{8\}$

Bill asked, "What about an open sentence like $8 + w = 2$? There isn't any whole-number replacement for w that will make a true sentence."

"Simple," said Susan. "The solution set is empty. We show it this way." She wrote:

$\{\ \}$

reference page

H-298

Developmental Experiences

▶ Write these sentences on the chalkboard.

$f + 3 = 7$	$2 = 3n$	$d + 9 = 9 + d$
$2y = y + y$	$24 \div h = 8$	$(4 \times 6) - 6 = 19$

Have the pupils find solutions for the open sentences. Encourage them to use a trial-and-success method.

Show the pupils how to write solution sets for open sentences using the set brackets.

$f + 3 = 7$	$\{4\}$
$2y = y + y$	$\{0, 1, 2, 3, \ldots\}$
$2 = 3n$	$\{\ \}$ (no solution)
$d + 9 = 9 + d$	$\{0, 1, 2, 3, \ldots\}$

Pages 299 and 300

● Assign page 299 for independent work. If any of the pupils have difficulty drawing set braces, allow them to just draw wavy lines. After the pupils have found the solution set for each equation, have them compare their answers and discuss how they reached the conclusions they did.

By this time some pupils may have discovered other ways to find solutions; but do not expect all pupils to make this kind of progress. It is important that the pupils use their own method in finding the solution set for open sentences. A trial-and-success method is desired for the majority of the class. Assign only as many exercises as will allow the pupils sufficient time to use the trial-and-success method. If any pupils have difficulty getting started, guide them to see that they can begin by using zero for the first trial, increasing the replacement number by one for each succeeding trial, until they find the solution. Eventually, they may see that often they can begin with greater numbers and find their solutions much faster.

The puzzle at the bottom of the page is intended to give practice in the basic combinations.

Name

Find the solution set for each open sentence.
Choose your replacements from the set of whole numbers.

1. $r + 7 = 6$ { }
2. $4 = t + 6$ { }
3. $21 - b = 21$ {0}
4. $4 = 5n$ { }
5. $m + 10 = 3$ { }
6. $k - 0 = 0$ {0}
7. $6 \times d = d \times 6$ {0, 1, 2, ...}
8. $5p + 8 = 48$ {8}
9. $3 + 6 = 3j$ {3}
10. $19 = 3g - 2$ {7}
11. $32 \div w = 8$ {4}
12. $5s = 3s + 2s$ {0, 1, 2, ...}

Find the standard numeral for each number to decode the secret message.

A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	2	3	4	5	6	7	8	9	10	11	12
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	14	15	16	17	18	19	20	21	22	23	24	25

	70 − 68 C	3 × 0 A	10 + 3 N		0 + 0 A	6 + 7 N	
36 ÷ 9 E	9 + 2 L	24 ÷ 6 E	5 × 3 P	42 ÷ 6 H	0 × 8 A	80 − 67 N	10 + 9 T
	12 + 7 T	9 + 8 R	0 − 0 A	9 + 12 V	28 ÷ 7 E	5 + 6 L	
8 + 14 W	93 − 85 I	48 − 29 T	63 ÷ 9 H	7 × 2 O	5 × 4 U	76 − 57 T	
17 − 17 A		15 + 4 T	35 − 18 R	9 + 11 U	42 − 29 N	2 × 5 K	?

H-299

● Before beginning page 300, write several open sentences on the chalkboard and have the pupils give the solution set for each open sentence. Use the set of whole numbers as the replacement set. Discuss the fact that some replacements for these open sentences result in false sentences and some replacements result in true sentences.

$2y + 7 = 35$ $\{14\}$
$p + 17 > 18$ $\{2, 3, 4, \ldots\}$
$18 - g > 7 + g$ $\{0, 1, 2, 3, 4, 5\}$

Write the following sentences on the chalkboard. Ask what is needed to make these sentences complete ($>$, $<$, or $=$ symbols). Have the pupils write $>$, $<$, or $=$ in the ○ to make each sentence true.

$2 \times 4 \bigcirc 4 \times 2$ $(=)$
$15 - 3 \bigcirc 13 - 5$ $(>)$
$16 \div 8 \bigcirc 16 \div 2$ $(<)$

Present a different kind of incomplete sentence. Ask what is needed to complete these sentences (operation signs). Have the pupils write $+$, $-$, $\times$, or $\div$ in the $\triangle$ to make each sentence true.

$7 \triangle 4 = 28$ $(\times)$ $30 \triangle 6 = 5$ $(\div)$
$7 \triangle 4 = 11$ $(+)$ $30 \triangle 6 = 24$ $(-)$

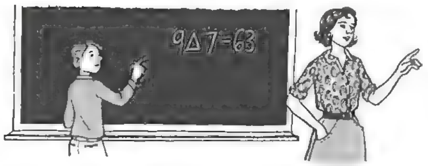
Have the pupils work independently to complete the sentences on the page. It may be desirable to assign only one column in each set of exercises at any one time.

Use >, <, or = in place of to make each sentence true.

1. $30 \times 5 = 50 + 50 + 50$
2. $649 + 17 = 666$
3. $58 < 4 \times 15$
4. $8 \times 4 > 81 \div 9$
5. $9 + 39 = 41 + 7$
6. $42 \div 7 = 36 \div 6$
7. $404 - 303 < 303 + 404$
8. $764 - 700 > 653 - 600$
9. $3 \times 9 < 33$
10. $9 \times 560 > 4500 + 54$
11. $78 < (8 \times 9) + 8$
12. $4 \times (6 + 9) = 36 + 24$
13. $3 \times (20 + 4) < 6 \times (10 + 4)$
14. $(8 + 3) \times 5 < 8 \times (3 + 5)$
15. $56 \div 7 < (7 \times 2) + 4$
16. $70 - 28 = 7 \times (10 - 4)$

Use +, −, ×, or ÷ in place of to make each sentence true.

17. $9 \times 7 = 63$
18. $64 \div 8 = 72 \div 9$
19. $3 \times 6 = 2 \times 9$
20. $120 - 105 = 20 - 5$
21. $24 - 6 = 40 - 22$
22. $42 \div 6 = 49 \div 7$
23. $35 + 7 = 7 \times 6$
24. $13 + 12 = 30 - 5$
25. $21 \div 3 = 16 - 9$
26. $10 + 0 = 50 \div 5$ or −



H-300

Supplemental Experiences

■ If more practice is necessary in finding solutions for open sentences, use these exercises.

$17 = (5 + 3) + s$	$(s = 9)$
$2 \times (n + 1) = 5 + n$	$(n = 3)$
$17 - (12 \div d) = 10 + d$	$(d = 3, 4)$
$2n + 3 = 9$	$(n = 3)$
$8 \times (w + 3) = 56$	$(w = 4)$
$35 = (4 \times 8) + y$	$(y = 3)$
$4a + 1 = 13$	$(a = 3)$
$2g + 1 = 21$	$(g = 10)$
$18 \div p = 7 + p$	$(p = 2)$
$3y + 1 = 56 \div 8$	$(y = 2)$

■ Write two different solution sets, such as the following, on the chalkboard.

$\{8\}$ $\qquad$ $\{0, 1, 2, \ldots\}$

Present several open equations and have the pupils tell which of the given solution sets will fit each.

$2 \times 4 = n$	$\{8\}$
$a \times 8 = 8 \times a$	$\{0, 1, 2, \ldots\}$
$14 - 6 = g$	$\{8\}$
$8 + b = b + 8$	$\{0, 1, 2, \ldots\}$

■ You may wish to use the suggested quiz, which was written for the following objectives:

The pupils should have an understanding of equations, inequalities, and open sentences. They should also be aware that open equations may have no solution, one solution, or more than one solution.

The pupils should be able to solve simple equations such as $43 - 34 = 72 \div s$ using their own method of finding the solution.

SUGGESTED QUIZ

1. Tell whether each equation is true or false.

$15 - 8 = 7$	TRUE
$17 \times 19 = 1700$	FALSE
$72 \div 9 = 64 \div 8$	TRUE
$324 + 666 = 950$	FALSE
$732 - 463 = 1195$	FALSE
$16 \times 15 = 15 \times 16$	TRUE

2. Find the solution set for each open sentence. Use the set of whole numbers for the replacement set.

$21 = (9 + 7) + s$	$\{5\}$
$4g + 1 = 25$	$\{6\}$

UNIT 21
DIFFERENCES: THE SET OF INTEGERS
Pages 301 Through 320

OBJECTIVE
To introduce the Set of Integers.

As a result of investigating the missing-addend equation, the pupil learns that differences of whole numbers are whole numbers or opposites of whole numbers. He learns that the set of differences of whole numbers is the Set of Integers. He learns that "the opposite of 2" is written $^{-}2$.

See Key Topics in Mathematics for the Intermediate Teacher: The Set of Integers.

KEY IDEAS
2 − 5 is a new kind of number.
2 − 5 is the opposite of 3.
5 + the opposite of 3 is 2.

CONCEPTS
same length	opposite
same direction	integer

KEY IDEA
2 − 5 is a new kind of number.

Scope
To introduce the set of differences.

Fundamentals
The solution to a missing-addend problem is a difference. If $a + \square = b$, then $\square$ is the difference $b - a$. If a and b are whole numbers, the missing-addend problem does not always have a whole-number solution. For example:

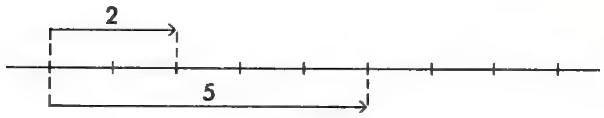
$2 + \square = 5$
$\square = 5 - 2$ (5 − 2 is 3, a whole number)

$5 + \square = 2$
$\square = 2 - 5$ (2 − 5 is not a whole number)

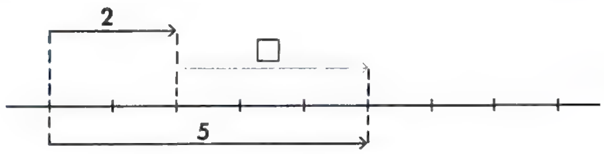
2 − 5 is, to the pupils, a new kind of number; 2 − 5 is the number which you add to 5 to get 2.

To investigate this new kind of number, we need a model for addition (differences are defined using addition) that works for addition of whole numbers and illustrates the new kind of number.

To introduce the model, we consider the question $2 + \square = 5$: what number is added to 2 to get 5? In the model, the numbers 2 and 5 are represented by arrows, above and below a scale. The scale should not be numbered.

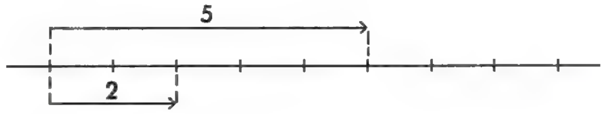


Then the answer to the question $2 + \square = 5$ is shown with a colored arrow, as in the diagram below.

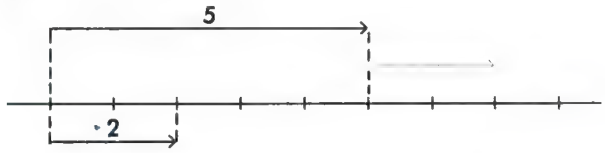


In this model for addition, the addends appear as two arrows above the scale and the sum as a single arrow below. $\square$ is the difference 5 − 2, or 3.

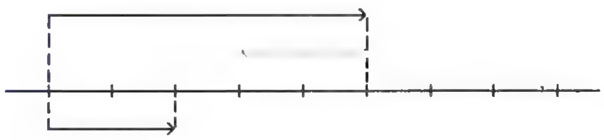
We can use this model to investigate the missing-addend question $5 + \square = 2$: what number is added to 5 to get 2? Again the addend, 5, and the sum, 2, are shown by arrows.



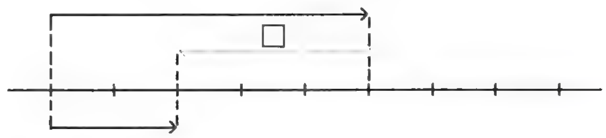
If we add to 5 by moving along the scale to the right, we will never reach the sum, 2.



Instead we must go in the opposite direction, to the left.



We extend the arrow until it reaches 2.

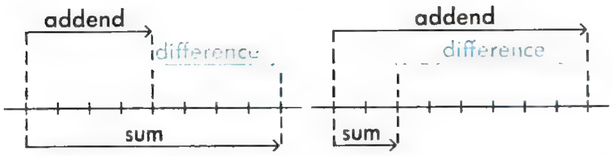


This arrow, shown in color in the preceding illustration, represents the number $\square$. What number is $\square$? It is the difference 2 − 5, a new kind of number.

This model for addition is chosen because it illustrates the following relationship:

addend + difference = sum

The addend is shown above the scale, and the sum is shown below. An arrow pointing in the appropriate direction shows the difference.



It is essential that the intervals on the scale remain unlabeled. A labeled number line would interfere with the perception of differences.

Developmental Experiences

masking tape
felt-tip pen
colored chalk
index cards (3″ × 5″)
rubber cement
cardboard arrows

for each child
$\frac{1}{2}$″ graph paper
colored pencil

► On the chalkboard write the following missing-addend problems:

$$2 + \square = 9 \qquad 7 + \square = 29$$
$$3 + \square = 9 \qquad 3 + \square = 25$$

Ask the pupils to determine $\square$ for each exercise, any way they can. Then tell the class that these are missing-addend problems, and point out that each of the solutions is a whole number. Then present the following missing-addend problem:

$$11 + \square = 3$$

Discuss this problem. The pupils will probably point out that $\square$ cannot be a whole number. Tell the class that $\square$ is a new kind of number.

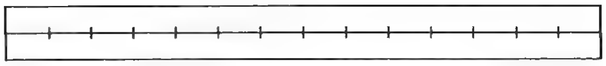
Write the following on the chalkboard:

$3 + \square = 11$
$\square$ is the difference $11 - 3$.
$11 - 3$ is a whole number.

$11 + \square = 3$
$\square$ is the difference $3 - 11$.
$3 - 11$ is a new kind of number.

Then have the pupils practice writing solutions to missing-addend problems as differences of whole numbers. Include problems that have whole-number solutions and those that do not.

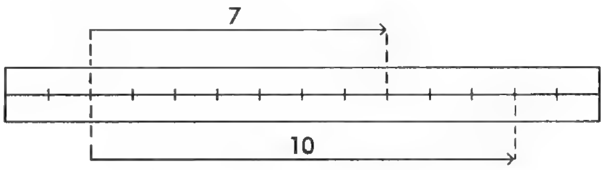
► Place a strip of masking tape or adhesive tape about 63 inches long in the middle of the chalkboard (or use a scale drawn on a transparency for the overhead projector). Using a felt-tip pen, mark the tape with dividing marks 5 inches apart, as indicated below. Do not number the scales.



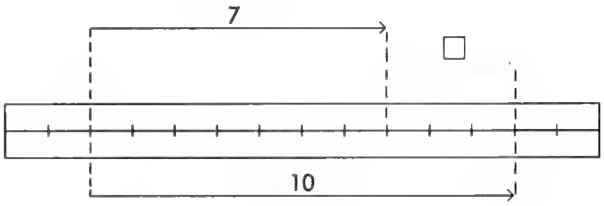
Give each pupil a sheet of $\frac{1}{2}$-inch graph paper and a colored pencil. Then write on the chalkboard the following missing-addend equation:

$$7 + \square = 10$$

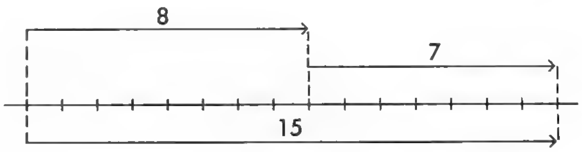
Also draw this picture on the chalkboard:



Ask the pupils to copy the picture on their graph paper. (Since they are using graph paper, they do not need to draw the scale.) Next ask the pupils to draw an arrow for the missing addend with a colored pencil. Draw the arrow yourself on the chalkboard. Label the missing addend $\square$.



Discuss the drawing with the class. Point out that the addends 7 and $\square$ are above the scale, the sum 10 is below the scale, and $7 + \square$ is 10. Next ask the pupils to illustrate the sum $8 + 7$ by drawing the addends end to end above a scale and the sum 15 below the same scale.



Let them discuss their results. Then have the pupils draw a picture on their graph paper for $6 + \square = 14$. Continue this procedure for the following missing-addend problems:

$$5 + \square = 15 \qquad 2 + \square = 11$$
$$9 + \square = 12 \qquad 3 + \square = 9$$

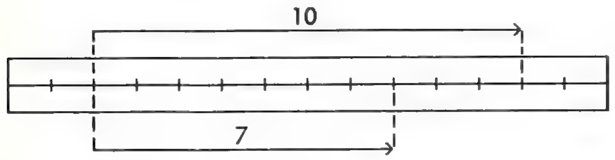
The procedure should be continued until the pupils understand how to use the arrows to show missing addends.

When the pupils have finished, remind them that the missing addends are differences. For example, in $5 + \square = 15$ the missing addend may be named this way: $\square = 15 - 5$. Once the pupils understand that this model for addition also shows differences, go on to the next step in the activity.

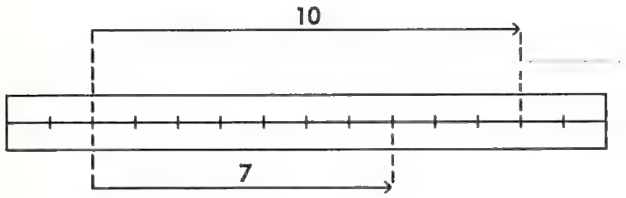
Write the following on the chalkboard:

$$10 + \square = 7$$
$$\square = 7 - 10$$

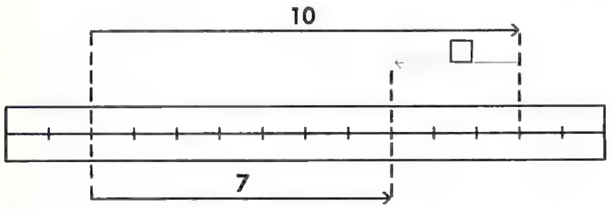
Represent the addend 10 and the sum 7 with arrows as shown below.



Then draw another arrow pointing to the right as indicated.



Ask the pupils whether the picture shows $10 + \square = 7$ (no). (You may need to remind the pupils that the addends are above the scale and the computed sum is below.) Since the picture does not show $10 + \square = 7$, erase the third arrow and draw it again. The arrow must point to the left.



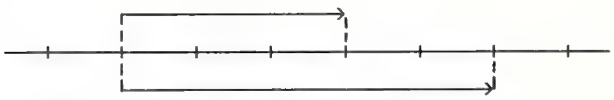
Label the arrow $\square$.

Tell the pupils that this shows the length and direction for the missing addend. Ask the class to name the missing addend as a difference ($\square = 7 - 10$). Then have the pupils draw and label the differences for these missing-addend problems:

$8 + \square = 3$	$(\square = 3 - 8)$
$2 + \square = 1$	$(\square = 1 - 2)$
$9 + \square = 5$	$(\square = 5 - 9)$
$15 + \square = 6$	$(\square = 6 - 15)$
$9 + \square = 10$	$(\square = 10 - 9)$
$8 + \square = 12$	$(\square = 12 - 8)$

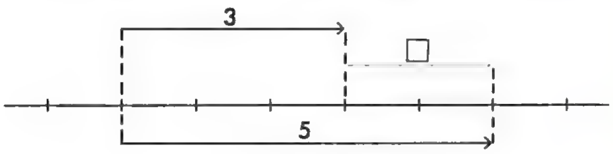
▶ Draw a line about 36 inches long in the middle of the chalkboard. Make dividing marks on the line about 5 inches apart so that it can be used as a scale.

Draw an arrow 3 units long above the scale, and have it pointing to the right. Draw another arrow 5 units long below the scale, and have it pointing to the right also.



Explain to the class that the picture you have just drawn represents a missing-addend sentence. The arrow below the scale represents the sum, and the arrow above the scale represents one of the addends.

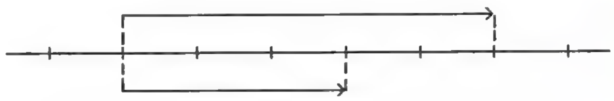
Call for a volunteer to draw an arrow above the scale to represent the missing addend. Remind him that the arrow will extend from the point of the arrow for the addend to the point of the arrow for the sum. Have him label the three arrows when he has finished.



Ask another pupil to write the missing-addend sentence on the chalkboard ($3 + \square = 5$). Then have the class name the missing addend as a difference, and write it.

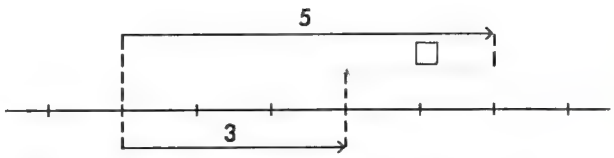
$$3 + \square = 5$$
$$\square = 5 - 3$$

Make another scale. Draw an arrow 5 units long above the scale and another 3 units long below the scale. Both arrows should point to the right.



Explain to the class that the picture you have drawn represents another missing-addend sentence. Remind them again that the arrow below the scale represents the sum, and the arrow above represents one of the addends.

Ask a pupil to draw an arrow above the scale to represent the missing addend. As before, the arrow will extend from the point of the addend arrow to the point of the sum arrow. Have him label the three arrows.



Ask another pupil to write the missing-addend sentence on the chalkboard ($5 + \square = 3$). Have the class name the missing addend as a difference, and write it.

$$5 + \square = 3$$
$$\square = 3 - 5$$

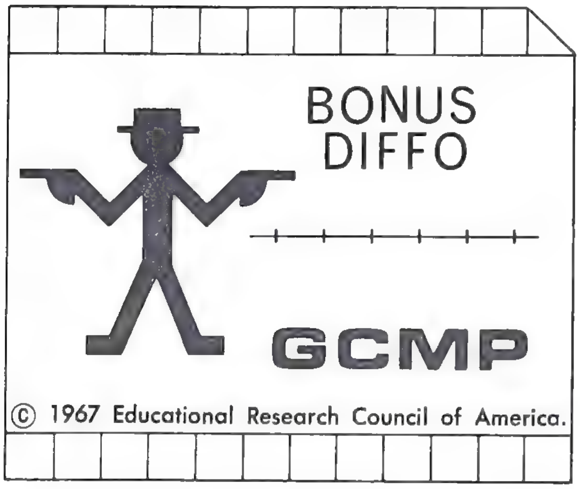
Repeat this activity to illustrate other missing-addend sentences, like the ones below.

$7 + \square = 4$ $\qquad$ $5 + \square = 2$

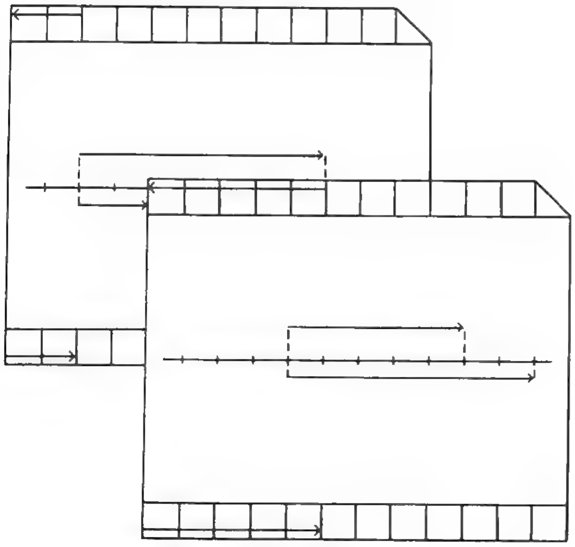
$4 + \square = 6$ $\qquad$ $4 + \square = 3$

► The game of Diffo*

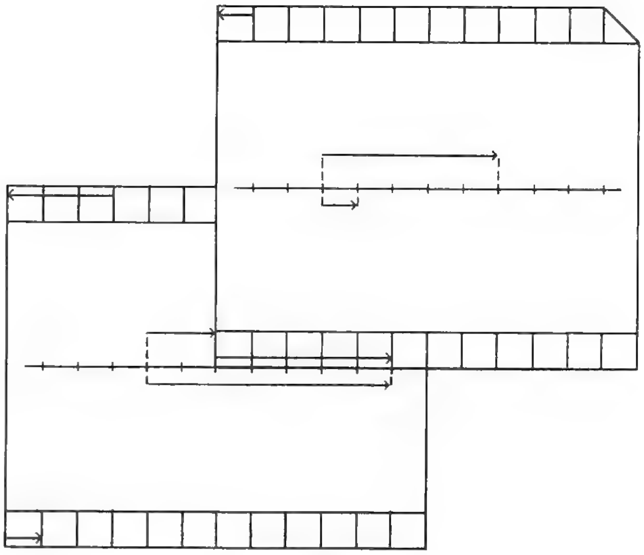
Make a deck of 25 Diffo cards, using rubber cement to paste on halves of 3 by 5 index cards a copy of each of the cards pictured at the back of this book. Cut off the upper right-hand corner of each card, to indicate the top of the card.



The upper and lower edges of each card show arrows for differences (missing addends). These differences are used to complete the arrow diagrams in the middle of other cards. A Diffo is scored when a pupil uses a difference as a missing addend, as shown below.



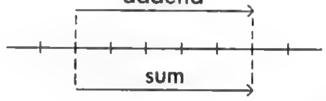
A Diffo.



Another Diffo.

Notice that the arrow for the sum is always below the scale.

Some cards show a difference of 0 by omitting an arrow from either edge or both edges. A difference of 0 is the solution to problems like the following:



Three or four pupils play with one Diffo deck. To play, one of the players (the dealer) deals all the cards and turns the last one up in the center of the playing area. Then the player on the dealer's left has a chance to make a Diffo by playing one of the cards in his hand on the card that is turned up. If he can't make a Diffo, the player on his left gets to try. When a player makes a Diffo, he leaves his card in the center and turns over the other card (the one previously in the center) in front of him. The next player plays on the new card.

One of the 25 Diffo cards is marked "Bonus Diffo." This is a wild card. The "Bonus Diffo" card can be played on any card that is in the center. And when it is in the center, any other card can be played on it.

A player's score is the number of cards turned over in front of him when the hand is over. The hand ends when nobody can play any more or when one player has no more cards left in his hand. The players record their points for the hand, and the game then continues with another hand.

*Copyright 1967, Educational Research Council of America.

Pages 301 through 305

● Discuss the statement and the missing-factor problems at the top of page 301 with the class. Then have the class consider the missing-addend problems. Be sure they see that 2 − 5 is not a whole number.

● Discuss the example at the top of page 302 with the class. Then have the pupils copy and complete the exercises on the graph paper in the lower right-hand corner of page 302 and on the right-hand side of page 303. Allow them to discuss and compare their results.

Name

UNIT 21 DIFFERENCES: THE SET OF INTEGERS

For Class Discussion

The missing-factor problem is solved.

5 × □ = 15	6 × □ = 29	12 × □ = 11
□ = 3	□ = $\frac{29}{6}$	□ = $\frac{11}{12}$

Is the missing-addend problem solved?

5 + □ = 15

□ = 15 − 5 The difference is a whole number.

16 + □ = 29

□ = 29 − 16 Is the difference a whole number? *yes*

2 + □ = 5

□ = 5 − 2 Is the difference a whole number? *yes*

5 + □ = 2

□ = 2 − 5 Is the difference a whole number? *no*

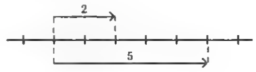
The solution of the missing-addend problem is a difference.

5 − 2 is a whole number.
2 − 5 is a new kind of number.

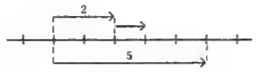
reference page

H-301

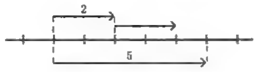
Show the arrow for □.



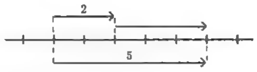
2 + □ = 5



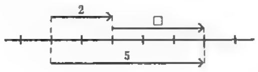
Not enough.



Not enough.

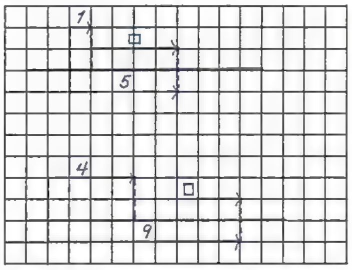


Just enough.

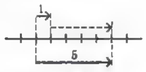


2 + □ = 5
□ = 5 − 2

Copy and complete. Show the arrow for □.

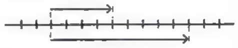


1. 1 + □ = 5



□ = *5* − *1*

2. 4 + □ = 9



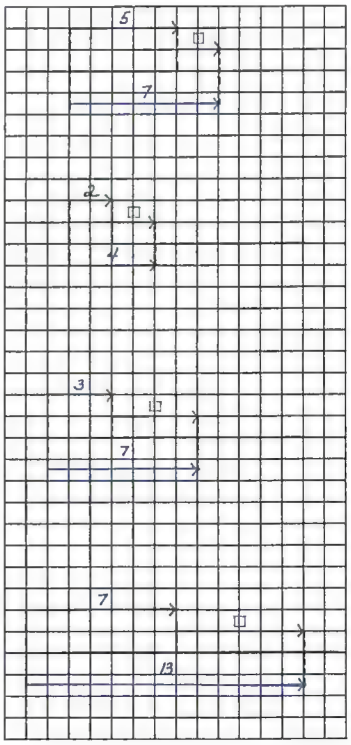
□ = *9* − *4*

reference page

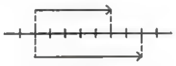
H-302

Name

Copy and complete. Show the arrow for □.

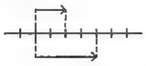


1. 5 + □ = 7



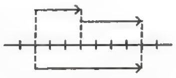
□ = *7* − *5*

2. 2 + □ = 4



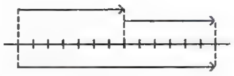
□ = *4* − *2*

3. 3 + □ = 7



□ = *7* − *3*

4. 7 + □ = 13

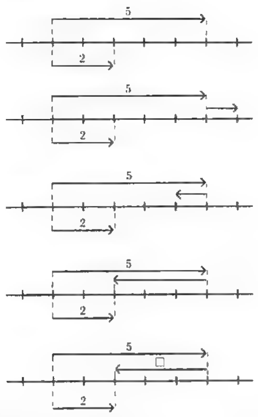


□ = *13* − *7*

H-303

● For pages 304 and 305, follow the same procedure suggested for the preceding pages.

Show the arrow for □.



5 + □ = 2

Wrong direction.

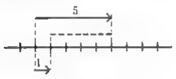
Not enough.

Just enough.

5 + □ = 2
□ = 2 − 5

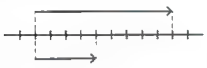
Copy and complete. Show the arrow for □.

1. 5 + □ = 1

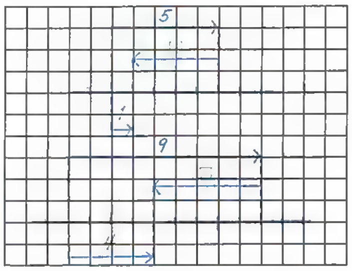


□ = 1 − 5

2. 9 + □ = 4



□ = 4 − 9



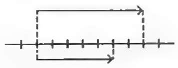
reference page

H-304

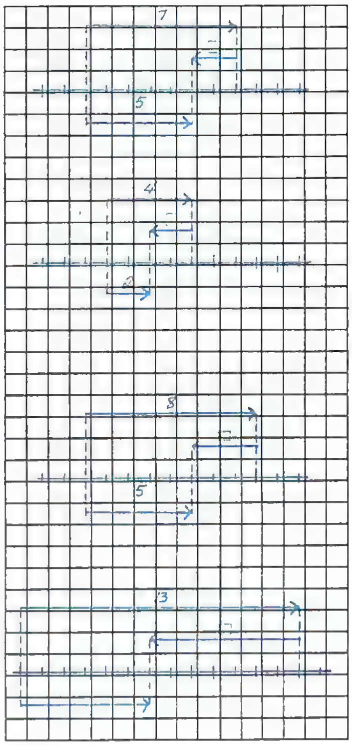
Name

Copy and complete. Show the arrow for □.

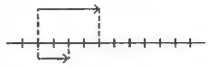
1. 7 + □ = 5



□ = 5 − 7

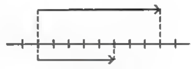


2. 4 + □ = 2



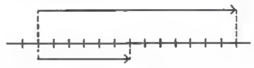
□ = 2 − 4

3. 8 + □ = 5



□ = 5 − 8

4. 13 + □ = 6



□ = 6 − 13

H-305

KEY IDEA

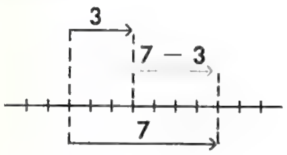
$2 - 5$ is the opposite of 3.

Scope

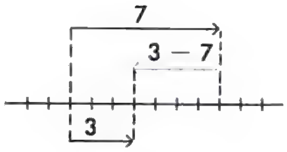
To investigate differences that are the same number.
To investigate differences that are opposites.

Fundamentals

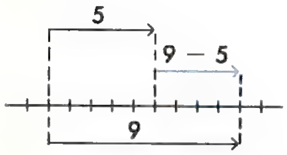
Differences represented by arrows of the same length and the same direction are the same number. For example:



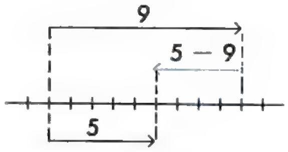
$3 + \square = 7$
$\square = 7 - 3$



$7 + \square = 3$
$\square = 3 - 7$



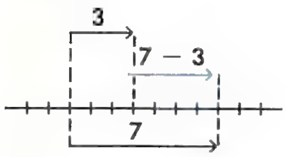
$5 + \square = 9$
$\square = 9 - 5$



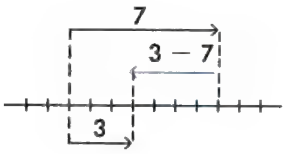
$9 + \square = 5$
$\square = 5 - 9$

$7 - 3$ and $9 - 5$ are the same number.
$3 - 7$ and $5 - 9$ are the same number.

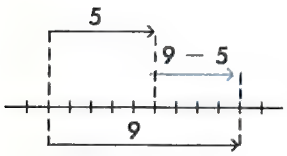
Differences represented by arrows that have the same length but opposite directions are opposites. For example:



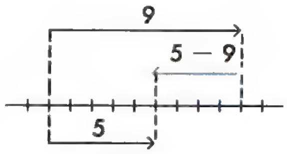
$3 + \square = 7$
$\square = 7 - 3$



$7 + \square = 3$
$\square = 3 - 7$



$5 + \square = 9$
$\square = 9 - 5$



$9 + \square = 5$
$\square = 5 - 9$

$3 - 7$ is the opposite of $7 - 3$ and of $9 - 5$.
$5 - 9$ is the opposite of $7 - 3$ and of $9 - 5$.
$9 - 5$ is the opposite of $3 - 7$ and of $5 - 9$.
$7 - 3$ is the opposite of $3 - 7$ and of $5 - 9$.

$9 - 5$ is a whole number. $5 - 9$ is the opposite of the whole number $9 - 5$. Differences of whole numbers are either whole numbers or opposites of whole numbers.

To indicate the opposite of 3, we write $^-3$. Therefore differences of whole numbers are either whole numbers $\{0, 1, 2, 3, 4, \ldots\}$ or opposites of whole numbers $\{0, ^-1, ^-2, ^-3, ^-4, \ldots\}$.

Readiness for Understanding

Ability to draw the solution to the missing-addend problem.

Knowledge of difference.

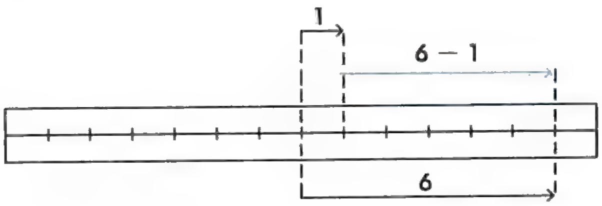
Developmental Experiences

masking tape	*for each child*
felt-tip pen	$\frac{1}{2}''$ graph paper
colored chalk	colored pencil

▶ Place a strip of masking tape or adhesive tape about 63 inches long in the middle of the chalkboard (or use a scale drawn on a transparency for overhead projector). Mark the tape with a felt-tip pen as indicated below. Use 5-inch dividing marks.

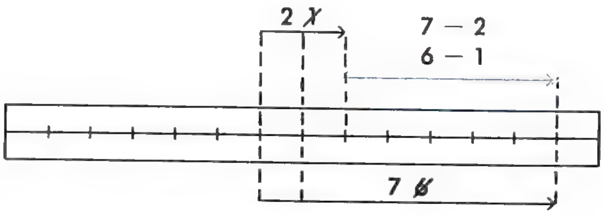
Give each pupil a sheet of $\frac{1}{2}$-inch graph paper and a colored pencil.

On the chalkboard write the equation $1 + \square = 6$. Draw the arrows for 1 and 6 to illustrate this sentence, and have the pupils copy the picture you have drawn. Then ask them to draw the arrow for the difference $6 - 1$. Use colored chalk to draw this arrow on the chalkboard.



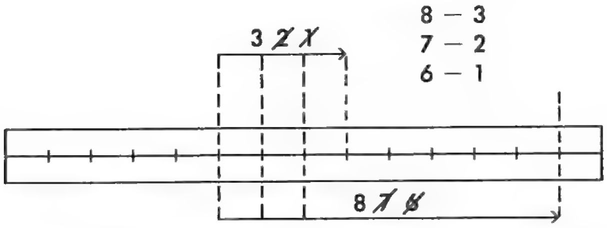
Be sure the pupils copy the picture as far to the right on their graph paper as possible.

Next extend the arrow for the addend from length 1 to length 2, and the arrow for the sum from 6 to 7, as indicated below. Label the addend 2 and the sum 7, and above the $6 - 1$ write $7 - 2$.



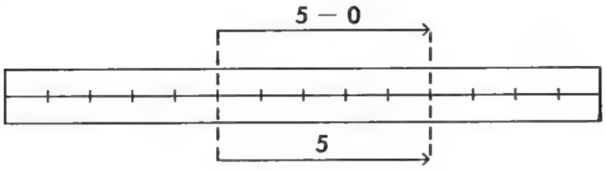
Have the pupils do the same on their graph paper. Tell them that $6 - 1$ is the same number as $7 - 2$.

Then ask them to extend the addend 2 to 3 and the sum 7 to 8 and to name the difference.

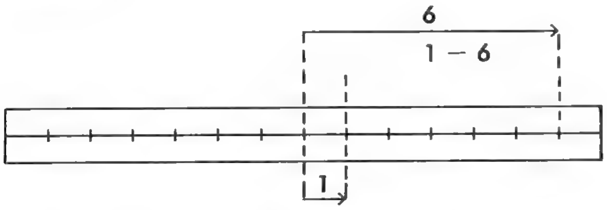


Ask the class to make a list of differences that are the same number as 6 − 1. Make the list on one side of the chalkboard (6 − 1, 7 − 2, 8 − 3, 9 − 4, 10 − 5, 11 − 6, . . .).

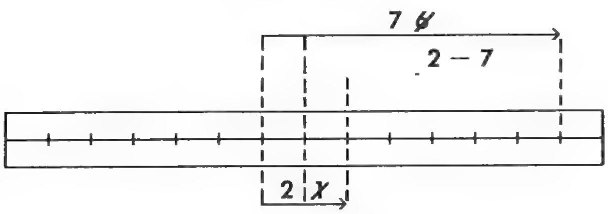
Tell the class that each of these differences is 5. Ask someone to show the difference 5 − 0. Ask whether 5 − 0 belongs in the list (yes). Ask how they know (it has the same length and direction).



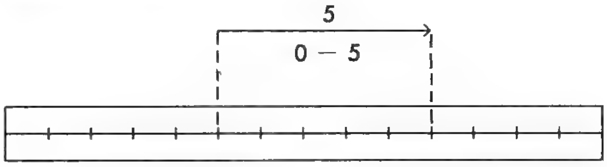
Next erase the material above and below the strip of tape, and write the equation 6 + □ = 1 on the chalkboard. Then draw the arrows shown below. Label the addend 6 and the sum 1, and have the pupils copy the picture on their graph paper. Ask a pupil to state the difference (1 − 6). Label the arrow for the difference.



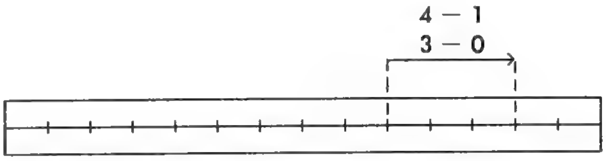
Then extend the arrow for the addend from 6 to 7, and the arrow for the sum from 1 to 2, as indicated below. Write 1 − 6 off to the side, and erase 1 − 6 from above the arrow. Ask the pupils to do the same on their graph paper. Then have someone state the difference (2 − 7). Label the difference.



Ask the class to make a list of differences equal to 2 − 7 and 1 − 6. Make the list off to one side of the chalkboard (1 − 6, 2 − 7, 3 − 8, 4 − 9, 5 − 10, . . .). Ask how they know that each difference is the same number (same length and direction). Ask the class whether anyone thinks 0 − 5 belongs in the list (yes). Ask how they know (same length and direction).

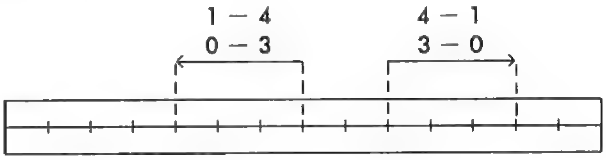


Erase the material above and below the strip of tape, and draw the arrow shown below. Label the arrow 3 − 0 and 4 − 1.



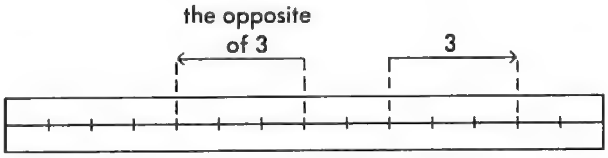
Ask the class to help you make a list of differences equal to 3 (3 − 0, 4 − 1, 5 − 2, 6 − 3, . . .).

Then draw another arrow, as indicated below, and label this arrow 0 − 3 and 1 − 4.



Ask the class to help you make a list of differences equal to 0 − 3 and 1 − 4 (0 − 3, 1 − 4, 2 − 5, 3 − 6, 4 − 7, . . .).

Tell the class that 0 − 3 is the opposite of 3. Ask the class what is the same about the two arrows (their lengths). Ask what is different (their directions). On the chalkboard change the labels of the arrows as shown below.



Then have the pupils draw arrows for 5 and the opposite of 5, and 7 and the opposite of 7.

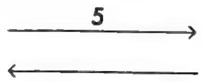
Write some missing-addend equations on the chalkboard, like the following:

9 + □ = 1	8 + □ = 12
5 + □ = 13	20 + □ = 13
10 + □ = 8	20 + □ = 27
15 + □ = 5	23 + □ = 14

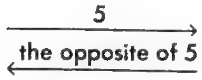
Ask the pupils to state each difference as a whole number or the opposite of a whole number. For example, in 9 + □ = 1 the difference is 1 − 9, or the opposite of 8. In 5 + □ = 13 the difference is 13 − 5, or 8.

The pupil pages introduce the opposite symbol; $^{-}7$ is read "the opposite of 7."

On the chalkboard draw an arrow and its opposite. Label one arrow 5.



Ask a pupil to state the number represented by the other arrow (the opposite of 5). Label the other arrow.



Then ask a pupil what number is the opposite of the opposite of 5 (5).

Erase the labels, and relabel in this way:

Have a pupil state the number shown by the unlabeled arrow (7). Then ask a pupil what number is the opposite of the opposite of the opposite of 7 (the opposite of 7).

Continue changing the numbers shown by the arrows. Always be sure the direction of the labeled arrow is correct. The following labels are suggested: 3, the opposite of 3; 4, the opposite of 4; 10, the opposite of 10; 21, the opposite of 21.

Present the following table on the chalkboard, the overhead projector, or a wall chart.

3 − 5	2 − 4	1 − 4
2 − 7	5	2
6 − 0	3	1
4 − 1	the opposite of 5	18 − 4
9 − 16	7	7 − 8
8 − 7	the opposite of 6	6 − 20
4 − 18	6	the opposite of 7

Match the first difference on the left (3 − 5) with a number on the right that is the same and with another number on the right that is the opposite. Then let each pupil complete the table working independently.

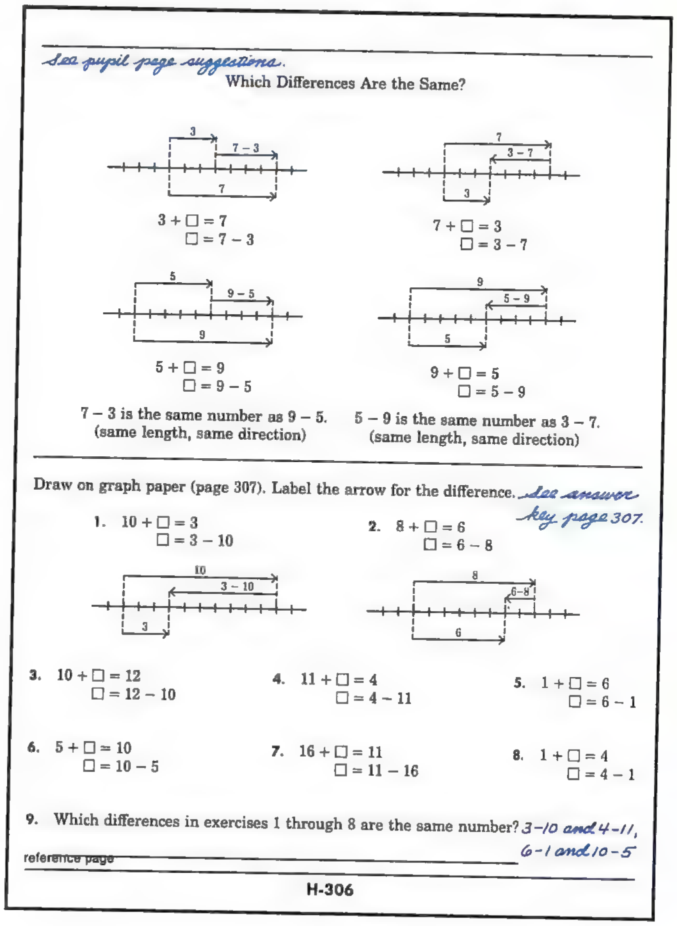
number	same number	opposite number
3 − 5	2 − 4	2
2 − 7	the opposite of 5	5
6 − 0	6	the opposite of 6
4 − 1	3	1 − 4
9 − 16	the opposite of 7	7
8 − 7	1	7 − 8
4 − 18	6 − 20	18 − 4

Allow the pupils to discuss their results.

Pages 306 through 313

Ask the pupils to compare 7 − 3 and 9 − 5 in the diagrams at the top of page 306. (They are the same length and point in the same direction.) Then ask a pupil to compare 3 − 7 and 5 − 9. (They, too, are the same length and have the same direction.) Tell the pupils that differences represented by arrows of the same length and direction are the same number.

Discuss exercises 1 and 2 with the class. Then assign exercises 3 through 8 for independent work. Tell the pupils to use the graph paper on page 307. Afterward, discuss the question at the bottom of the page.



See pupil page suggestions.

Which Differences Are the Same?

3 + □ = 7
□ = 7 − 3

7 + □ = 3
□ = 3 − 7

5 + □ = 9
□ = 9 − 5

9 + □ = 5
□ = 5 − 9

7 − 3 is the same number as 9 − 5. (same length, same direction)

5 − 9 is the same number as 3 − 7. (same length, same direction)

Draw on graph paper (page 307). Label the arrow for the difference. See answer key page 307.

1. 10 + □ = 3
   □ = 3 − 10
2. 8 + □ = 6
   □ = 6 − 8
3. 10 + □ = 12
   □ = 12 − 10
4. 11 + □ = 4
   □ = 4 − 11
5. 1 + □ = 6
   □ = 6 − 1
6. 5 + □ = 10
   □ = 10 − 5
7. 16 + □ = 11
   □ = 11 − 16
8. 1 + □ = 4
   □ = 4 − 1
9. Which differences in exercises 1 through 8 are the same number? 3−10 and 4−11, 6−1 and 10−5

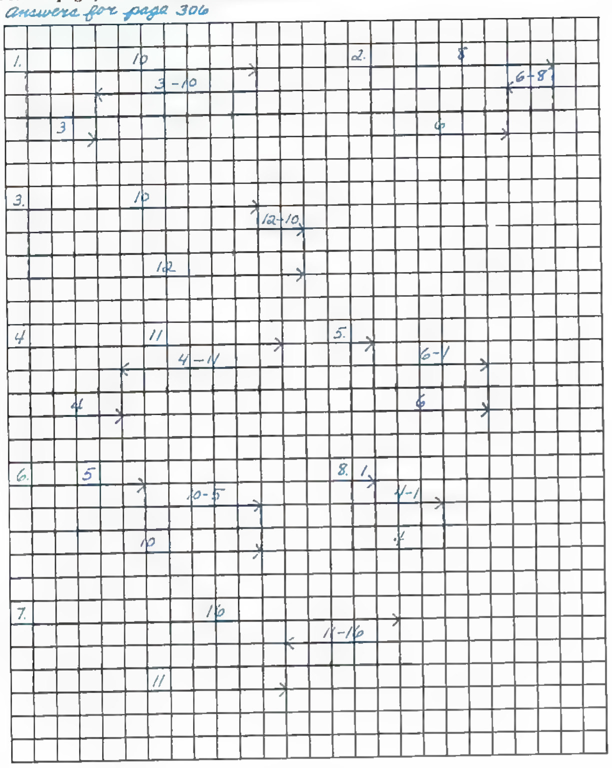
reference page

H-306

Name

Use this page for the exercises on page 306.

Answers for page 306



H-307

See pupil page suggestions.

Which Differences Are Opposites?

$3 + \square = 7$
$\square = 7 - 3$

$7 + \square = 3$
$\square = 3 - 7$

$3 - 7$ is the *opposite* of $7 - 3$. (same length, opposite direction)
$7 - 3$ is the opposite of $5 - 9$. (same length, opposite direction)

$5 + \square = 9$
$\square = 9 - 5$

$9 + \square = 5$
$\square = 5 - 9$

$9 - 5$ is the opposite of $5 - 9$. Why? *See pupil page*
$3 - 7$ is the opposite of $9 - 5$. Why? *suggestions.*

Draw on graph paper (page 309). Label the arrow for the difference. *See answer key page 309.*

1. $10 + \square = 5$
   $\square = 5 - 10$
2. $5 + \square = 10$
   $\square = 10 - 5$

3. $5 + \square = 9$
   $\square = 9 - 5$
4. $11 + \square = 16$
   $\square = 16 - 11$
5. $6 + \square = 1$
   $\square = 1 - 6$
6. $3 + \square = 5$
   $\square = 5 - 3$
7. $8 + \square = 6$
   $\square = 6 - 8$
8. $10 + \square = 8$
   $\square = 8 - 10$
9. Which differences are opposites? *5−10 and 10−5, 5−10 and 16−11, 10−5 and 1−6, 16−11 and 1−6, 5−3 and 8−10, 5−3 and 6−8*
10. Which differences are the same? *5−10 and 1−6, 10−5 and 16−11, 6−8 and 8−10*

reference page

H-308

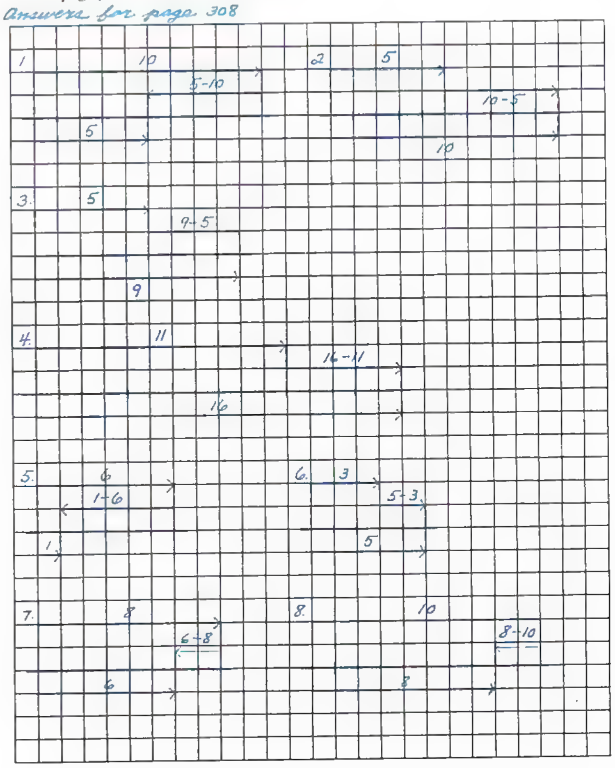
● Ask the pupils to compare 7 — 3 and 3 — 7 in the diagrams at the top of page 308. (The arrows are the same length, but they go in opposite directions.) Now have the pupils compare 7 — 3 and 5 — 9 in the same diagrams. (The arrows are the same length, but they go in opposite directions.) Follow this procedure with the diagrams for 9 — 5 and 5 — 9 and the diagrams for 3 — 7 and 9 — 5. (In both comparisons, the arrows have the same length, but they go in opposite directions.)

Discuss with the class exercises 1 and 2 before assigning exercises 3 through 8 for independent work. Tell the pupils to use the graph paper on page 309 for exercises 3 through 8. Allow the class to discuss the questions at the bottom of the page.

Name

Use this page for the exercises on page 308.

Answers for page 308

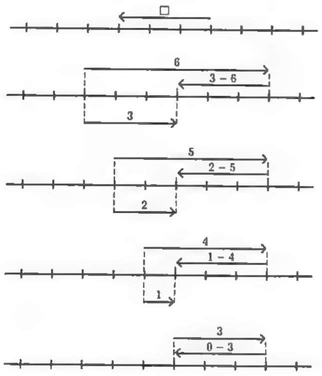


H-309

● Have the pupils compare the arrows for the differences illustrated in the example on page 310. Then point out that all these differences are the same number. Ask a pupil to explain why this is true (same length, same direction). Have another pupil explain how he knows that 1 − 4 is the opposite of 3 (same length, opposite direction). Discuss these answers with the class. Pupils may wish to use the graph paper at the bottom of page 310 to draw arrow diagrams for more differences which are the same number as the opposite of 3. Then have the pupils copy and complete the exercise on page 311, using the graph paper at the bottom of the page.

● Have the pupils copy and complete exercise 1 at the top of page 312, using the symbol ⁻3 for the opposite of 3. (Note: no measurement is necessary—lengths may be compared visually.) Pupils should use the graph paper on page 313. Let the class discuss exercises 2 through 7 and work independently on exercises 8 through 12. Then discuss the statement at the bottom of the page. Ask the children to continue the list of whole numbers and the list of opposites of whole numbers.

What number is □?



6 + □ = 3
□ = 3 − 6

5 + □ = 2
□ = 2 − 5

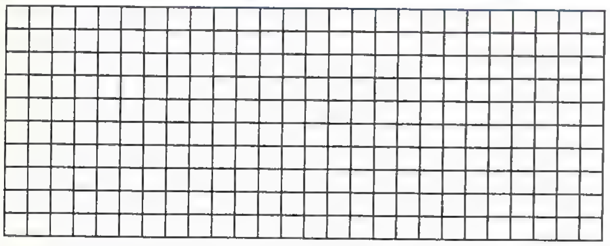
4 + □ = 1
□ = 1 − 4

3 + □ = 0
□ = 0 − 3

0 − 3
1 − 4
2 − 5
3 − 6

Each of these differences is the same number.
How do you know? *They are the same length and the same direction.*

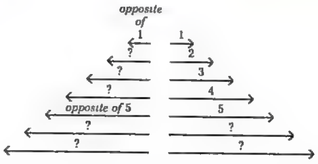
Each number is the opposite of 3. Why? *They are the same length but opposite direction from 3.*

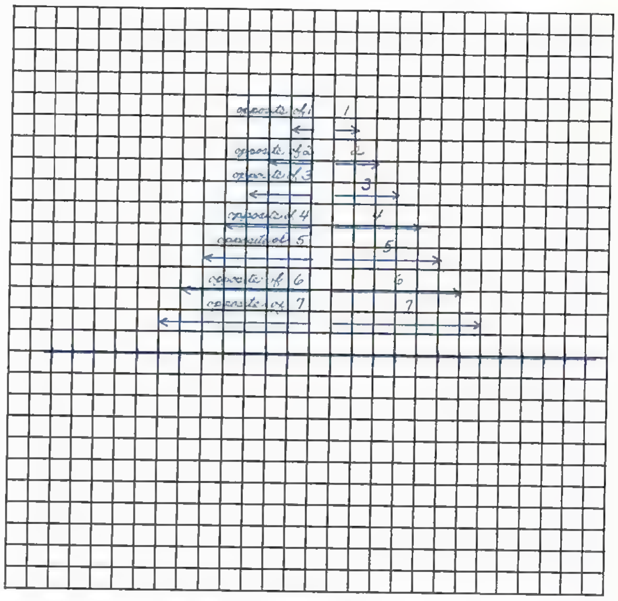


H-310

Name

Copy and complete.

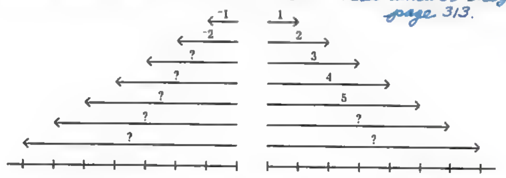




H-311

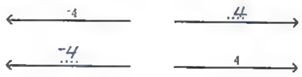
⁻2 is read "the opposite of 2."
The opposite sign ⁻ is higher than the minus sign −.

1. Use graph paper (page 313). Copy and complete. *See answer key page 313.*



2. ⁻4 is *the opposite of* *4*.

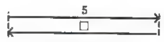
3. *The opposite of* 4 is *⁻4*.



What number is □?

4. 

□ = 3

5. 

□ = ⁻5

6. 

□ = ⁻4

7. 

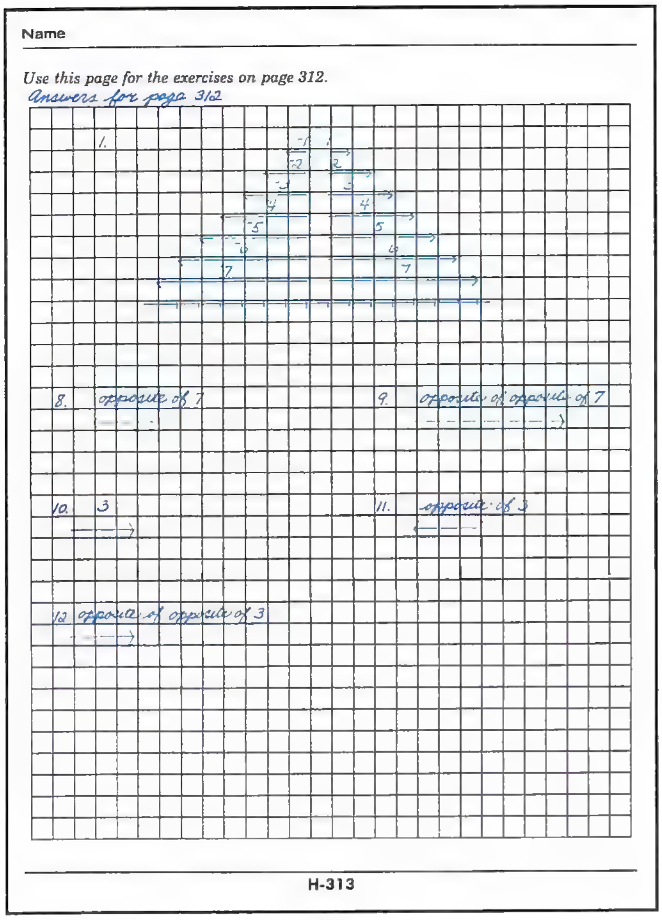
□ = ⁻3

8. Draw the arrow for *the opposite of* 7 on graph paper. *See answer key page 313.*
9. Draw the arrow for *the opposite of* the opposite of 7.
10. Draw the arrow for 3.
11. Draw the arrow for *the opposite of* 3.
12. Draw the arrow for *the opposite of* the opposite of 3.

Differences of whole numbers are
whole numbers {0, 1, 2, 3, 4, . . .}
or opposites of whole numbers {0, ⁻1, ⁻2, ⁻3, ⁻4, . . .}.

reference page

H-312



KEY IDEA

5 + the opposite of 3 is 2.

Scope

To introduce the set of integers.

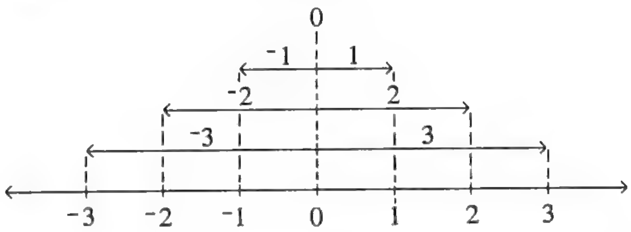
To use whole numbers and opposites of whole numbers to solve missing-addend problems.

Fundamentals

Differences of whole numbers are integers. The set of integers has three parts: positive integers, negative integers, and zero.

Positive integers:	$\{1, 2, 3, 4, \ldots\}$
Negative integers:	$\{^-1, ^-2, ^-3, ^-4, \ldots\}$
Zero:	$\{0\}$
The set of integers:	$\{\ldots, ^-4, ^-3, ^-2, ^-1, 0, 1, 2, 3, \ldots\}$

The picture below shows how differences of whole numbers are used to extend the number line to show opposites of whole numbers.



Any number can be considered in several ways, and integers are no exception. For example, $5 - 2$ is a difference of two whole numbers and is itself a whole number, 3. Also, $2 - 5$ is a difference of two whole numbers. $2 - 5$, however, is not a whole number itself but the opposite of a whole number, $^-3$. We see that $2 - 5$ is an integer, the opposite of 3, the opposite of $5 - 2$, and $^-3$.

The pupil returns to the missing-addend problem and sees that if a and b are whole numbers, the equation $a + \square = b$ always has a solution in the set of integers. For example:

$$3 + \square = 7$$
$$\square = 4$$
$$7 + \square = 3$$
$\square$ is the opposite of 4

Notice that the unit starts and ends with the missing-addend problem. The objective of the unit is to introduce opposites and to develop the understanding that integers are solutions to missing-addend problems. The objective is not to develop computational skill with sums of integers.

The suggested quiz at the end of the unit is meant to be simple and short.

Readiness for Understanding

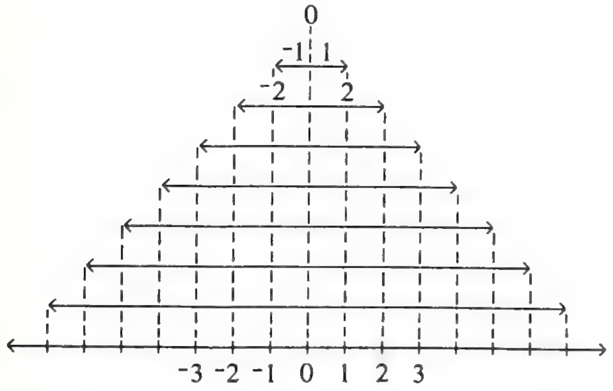
Knowledge of opposites of whole numbers.

Knowledge that the missing addend is a difference.

Developmental Experiences

► Introduce the set of integers by showing the fol-

lowing figure on the chalkboard. (The same diagram on a transparency for the overhead projector would be even more convenient.)



Ask the pupils to complete the figure. Help the class see that the set of differences of whole numbers includes all the whole numbers and their opposites. Explain that the set of integers is the set of differences of whole numbers and that it has three parts: positive integers, negative integers, and zero.

Positive integers	$\{1, 2, 3, 4, 5, \ldots\}$
Negative integers	$\{^-1, ^-2, ^-3, ^-4, \ldots\}$
Zero	$\{0\}$

Remind the class that we began with the missing-addend problem. Write the following on the chalkboard:

$8 + \square = 10$
$\square = 10 - 8$
$\square = \underline{?}$ (2)
$10 + \square = 8$
$\square = 8 - 10$
$\square = \underline{?}$ (the opposite of 2)

Ask the pupils to complete the sentences with a whole number or the opposite of a whole number. Similarly, have them complete the following equations:

$3 + \square = 10$	$4 + \square = 19$	$7 + \square = 2$
$8 + \square = 1$	$5 + \square = 25$	$9 + \square = 9$
$5 + \square = 0$	$3 + \square = 0$	$175 + \square = 175$

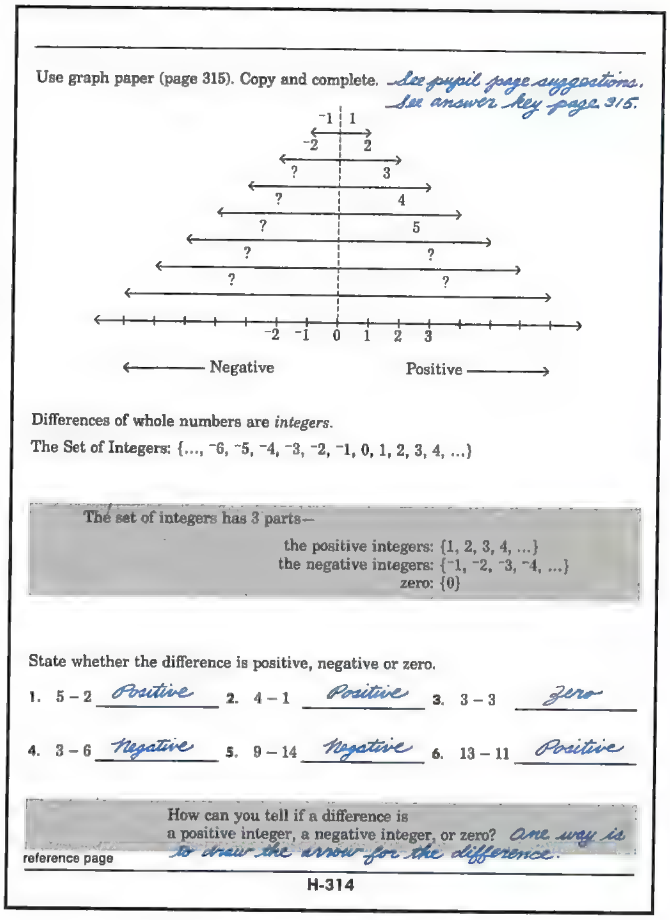
Have the pupils name the differences below with whole numbers or opposites of whole numbers.

$\square = 5 - 2$	$\square = 3 - 3$	$\square = 0 - 1$
$\square = 10 - 1$	$\square = 5 - 5$	$\square = 2 - 0$
$\square = 1 - 10$	$\square = 2 - 10$	$\square = 0 - 5$
$\square = 3 - 1$	$\square = 1 - 0$	$\square = 3 - 14$

Help the pupils see that the $\square$ in $\square = 0 - 5$ is the number that answers the question, "What number do you add to 5 to get 0?"

Pages 314 through 320

● Ask the pupils to copy and complete the exercise at the top of page 314, using graph paper on page 315. Then, with the class, discuss the fact that the set of integers is the set of differences. Let the pupils discuss exercises 1 through 6. Note that $^-1$ is read "the opposite of 1."



Use graph paper (page 315). Copy and complete. See pupil page suggestions. See answer key page 315.

Differences of whole numbers are *integers*.

The Set of Integers: $\{\ldots, ^-6, ^-5, ^-4, ^-3, ^-2, ^-1, 0, 1, 2, 3, 4, \ldots\}$

The set of integers has 3 parts—
the positive integers: $\{1, 2, 3, 4, \ldots\}$
the negative integers: $\{^-1, ^-2, ^-3, ^-4, \ldots\}$
zero: $\{0\}$

State whether the difference is positive, negative or zero.

1. 5 − 2 Positive 2. 4 − 1 Positive 3. 3 − 3 Zero
4. 3 − 6 Negative 5. 9 − 14 Negative 6. 13 − 11 Positive

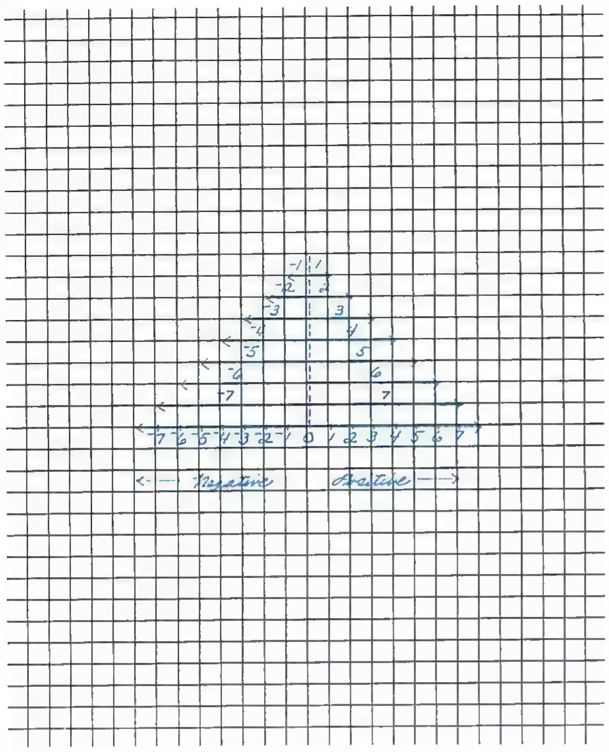
How can you tell if a difference is a positive integer, a negative integer, or zero? One way is to draw the arrow for the difference.

reference page

H-314

Name

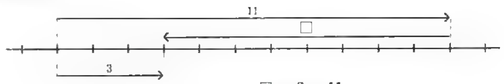
Use this page for the exercises on page 314.



H-315

For Class Discussion

11 + □ = 3

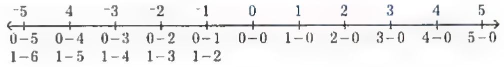


□ = 3 − 11

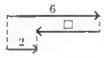
3 − 11 is the opposite of 8. How do you know? *

11 + *opposite of* 8 is 3. How do you know? *

1. Copy on graph paper (page 317) and complete by showing 5 differences for each integer. *See answer key page 317*

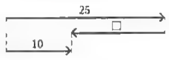


2. What number is □?



6 + □ = 2
□ = 2 − 6
□ is the opposite of 4. Why? *
6 + *opposite of* 4 = 2

3. What number is □?



25 + □ = 10
□ = 10 − 25
□ is the opposite of 15. Why? *
25 + *opposite of* 15 = 10

It's too much work to write *opposite of.*

4. 11 + □ = 3
□ = 3 − 11
□ = ⁻8
11 + ⁻8 = 3

5. 19 + □ = 13
□ = 13 − 19
□ = ⁻6
19 + ⁻6 = 13

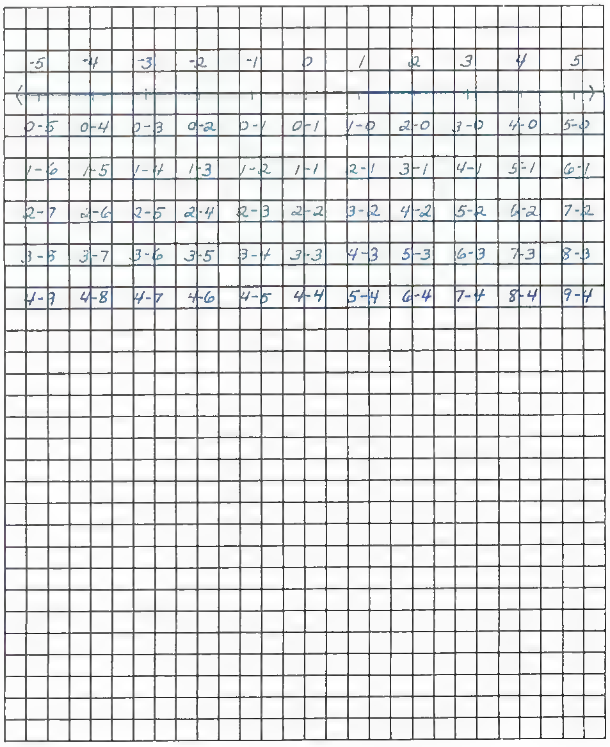
6. 13 + □ = 1
□ = 1 − 13
□ = ⁻12
13 + ⁻12 = 1

H-316

*See pupil page suggestions.

Name

Use this page for the exercises on page 316.



H-317

● Discuss the figure at the top of page 316 with the class. The pupils should realize that 3 − 11 is the opposite of 8 because the arrows that represent 3 − 11 and 8 have the same length, but they go in opposite directions. Also, 11 + *opposite of* 8 is 3; the arrows show that this is true.

Have the pupils copy and complete exercise 1, using graph paper on page 317. Ask the pupils to verify that 4 − 5 is the opposite of 1.

Discuss exercises 2, 3, and 4 with the class. "Why?" in exercise 2 may be answered as "same length as 4 but opposite direction." "Why?" in exercise 3 may be answered as "same length as 15 but opposite direction." Assign exercises 5 and 6 for independent work, followed by discussion.

● With the class, discuss the example and exercises 1 through 3 on page 318. Then assign the remaining exercises for independent work. Let the pupils discuss their results.

● Pages 319 and 320 provide applications of length and direction concepts. Allow the pupils to answer the questions in their own words and use their own ideas about opposites.

The Missing-Addend Problem Is Solved.

$2 + \square = 5$	$5 + \square = 2$
$\square$ is 3	$\square$ is the opposite of 3; $\square = {}^-3$.
$2 + 3 = 5$	$5 + {}^-3 = 2$

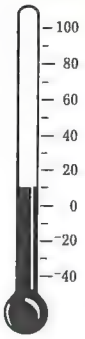
Solve the missing-addend problems. Complete.

1. $1 + \square = 2$ $\square =$ 1
2. $2 + \square = 1$ $\square =$ $^-1$
3. $8 + \square = 9$ $\square =$ 1
4. $7 + \square = 1$ $\square =$ $^-6$
5. $16 + \square = 7$ $\square =$ $^-9$
6. $17 + \square = 17$ $\square =$ 0
7. $5 + \square = 0$ $\square =$ $^-5$
8. $2 + \square = 25$ $\square =$ 23
9. $3 + \square = 2$ $\square =$ $^-1$
10. $100 + \square = 200$ $\square =$ 100
11. $300 + \square = 200$ $\square =$ $^-100$
12. $6 + \square = 5$ $\square =$ $^-1$
13. $17 - 3 =$ 14
14. $3 - 17 =$ $^-14$
15. $21 - 21 =$ 0
16. $1 - 1 =$ 0
17. $25{,}105 - 25{,}100 =$ 5
18. $7 + {}^-6 =$ 1
19. $2 + {}^-1 =$ 1
20. $25{,}100 - 25{,}105 =$ $^-5$
21. $16 + {}^-9 =$ 7
22. $5 + {}^-5 =$ 0
23. $47 + {}^-47 =$ 0
24. $48 + {}^-47 =$ 1

$2 - 5$ is a new kind of number; $2 - 5$ is the opposite of 3, or $^-3$.

H-318

Find a Lazy Way.



1. What temperature is shown on this thermometer? 10°
   What will the reading be if the temperature goes
   up 10°, then down 10°,
   then up 15°, then down 25°,
   then up 100°, then down 50°,
   then up 10°, then down 70°,
   then up 10°, then down 20°? $^-20°$

2. The lowest temperature recorded in the United States was $^-70°$ at Rogers Pass, Montana on January 20, 1954. The highest temperature recorded in the United States was 134° at Greenland Ranch, California on July 10, 1913. What is the difference in the temperatures? 204°

3. One winter afternoon the temperature was $^-3°$. By early evening it had dropped 9°. What was the temperature in the evening? $^-12°$

4. Add. Is there a lazy way?

$50 + {}^-49 + 48 + {}^-47 + 46 + {}^-45 + 44 + {}^-43 + 42 + {}^-41 +$

$40 + {}^-39 + 38 + {}^-37 + 36 + {}^-35 + 34 + {}^-33 + 32 + {}^-31 +$

$30 + {}^-29 + 28 + {}^-27 + 26 + {}^-25 + 24 + {}^-23 + 22 + {}^-21 +$

$20 + {}^-19 + 18 + {}^-17 + 16 + {}^-15 + 14 + {}^-13 + 12 + {}^-11 +$

$10 + {}^-9 + 8 + {}^-7 + 6 + {}^-5 + 4 + {}^-3 + 2 + {}^-1 =$ 25

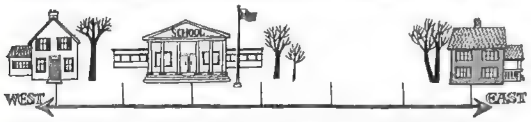
H-320

Name

1. Each day a frog jumps 2 feet up a muddy hill and slides back 1 foot. How far up the hill is he at the end of the first day? 1 foot

   At the end of the third day, he is 3 feet up the hill, how many more days will he take to get up the hill (and stay up)? 6 days

2. Sam lives 2 blocks west of school. Carl lives 4 blocks east of school.



How far does Carl walk to school? Which direction? If Sam walks from his house to Carl's house, how far does he walk? Which direction?

Carl walks four blocks west. Sam walks six blocks east.

One morning Sam met Carl and they walked to school together. When they got to school, they found that each boy had walked the same distance from home to school. How did they do this? Sam walked three blocks east. Carl walked three blocks west.

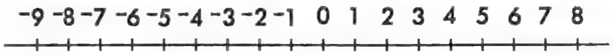
3. Paul measured the pressure of his bicycle tire. It was 3 pounds low. He added some air and measured the pressure again. It was then 6 pounds too high. What was the difference in the pressures? 9 pounds

H-319

Supplemental Experiences

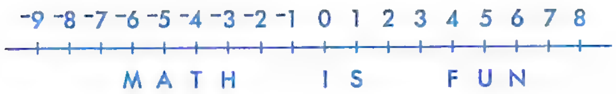
■ Play Diffo. (See Developmental Experiences for Key Idea: 2 − 5 is a new kind of number.)

■ Distribute copies of the following puzzle to the pupils. Explain that they are to use the number line to solve the puzzle. Then have the pupils complete the activity independently.

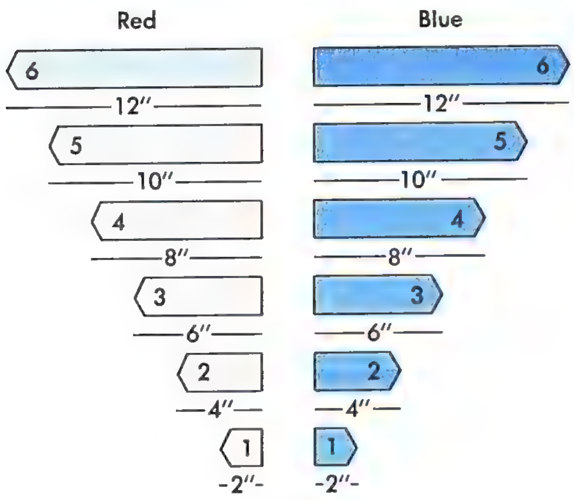


1. Find the integer that is one greater than 3. Label it F.
2. Find the integer that is one less than 2. Label it S.
3. Find the integer that is three greater than 3. Label it N.
4. Find the integer that is three less than $^-1$. Label it T.
5. Find the integer that is one greater than $^-1$. Label it I.
6. Find the integer that is two greater than $^-8$. Label it M.

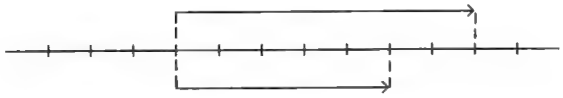
7. Find the integer that is six greater than $^-1$. Label it U.
8. Find the integer that is one less than $^-4$. Label it A.
9. Find the integer that is five less than 2. Label it H.



■ Prepare a set of 6 red cardboard arrows and a set of 6 blue cardboard arrows. They should be 1 inch wide and 2, 4, 6, 8, 10, and 12 inches long. Label the arrows from shortest to longest with the numbers 1, 2, 3, 4, 5, and 6 respectively.



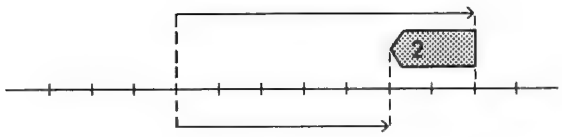
Draw a scale 36 inches long in the middle of the chalkboard. Make the dividing marks 2 inches apart. Draw an arrow 5 units long below the scale pointing to the right, and draw another arrow 7 units long above the scale pointing in the same direction.



Tell the pupils they are going to play a game. Divide the class into two teams. Then show them the two sets of arrows, and have them look at the picture you have drawn on the chalkboard. Explain to the pupils that the arrow below the scale represents the sum and the one above represents one of the addends in an addition sentence. They are to choose from the set of cardboard arrows the arrow having the appropriate length and color to represent the missing addend in the diagram on the board. Explain that red arrows are used to point to the left and blue arrows point to the right.

Now work the example on the board with the class. The arrow for the missing addend is two units long and points to the left. It must extend from the tip of the addend arrow to the tip of the sum arrow.

Take the red arrow marked 2 and put it in place above the scale.



Help the pupils decide that the missing-addend sentence illustrated by this diagram is $7 + \square = 5$. The difference is $5 - 7$.

Work one or two more examples with the class. Then let them play the game. In drawing a diagram, always draw the arrow for the addend above the scale and the arrow for the sum below it. When the diagram is drawn, ask a member of one team to give the number and color of the arrow needed to extend from the tip of the addend arrow to the tip of the sum arrow. If he cannot answer, or if he answers incorrectly, ask a member of the other team the same question. A team scores one point for each correct answer.

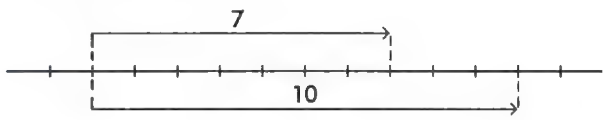
The game can be varied by asking the pupils to give the missing-addend sentence illustrated by a given diagram. Then the team scores one point for each arrow correctly identified and another point for identifying the missing-addend sentence.

■ You may wish to give the following quiz. Notice that this is a concept test, not a test in computational skill.

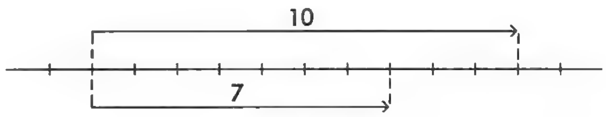
SUGGESTED QUIZ

Copy and complete. Show $\square$ as a difference.

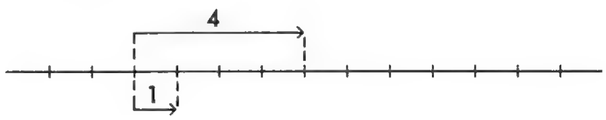
1. $7 + \square = 10$



2. $10 + \square = 7$



3. $4 + \square = 1$



Answer with a whole number or the opposite of a whole number.

4. $5 + \square = 10$
   $\square = \underline{?}$ (5)

5. $10 + \square = 5$
   $\square = \underline{?}$ (the opposite of 5)

6. If $\square = 6 - 9$, then $9 + \square = 6$
   $\square = \underline{?}$ (the opposite of 3)

7. The opposite of the opposite of 6 is $\underline{?}$ (6)

UNIT 22
GEOMETRY: POINTS, LINES, AND PLANES

Pages 321 Through 332

OBJECTIVE

To develop a perception of plane.

The pupil learns that a plane is unbounded; it extends as far as he wishes. He observes that four points either lie in one plane or determine four planes. He learns a test to show whether a fourth point is in the same plane as three given points. He explores intersecting and parallel planes.

See Key Topics in Mathematics for the Intermediate Teacher: Geometry.

KEY IDEA

Four points may determine four planes.

CONCEPTS

plane
point of intersection

KEY IDEA

Four points may determine four planes.

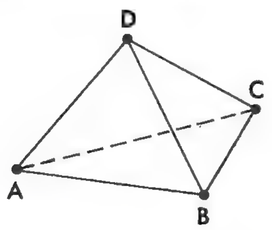
Scope

To explore planes.

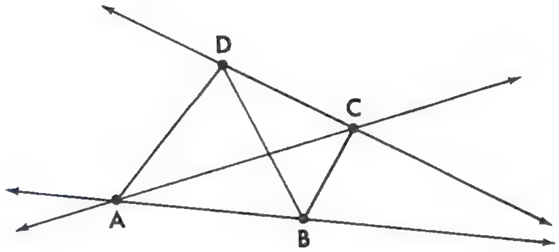
Fundamentals

A plane is a primitive mathematical concept—that is, a concept that is not formally defined. A plane is determined by three points not on a line. Think of the vertices of a triangle as the three points determining a plane, and then think of the plane as an endless extension of the interior of this triangle in all lateral directions. This property of planes might be described as "flatness."

A question that is considered in this unit is whether a fourth point is in a given plane. Consider three points, A, B, and C, in the plane of this paper and a fourth point, D, above the paper.



Observe that lines AC and BD do not intersect. Imagine the point D moving toward the plane of the paper and finally touching the paper as shown below.



When D is in the plane of the paper, the diagonals AC and BD intersect and determine a new point, E. This leads to a procedure for testing whether a point D is in the plane of points A, B, and C; count points of intersection. If the lines determined by the four points have only 4 points of intersection, then D is not in the plane. On the other hand, if there are more than 4 points of intersection, then points A, B, C, and D are all in the same plane.

The spirit of this unit should be one of investigation. While it is important for the pupils to develop an understanding of space, this understanding should not be forced. Rather they should enjoy learning how to perceive planes suggested by physical objects in their environment.

Readiness for Understanding

Perception of point and line.

Developmental Experiences

string	*for each child*
masking tape	index card (3″ × 5″)
magnets	
sheet metal	
tagboard	
cardboard	
marbles	
erasers	

▶ Introduce the idea of a plane by referring to physical models of parts of planes. Point out that your desk top represents part of a plane. Then have the pupils imagine that it extends endlessly in all lateral directions. This endless extension would be a plane.

Use index cards or sheets of tagboard of different sizes, such as 3 × 5, 6 × 8, and 9 × 12, to show that it is always possible to have a larger part of a plane. Lead the pupils to discover that no matter how large our physical model of a plane is, there is always more of the plane outside that part. The pupils should discover that a plane can have no boundaries. Many illustrations will help the pupils understand this concept.

Ask a pupil to see whether he can balance a card on the end of one finger. Ask whether the card is stable, or well balanced. Then ask him to try to balance the card on the ends of two fingers. He will observe the same difficulty—the card is not well balanced, or

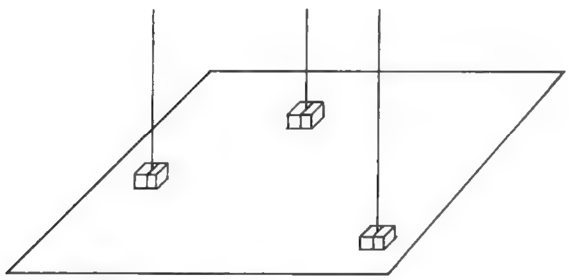
stable. Ask the pupils to discuss what would happen with a one-legged or a two-legged chair.

Have two pupils, each holding a sharpened pencil with the point up, try to balance a sheet of cardboard on the pencil points. They will not be able to balance the sheet. However, if another pupil holds a third pencil near the other two, at the same height but not in line with them, the cardboard will rest steadily on the three pencils. If the cardboard represents a plane and the pencil points represent points, the pupils will readily see that three points not in a straight line are required to determine a plane.

This fact can be reinforced by comparing the stability of a two-legged stool with that of a three-legged stool.

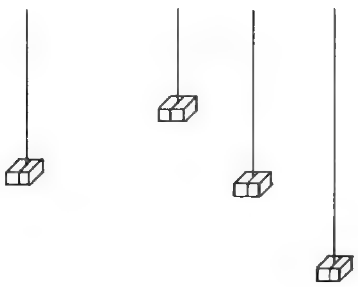
Ask the class how many planes can pass through three given points not in the same line. Let pupils use 3 by 5 index cards and pencils to investigate this. Since any cards (representing planes) that rest on the same three pencil points (representing points) will represent the same plane, pupils should conclude that three points not in the same line determine exactly one plane.

► Use magnets and sheet metal to illustrate that three points determine a plane. Suspend three small magnets at the end of strings, as illustrated. Then attach a flat piece of sheet metal (such as the lid of a coffee can or a metal desk drawer divider) to the magnets.



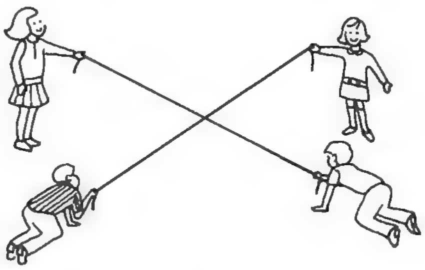
Vary the lengths of the strings to alter the plane of the sheet metal. Will it ever be necessary to bend the sheet metal to have it contact all three magnets? (no) This illustrates that three points determine a plane.

Now suspend a fourth magnet. Place it so that it is not in the plane of the other three.



Again attach the sheet of metal to the magnets. Then ask a pupil to show how many different planes can be illustrated using only three of the magnets at a time.

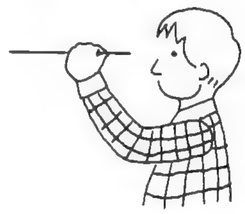
► Have four pupils hold two taut strings so that they intersect.



Have other pupils locate points in the plane of the strings and other points not in the plane.

► Have a pupil locate three points in the room and describe the plane determined by these points. Then ask another pupil to show a fourth point in this plane. His location should be acceptable to the pupil who designated the plane.

► Have the pupils each hold a piece of cardboard horizontal at eye level.



As they look at the cardboard, ask what geometric figure they see. Some pupils may observe, "If you slice a plane, it looks like a line."

Have the children locate a point level with the cardboard but to the right of it. Ask them, "Is this point in the plane? Why?" Then have the pupils locate a point level with the cardboard but to the left of it. Ask, "Is this point in the plane?" Ask whether they can locate still another point that is in the plane of their cardboard. Then ask whether there are other points beyond these that are in the same plane. This will develop the idea that a plane extends on and on without bound.

Pages 321 through 325

- Discuss the models of planes that are shown on page 321. The pupils should understand that a plane is flat and extends endlessly. Note that the last part of question 2 introduces the phrase " of the same plane." Answers for this question should not be formally tested. If the suggested object is level with the page, let that be sufficient. The third exercise helps to motivate a discussion of what is meant by both "plane" and "edge." The pupils who say that a plane has an edge may be asked whether there could be anything in the same plane but beyond the edge. If so, is there really an edge?

- Ask the pupils to use rulers and colored pencils to draw the lines determined by points A, B, and C on page 322. Provide cardboard, string, and tape for the pupils to use in constructing the model described on the page. When the pupils have constructed their models, let them discuss the questions at the bottom of the page.

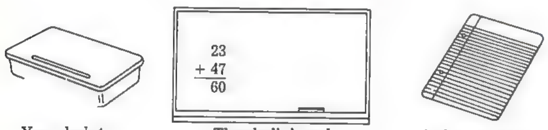
- Use page 323 for class discussion. This page extends the concept of plane and develops the idea that three points not in a line determine a plane. Notice that the condition that the three points not be in a line is not stated explicitly. Rather this idea should be brought out through discussion of question 1.

Name

UNIT 22 GEOMETRY: POINTS, LINES, AND PLANES

For Class Discussion See pupil page suggestions.

What is a plane?
Look at parts of planes. Parts of planes are illustrated by flat surfaces.

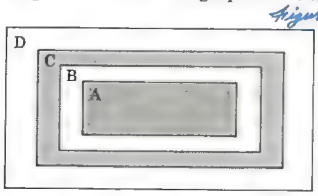


Your desk top | The chalk board | A sheet of paper

These examples have boundaries. But a plane, like a line, extends on and on without end.

1. Does the outside of a ball show part of a plane? Does an orange? An egg? A piece of chalk? Why? No, because they do not have flat surfaces.*

2. Figure A shows part of a plane. Figure B shows a larger part of the same plane. Which figures show an even larger part of the same plane? figures C and D



3. Does a plane have an edge? Explain. No, because a plane extends on and on without end.*

*answers may vary.

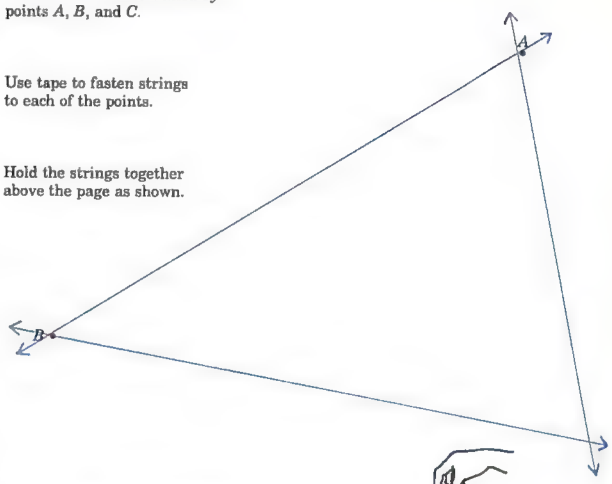
H-321

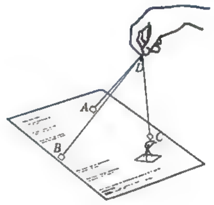
For Class Discussion

Draw the lines determined by points A, B, and C.

Use tape to fasten strings to each of the points.

Hold the strings together above the page as shown.





How many lines are determined by points A, B, and C? 3

How many lines are determined by points A, B, C, and D? 6

How many planes are determined by points A, B, C, and D? 4

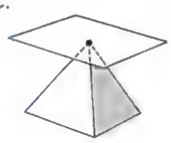
reference page

H-322

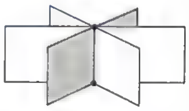
Name

For Class Discussion See pupil page suggestions.

How many points determine a plane?



Many planes can contain one given point.
One point does not determine one plane.

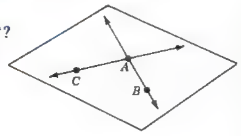


Many planes can contain two given points.
Two points do not determine one plane.

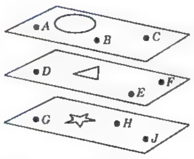
Exactly one plane can contain three given points.
Three points determine exactly one plane.

1. Can three points be arranged so that they are in more than one plane? Yes, if all three points are in one line

2. How many planes contain both lines AC and AB? One plane



3. Does plane ABC contain a triangle? No*
In which plane is the circle? Plane ABC
Where is the star? Plane GHT



*answers may vary.

H-323

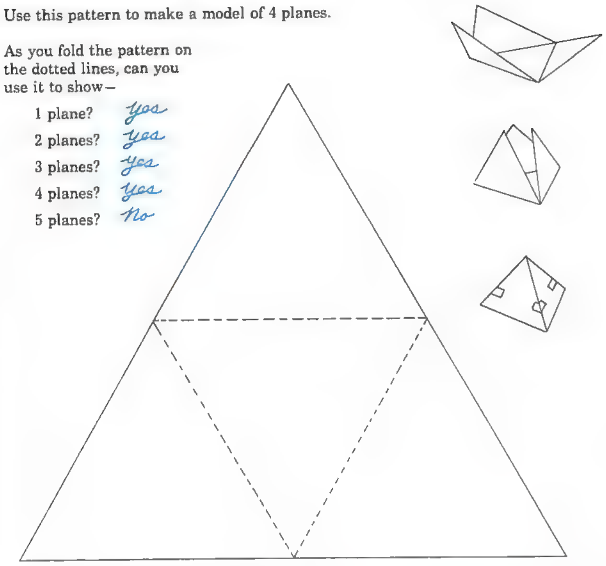
● Ask the pupils to investigate the questions at the top of page 324 as they construct the model. Have them use the completed model to investigate the questions at the bottom of the page.

For Class Discussion

Use this pattern to make a model of 4 planes.

As you fold the pattern on the dotted lines, can you use it to show—

1 plane? *yes*
2 planes? *yes*
3 planes? *yes*
4 planes? *yes*
5 planes? *no*



Examine your model of four planes.

1. What do you find where the edges meet? *a point or corner*
2. Can you find two edges that do not meet? *yes*
3. What do you find where two planes meet? *an edge or line*
4. Can you find two planes that do not meet? *no*

reference page

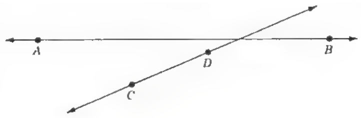
H-324

● Page 325 presents the idea that more than three points may be contained in one plane. Ask the pupils to read the page and then discuss the illustrations. Encourage them to make string and index-card models to illustrate the situations depicted.

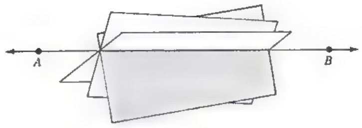
Name

For Class Discussion

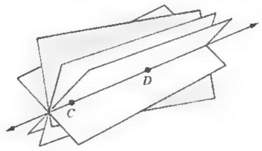
Three points determine one plane. What about four points?



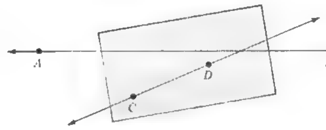
Lines AB and CD intersect in a point.



Line AB is contained in many planes.



Line CD is contained in many planes.



One of the planes that contains line AB also contains line CD.

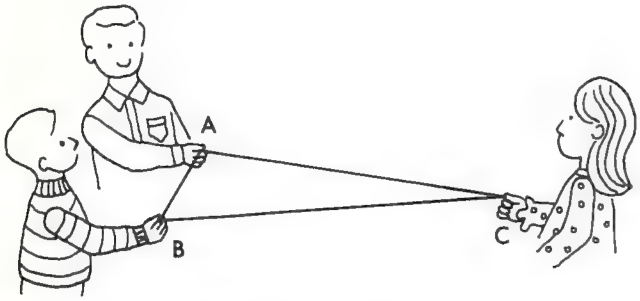
Since A, B, C, and D determine a pair of intersecting lines, the four points are in one plane.

Find 4 points in your classroom. Are they in one plane? Do they determine a pair of intersecting lines? *If the four points are in one plane, they may determine a pair of intersecting lines.*

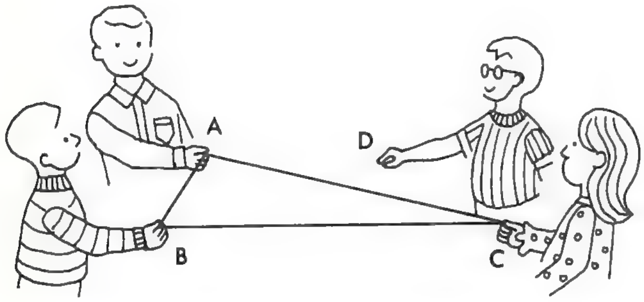
H-325

Developmental Experiences

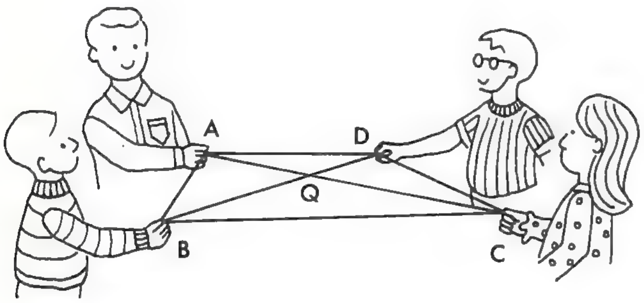
► Have three pupils stretch taut a piece of string, holding it at points A, B, and C as shown.



Have a fourth child choose another point D. Ask the class whether D is in the plane determined by A, B, and C.

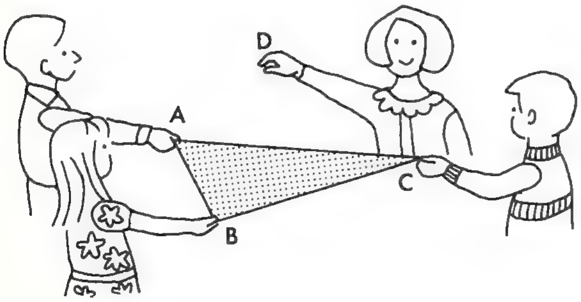


To determine whether D is in the plane, suggest this test.

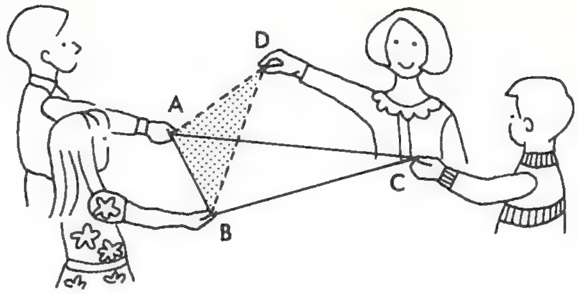


Do lines AC and BD intersect? If they do, then point D is in the same plane. If they do not, then point D is not in the same plane as A, B, and C, and the four points determine four distinct planes:

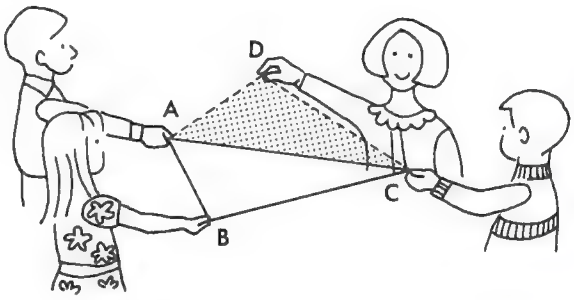
1. The plane of A, B, and C



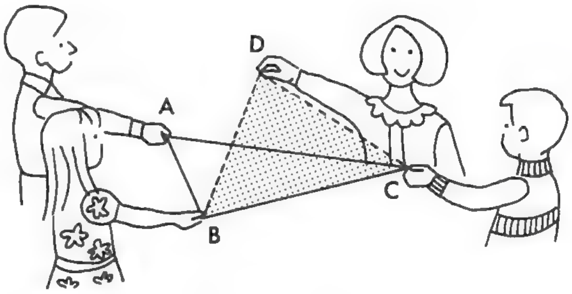
2. The plane of A, B, and D



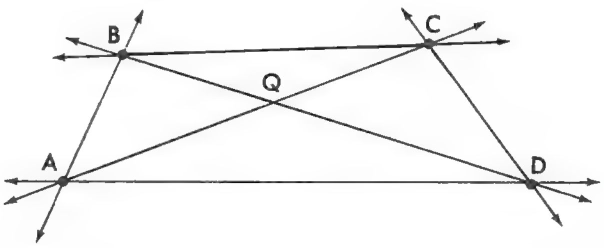
3. The plane of A, C, and D



4. The plane of B, C, and D



Locate three points A, B, and C (not in a line) on the chalkboard. Draw the lines AB, BC, and AC. Then choose a fourth point D, not on any of these lines, as illustrated. Draw lines AD, BD, and CD on the chalkboard. A new point, Q, is determined by the intersection of AC and BD.



Explain that since AC and BD intersect, the four points, A, B, C, and D, are in one plane.

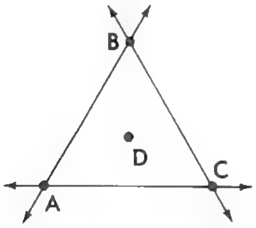
Now locate three points A, B, and C (not in a line) on the chalkboard. Draw the lines AB, BC, and AC. This time choose a fourth point a few feet in front of the chalkboard. Use masking tape and string to con-

nect the fourth point D to the points on the chalkboard. A, B, C, and D are points of intersection of these lines. Ask if there are any other points of intersection (no). Ask the class whether D is in the plane of A, B, and C (no). Explain that when the fourth point does not create another point of intersection, it is not in the plane of the other three points.

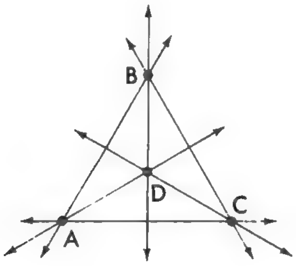
Now allow point D to move toward the chalkboard. Observe that a new point of intersection is determined when D reaches the plane of the chalkboard.

Pages 326 through 330

● Ask the pupils to follow the directions on page 326, using pencil and paper. As they complete the activity, they will see that at least one more point of intersection is determined. Some pupils may need help in finding another point of intersection. For example, some may select a fourth point D as follows:



When this happens, help the pupils see that there are actually three new points of intersection.



● Use page 327 for class discussion. This page begins by asking each pupil to choose three points on his paper and one point above the paper.

After the pupils have located their four points, ask them how many lines are determined. Have some of the pupils tell how many lines they can see. Write the name for each different suggestion on the chalkboard. Ask the pupils to specify which lines they can see (AB, BC, AC, EB, EA, EC).

Next have the pupils identify each point of intersection. They will see that the four points do not determine another point of intersection. Thus they are not in a plane.

In the third illustration, point N is above the paper, and no new point of intersection is determined.

In the fourth illustration, either E *is* on the paper or AB *does not* intersect EC. Encourage the class to make string models to help them visualize the answers to the questions on this page.

● As the pupils construct the model of a box, have them discuss the questions at the top of page 328.

The completed models should be saved and used in the investigations on page 329.

● Ask the pupils to use their box models to investigate the questions on page 329. You may wish to have the pupils arrange boxes on a tabletop as they investigate the picture at the bottom of the page. The colored planes will not meet. If both planes are extended, another box can always be placed between them.

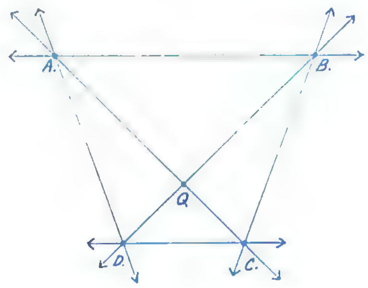
● Ask the pupils to locate three points, A, B, and C as requested on page 330. They may mark the points with pencil. Then for each exercise they may use a pencil or fingers to show the fourth point. Let the children work in teams of five so that those sitting near each other can check to see that the points all lie in the planes. When differences of opinion arise, remind the pupils how to test whether four points are in a plane.

For exercises 4 and 5, the pupils should be encouraged to locate points beyond any shown so far.

For Class Discussion See pupil page suggestions.

Make this drawing at the bottom of the page.

1. Mark three points *A*, *B*, and *C*, not on a line.
2. Draw the three lines determined by these three points.
3. The lines intersect at *A*, *B*, and *C*.
Is another point of intersection determined? no
4. Now mark a fourth point *D*, not on any of the lines.
5. Draw the three additional lines determined by the fourth point.
6. Is another point of intersection determined? yes, point Q is another point of intersection.



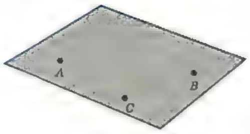
Points *A*, *B*, *C*, and *D* are in one plane.

H-326

Name

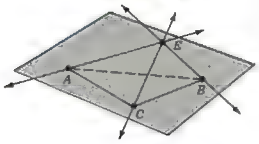
For Class Discussion *See pupil page suggestions.*

Mark 3 points A, B, and C on your paper.



What happens if point E is above the paper?

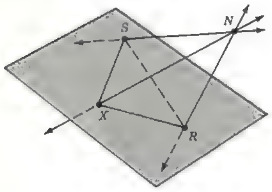
Is another point of intersection determined? *No*



Does line EC intersect line AB? *No*

Does line EA intersect line BC? *No*

Does line EB intersect line AC? *No*

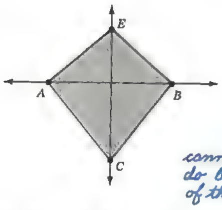


Points S, R, and X are on a paper.

Point N is above the paper.

Is another point of intersection determined? *No*

Explain. *Two lines must be in the same plane to intersect.*



Points A, B, and C are on a paper.

Point E is not on the paper.

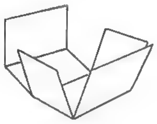
Line AB intersects line EC.

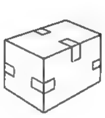
What is wrong? *AB and EC cannot intersect. They look as if they do because the picture is flat. One of the statements must be false.*

H-327

For Class Discussion

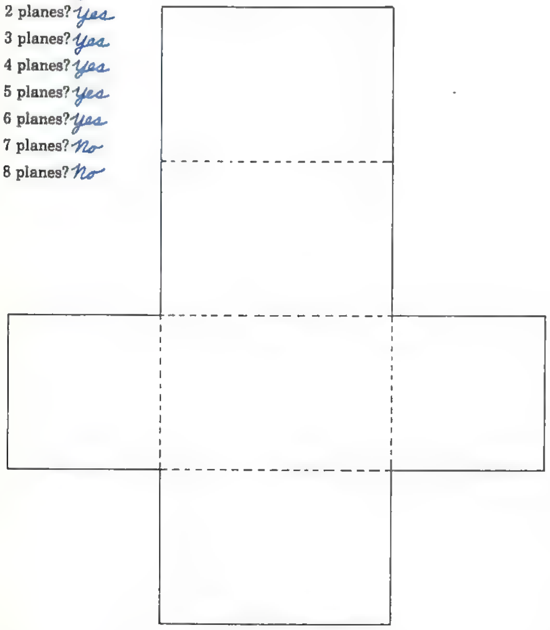
Use this pattern to make a box.





As you fold the pattern on the dotted lines, can you use it to show—

1 plane? *Yes*
2 planes? *Yes*
3 planes? *Yes*
4 planes? *Yes*
5 planes? *Yes*
6 planes? *Yes*
7 planes? *No*
8 planes? *No*

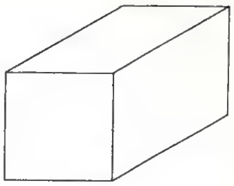


H-328

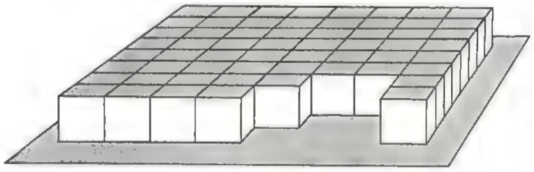
Name

For Class Discussion

Examine your model of a box.



1. What do you find where two edges meet? *A point or corner*
2. Can you find two edges that do not meet? *Yes*
3. What do you find where two planes meet? *An edge or line*
4. Can you find two planes that do not meet? *Yes*

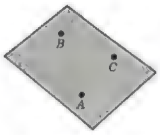


5. Will the two colored planes ever meet? *See pupil page suggestions.*
   How do you know?

H-329

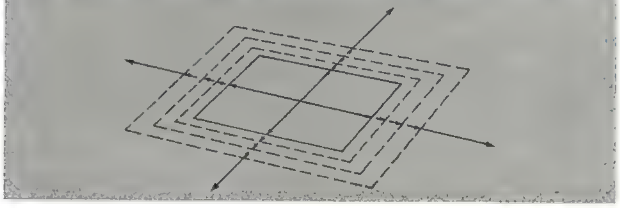
For Class Discussion

On a sheet of paper locate 3 points A, B, and C, not on a line.



1. With your finger locate point D to the right of point A. Is point D in the plane of the paper? How do you know? *
2. Now locate point E near the lower right corner of the paper. Is point E in the plane? *
3. Locate a point F not on the paper, in the plane with points A, B, and C. How do you know that F is in the plane? *
4. Can you locate a point beyond these 6 points that is in the plane? How do you know? *Yes. A plane extends in every direction.*
5. Is every point on the paper in the plane? Are these the only points in the plane? *Yes. Every point on the paper is in the plane. These are not the only points in the plane.*

A plane extends as far as you wish.



** Answers depend on location of point. Accept answers based on visual determinations.*

H-330

Developmental Experiences

Have the pupils locate points in various planes. The following suggestions may be useful:

Locate a point in the plane of the chalkboard but not on the chalkboard.

Locate a point in the plane of a picture but not on the picture.

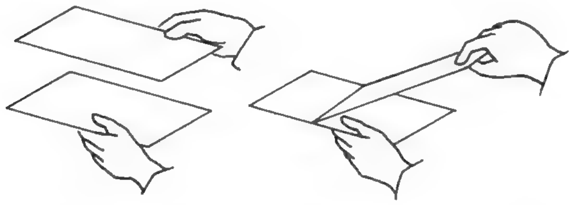
Locate a point on a plane that is shown by a sheet of paper parallel to the floor.

Locate a point that is on two different planes. Describe both planes.

Locate a point that is in the plane of the top of a book, but not on the book.

Hold up a sheet of notebook paper. Locate two points in the plane of the paper but not on the paper. Is the line connecting these two points in the plane of the paper?

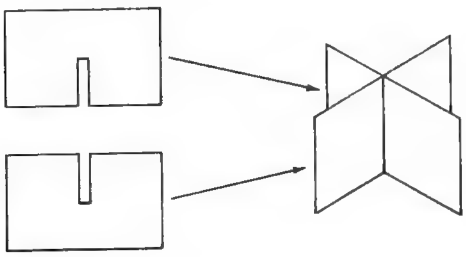
Use index cards or cardboard sheets to illustrate parallel and intersecting planes. Hold the cards the same distance apart, one above the other, to illustrate parallel planes.



Two planes are parallel if they do not intersect. Ask the pupils to look at the plane of the ceiling and the plane of the floor. These planes are parallel. Find other parallel planes in the classroom; also ask pupils to describe parallel planes that they have observed outside the classroom. Opposite sides of boxes are good examples.

Discuss intersecting planes and examples of intersecting planes that pupils can "see." Two adjacent walls of a room, two pages in a partly opened book, and many other examples should be discussed. Some pupils may tell about things in their homes that illustrate two intersecting planes.

The teacher should give the pupils index cards or sheets of cardboard to help them visualize intersecting planes.



Investigation will show that two planes either do not intersect or have exactly one line of intersection.

Ask three pupils each to locate a point in the classroom. Let each one hold a marble to indicate the location of his point. Now blindfold a fourth pupil, and ask him to hold a marble in the plane determined by the three points. The class can help him by responding to his attempts with "You're getting warmer," or "You're getting colder," "Up," "Down," "Right," or "Left." When the pupil succeeds, he takes off the blindfold, and another team of pupils may play the game.

Give two pupils each a piece of cardboard. Ask them to stand facing each other at the front of the room and to hold the two pieces of cardboard so that two distinct planes are represented. Point out to the class that each pupil's cardboard represents a plane.

Ask a third pupil to come forward and locate points in the first pupil's plane that are not on the cardboard. A fourth pupil should do this with the second plane.

Now have two other pupils each locate a point that is in neither plane. Each point should be indicated by holding a marble, an eraser, or some other object in the desired location.

Ask the first pupil to tilt his piece of cardboard so that it determines a plane containing the two points. Then ask the second pupil to do the same.

Ask whether anyone in the class can find a third point that is in both planes. Guide the class in discussion

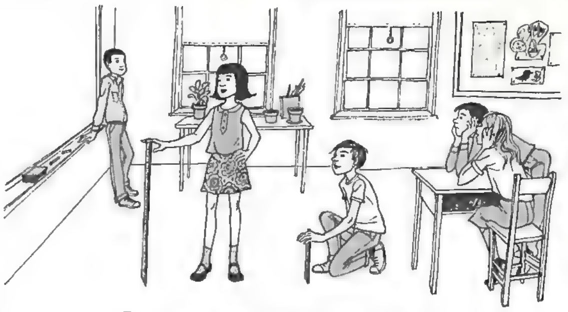
to see that when two planes intersect, they intersect in a line.

You can emphasize this result by repeating the experiment. This time, however, ask two pupils to hold a string taut. Then have the cardboards tilted so that each plane contains the line of the string.

► Give two pupils each a piece of cardboard, and have them hold their pieces of cardboard so as to determine two planes. Ask a third child to use a marble to locate a point that is in both planes. Encourage the other members of the class to suggest ways of proceeding.

Next have the pupils holding the pieces of cardboard try to tilt their planes so that it is not possible to locate a point in both planes. Have the class check to make sure the planes do not intersect. Explain that nonintersecting planes are parallel. Two planes are parallel if there is no point that is in both planes.

Name



Do you see a plane along the window wall?
Along the table's two legs?
Across the floor?
Along the desk surface?

1. Consider a plane that is 3 feet above the floor on the left side of the room and 1 foot above the floor at your desk. Locate some points that are in this plane. Does this plane intersect the plane of the floor? *yes*

Locate some points that are in both planes.
Describe all the points that are in both planes.
The points in both planes occur at the intersection of the two planes; a line.

2. Locate some points 2 feet above the floor of the classroom. Are these points in the same plane? *yes*

Are there any points that are in this plane and also in the plane of the floor? *no*

If a plane is everywhere 2 feet above the floor, it is *parallel* to the plane of the floor.

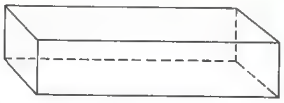
H-331

Two planes are parallel, or they intersect in a line.

Parallel planes have no points in common.
Figures W and R show parallel planes.

1. What examples of parallel planes can you see in your classroom?
*Opposite walls and the ceiling and floor show parallel planes.**

2. What parallel planes are shown by the sides of a box?

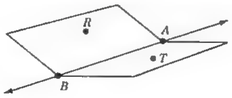


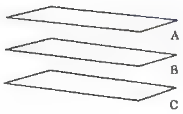
The front and back, the top and bottom, and the two ends show parallel planes.

3. Fold a piece of cardboard. How many planes are shown by the folded cardboard? How would you describe the intersection of these planes? *At least 2 planes are shown. The planes intersect in a line.*

4. Describe some pairs of intersecting planes in your classroom. Identify their lines of intersection. *A wall and the floor, two adjacent walls, and a wall and the ceiling show examples of intersecting planes.**

5. Line AB is in plane ABR. It is also in plane *ABT*.





6. Three planes A, B, and C, are parallel. Somewhere (not shown in this model) there is a line that intersects planes A and B. Must that line intersect plane C? Explain.
See pupil page suggestions.

* *Answers may vary.*

H-332

Pages 331 and 332

● Ask the pupils to look at the classroom pictured on page 331 or at their own classroom and locate the planes described in the questions at the top of the page.

In exercise 1 the pupils may show with string the location of the indicated plane. They will see that this plane and the plane of the floor intersect in a line. The pupils may use string to show the line of intersection of the two planes.

Exercise 2 can be discussed in terms of the pupils' classroom. Ask pupils in various parts of the room to help locate the plane that is 2 feet above the floor. Several points in this plane can be located. This exercise will demonstrate that parallel planes are the same distance apart at all points.

● Page 332 can be used for class discussion. Discuss each question with the pupils after they have had a chance to decide on their own answers. It may be effective to make a model for the last question using index cards and string, and to show the intersection of the line with the third plane, using a marble to represent the intersection. The pupils may suggest other models of their own.

UNIT 23
USING THE RULER
Pages 333 Through 340

OBJECTIVE
To develop skill in the use of a ruler.

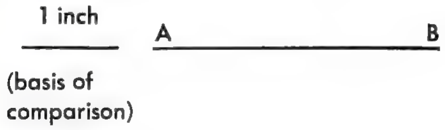
The pupil reviews measurement as comparison. He learns that measurement in standard units requires a universally or legally accepted basis of comparison. He practices measuring lengths to the nearest inch, $\frac{1}{2}$ inch, $\frac{1}{4}$ inch, and $\frac{1}{8}$ inch. One of the basic objectives of this unit is to have the pupil perceive a number of inches as a length rather than as a mark on a ruler.
See Key Topics in Mathematics for the Intermediate Teacher: Measurement.

KEY IDEA
The inch is a basis of comparison.

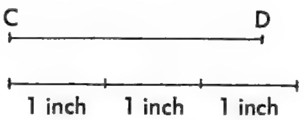
KEY IDEA
The inch is a basis of comparison.

Scope
To practice using a ruler.

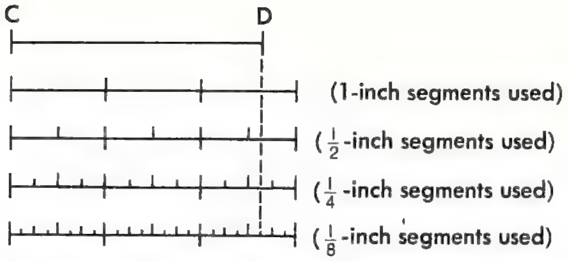
Fundamentals
It should be recalled that measurement is comparison. The standard inch is one basis of comparison, and the ruler is a device that helps to compare lengths with the inch. For example:



A 12-inch ruler shows that the line segment AB is three times the basis of comparison, or 3 inches in length.
Whenever the length of one object is used as a unit to measure the length of another object, the measurement is always an approximation of that length.
For example, let's measure line segment CD. By placing three 1-inch segments along the line, we see that CD is more than 2 inches, but less than 3 inches, in length.



By using smaller units of measure, we obtain measurements for CD that give better approximations of its length.

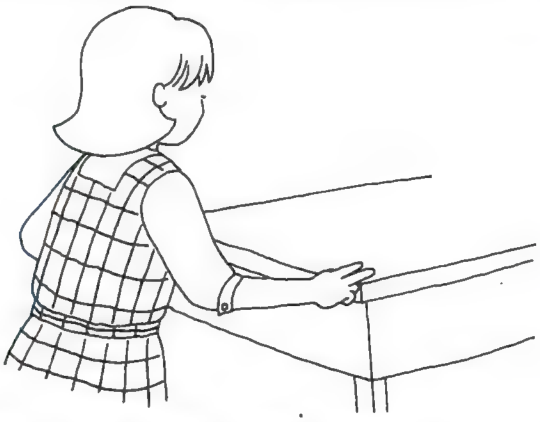


The measurement of CD to the nearest $\frac{1}{2}$ inch is $\frac{5}{2}$ inches, or $2\frac{1}{2}$ inches; to the nearest $\frac{1}{4}$ inch it is $\frac{11}{4}$ inches, or $2\frac{3}{4}$ inches; and to the nearest $\frac{1}{8}$ inch it is $\frac{21}{8}$ inches, or $2\frac{5}{8}$ inches.

Readiness for Understanding
Knowledge of measurement.

Developmental Experiences
for each child
12-inch ruler
tagboard strips (1″ × 1″, 1″ × 2″, 1″ × 3″, 1″ × 4″, 1″ × 5″, 1″ × 6″, and 1″ × 9″)
3 paper strips (1″ × 1″)

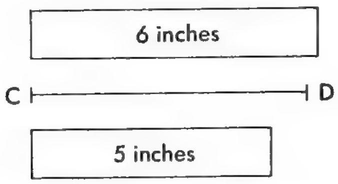
▶ Explain to the children that they can measure lengths using the width of the two fingers nearest their thumb.



Point out that many objects cannot be measured to an exact number of two-finger widths.
Then have each pupil measure the length of a table (or desk) in the room to the nearest two-finger width. Record each measurement on the chalkboard. The pupils will observe that not all the measurements for the length of the table are the same. Encourage them to suggest some reasons why this happened. They might suggest that the width of two fingers differs from one child to another. Or they might suggest that the way in which the fingers are placed along the edge of the table can affect the measurement.

▶ Prepare a set of 6 tagboard strips for each child. The strips should be 1 inch wide and 1, 2, 3, 4, 5, and 6 inches long, respectively.
Tell the pupils that the shortest strip is 1 inch long. Have them label this strip "1 inch." Then tell the pupils to use the 1-inch strip to measure the lengths of the others. Have the length written in inches on each strip.

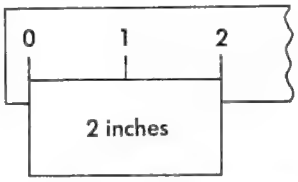
Draw a $5\frac{3}{4}$-inch line segment on the chalkboard, and label it CD. Ask a child to measure this line segment for the class. By placing both a 6-inch strip and a 5-inch strip along the segment, the pupil can show that the length of CD is less than 6 inches but greater than 5 inches.



Ask the pupils whether the length is closer to 6 or 5 inches (6 inches).

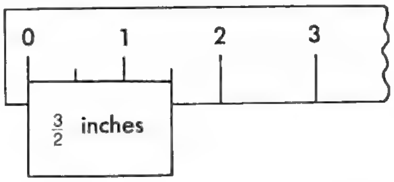
Ask a pupil to look at his 5-inch strip and find an object in the room that he thinks is about 5 inches long. Have him use his 5-inch strip to measure the length of that object. Then have different pupils repeat the activity—examining a strip, choosing an object in the room that is the same length, and comparing the length of the strip and the length of the object.

► Give each pupil a strip of tagboard 1 inch wide and 9 inches long. Instruct the pupils to mark a starting point near the left end of the strip and label it 0. Then have them use their tagboard strips from the previous exercise to mark points that are 1, 2, 3, 4, 5, and 6 inches from 0. Each length should be labeled above the mark, as shown.



Next have the pupils use their tagboard rulers to measure the lengths of different objects in the classroom. In each case the length should be measured to the nearest inch.

Now ask the pupils to fold their 1-inch tagboard strips into 2 equal lengths. Have someone tell the length of each part (1 inch *divided by* 2, or $\frac{1}{2}$ inch). Similarly, have each of the other tagboard strips folded into 2 equal lengths, and have each length identified ($\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, and $\frac{6}{2}$ inches). Tell the pupils to find these lengths on their tagboard rulers, measuring from 0. Have them mark the distances (without labeling) as shown. Tell the children they can use the folded 1-inch strip to divide each of the remaining inches on the ruler into 2 equal parts.



Let the pupils use these rulers to measure the lengths of several objects first, to the closest inch and then to the closest $\frac{1}{2}$ inch. For example, to the closest inch, a pencil may be 4 inches long; and to the closest $\frac{1}{2}$ inch, the length of the same pencil may be $\frac{7}{2}$, $\frac{8}{2}$, or $\frac{9}{2}$ inches.

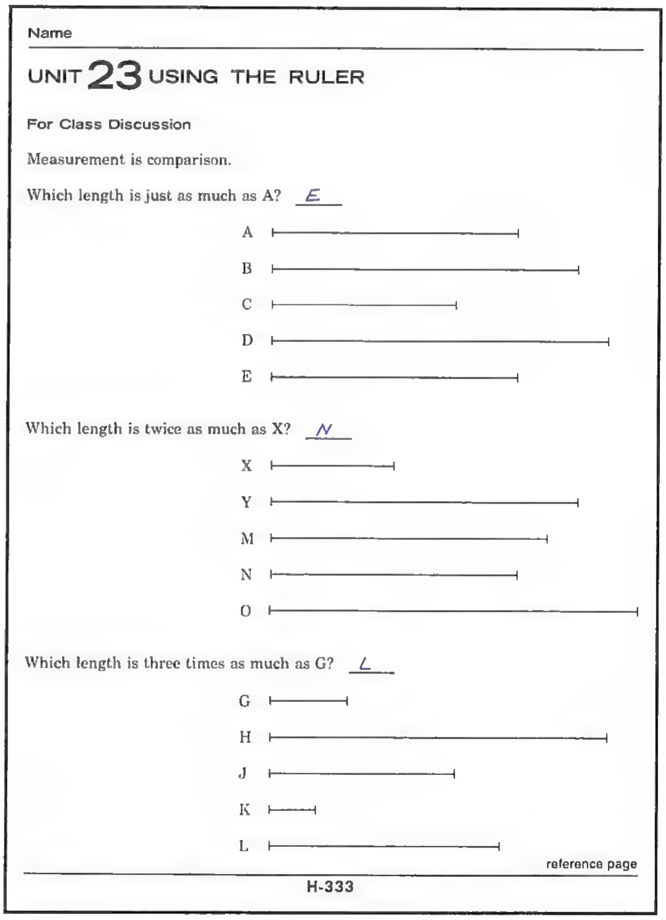
Pages 333 through 335

● Page 333 provides practice in comparing lengths. A visual comparison is good enough. Assign the page for independent work then discuss the results.

● On page 334, the children use the inch as a unit of measure. Ask them to use their 1-inch through 6-inch tagboard strips to measure the length of each segment. When work has been completed, discuss the results.

● Page 335 provides practice in measuring lengths to the nearest inch and to the nearest $\frac{1}{2}$ inch. Tell the pupils to use their tagboard rulers and to express the measurements to the nearest $\frac{1}{2}$ inch as quotients ($\frac{3}{2}$ inches, $\frac{8}{2}$ inches, and so on). Assign the exercises for independent work.

As the results of these exercises are discussed, ask the pupils to give a mixed fraction for each of the quotients. Direct the children to inspect their rulers to see whether $\frac{5}{2}$ inches is 2 inches + $\frac{1}{2}$ inch.



Name

UNIT 23 USING THE RULER

For Class Discussion

Measurement is comparison.

Which length is just as much as A? E

A
B
C
D
E

Which length is twice as much as X? N

X
Y
M
N
O

Which length is three times as much as G? L

G
H
J
K
L

reference page

H-333

Measurement is comparison.

We can measure by comparing with a standard unit.

One standard unit often used is the *inch*.

A ——— *B*

One inch is the length of *AB*.

Measure the lengths in inches.

1. *C* ——— *D*
   How long is *CD*? 3 inches
2. *E* ——— *F*
   How long is *EF*? 4 inches
3. *G* ——— *H*
   How long is *GH*? 2 inches
4. *J* ——— *K*
   To the nearest inch, how long is *JK*? 4 inches
5. *L* ——— *M*
   To the nearest inch, how long is *LM*? 5 inches
6. To the nearest inch, how long is your pencil? Answers will vary.

reference page

H-334

Name

For Class Discussion

1 inch ———

$\frac{1}{2}$ inch ——

Measure the lengths.

1. *A* ——— *B*
   To the nearest inch, how long is *AB*? 3 inches
   To the nearest $\frac{1}{2}$ inch, how long is *AB*? $\frac{5}{2}$ or $2\frac{1}{2}$ inches
2. *C* ——— *D*
   To the nearest inch, how long is *CD*? 3 inches
   To the nearest $\frac{1}{2}$ inch, how long is *CD*? $\frac{7}{2}$ or $3\frac{1}{2}$ inches
3. *E* ——— *F*
   To the nearest inch, how long is *EF*? 5 inches
   To the nearest $\frac{1}{2}$ inch, how long is *EF*? $\frac{10}{2}$ or $5\frac{0}{2}$ inches
4. *G* ——— *H*
   To the nearest inch, how long is *GH*? 5 inches
   To the nearest $\frac{1}{2}$ inch, how long is *GH*? $\frac{11}{2}$ or $5\frac{1}{2}$ inches
5. *K* ——— *L*
   To the nearest inch, how long is *KL*? 5 inches
   To the nearest $\frac{1}{2}$ inch, how long is *KL*? $\frac{9}{2}$ or $4\frac{1}{2}$ inches
6. How long is your book, to the nearest inch? Answers may vary.
   To the nearest $\frac{1}{2}$ inch? Answers may vary.

H-335

Developmental Experiences

► Give each child two paper strips 1 inch wide and 1 inch long. Tell the pupils that the length of each strip is 1 inch.

Ask how many $\frac{1}{2}$ inches there are in 1 inch. The children should know there are 2. Let them suggest a way to fold a 1-inch strip to show a $\frac{1}{2}$-inch length. Someone should suggest folding the strip in the middle. Tell the children to do this. Then have a pupil show a $\frac{1}{2}$-inch length on his folded strip.

Next ask how many $\frac{1}{4}$ inches there are in 1 inch (since $4 \times \frac{1}{4}$ is 1, there must be four $\frac{1}{4}$ inches in one inch). Tell the children to fold the second of their 1-inch strips into four equal parts. Then have a pupil show a $\frac{1}{4}$-inch length on his folded strip.

Direct the children to open the papers they have just folded and use them to mark their tagboard rulers with $\frac{1}{4}$-inch lengths. Then have the rulers used to measure the lengths of various objects to the nearest $\frac{1}{4}$ inch. Tell the pupils that when lengths are described as $\frac{2}{2}$ inches, $\frac{3}{2}$ inches, and so on, this indicates measurement to the nearest $\frac{1}{2}$ inch. When lengths are described as $\frac{2}{4}$ inches, $\frac{3}{4}$ inches, and so on, measurement to the nearest $\frac{1}{4}$ inch is indicated.

► Give each child a paper strip 1 inch wide and 1 inch long. Tell the class that the length of each strip is 1 inch. Ask the children to fold their strips to show a length of $\frac{1}{2}$ inch and then to fold the strips again to show a length of $\frac{1}{4}$ inch. Then have the strips opened, and ask how the length could be divided into eight equal parts. When the class agrees on how this is to be done, have the last fold made. The children should notice that the lengths may not look the same because the folding was difficult. Each part, however, should be about $\frac{1}{8}$ inch long.

Tell the pupils to place their strips along the edges of their 12-inch rulers to find a distance of $\frac{1}{8}$ inch, measuring from 0. Notice that the $\frac{1}{8}$-inch length on the ruler can be used to verify the accuracy of the folding. Then have the pupils find other distances, still measuring from 0: $\frac{2}{8}$ inches, $\frac{3}{8}$ inches, and so forth to $\frac{16}{8}$ inches.

Next have the class use their rulers to measure the lengths of various objects to the nearest $\frac{1}{8}$ inch.

Pages 336 and 337

● Page 336 provides practice in measuring lengths to the nearest inch, $\frac{1}{2}$ inch, and $\frac{1}{4}$ inch. Tell the pupils to use their tagboard rulers. Assign the page for independent work, and discuss the results.

As the results are discussed, let the pupils explore the idea that $\frac{9}{4}$ inches is also 2 inches $+ \frac{1}{4}$ inch. Have someone give a mixed fraction for each fraction.

● Use page 337 as a class activity. Let the children use their 12-inch rulers. Help them identify lengths of $\frac{1}{2}$ inch, $\frac{2}{2}$ inch, $\frac{3}{2}$ inches, and so forth, on their rulers. Then have them measure each segment in the exercises to the nearest inch and to the nearest $\frac{1}{2}$ inch.

Similarly, help the children identify lengths on their rulers that are multiples of $\frac{1}{4}$ inch. Then have them measure each segment in the exercises to the nearest $\frac{1}{4}$ inch. In the same manner, have the children measure to the nearest $\frac{1}{8}$ inch.

---

1 inch

$\frac{1}{2}$ inch

$\frac{1}{4}$ inch

Measure the lengths.

1. A —— B
   How long is AB, to the nearest inch? 3 inches
   To the nearest $\frac{1}{2}$ inch? $\frac{7}{2}$ or $3\frac{1}{2}$ inches
   To the nearest $\frac{1}{4}$ inch? $\frac{13}{4}$ or $3\frac{1}{4}$ inches

2. C —— D
   How long is CD, to the nearest inch? 4 inches
   To the nearest $\frac{1}{2}$ inch? $\frac{7}{2}$ or $3\frac{1}{2}$ inches
   To the nearest $\frac{1}{4}$ inch? $\frac{14}{4}$ or $3\frac{3}{4}$ inches

3. E —— F
   How long is EF, to the nearest inch? 5 inches
   To the nearest $\frac{1}{2}$ inch? $\frac{9}{2}$ or $4\frac{1}{2}$ inches
   To the nearest $\frac{1}{4}$ inch? $\frac{19}{4}$ or $4\frac{3}{4}$ inches

4. G —— H
   How long is GH, to the nearest inch? 5 inches
   To the nearest $\frac{1}{2}$ inch? $\frac{10}{2}$ or $5\frac{0}{2}$ inches
   To the nearest $\frac{1}{4}$ inch? $\frac{20}{4}$ or $5\frac{0}{4}$ inches

5. How wide is your desk, to the nearest $\frac{1}{4}$ inch? Answers will vary.

6. How long is this page, to the nearest $\frac{1}{4}$ inch? $\frac{40}{4}$ or $10\frac{0}{4}$ inches

H-336

Name

1 inch

$\frac{1}{2}$ inch

$\frac{1}{4}$ inch

$\frac{1}{8}$ inch

Measure the lengths.

1. A —— B
   To the nearest inch, how long is AB? 2 inches
   To the nearest $\frac{1}{2}$ inch, how long is AB? $\frac{4}{2}$ or $2\frac{0}{2}$ inches
   To the nearest $\frac{1}{4}$ inch, how long is AB? $\frac{9}{4}$ or $2\frac{1}{4}$ inches
   To the nearest $\frac{1}{8}$ inch, how long is AB? $\frac{17}{8}$ or $2\frac{1}{8}$ inches

2. C —— D
   To the nearest inch, how long is CD? 4 inches
   To the nearest $\frac{1}{2}$ inch, how long is CD? $\frac{7}{2}$ or $3\frac{1}{2}$ inches
   To the nearest $\frac{1}{4}$ inch, how long is CD? $\frac{14}{4}$ or $3\frac{2}{4}$ inches
   To the nearest $\frac{1}{8}$ inch, how long is CD? $\frac{29}{8}$ or $3\frac{5}{8}$ inches

3. E —— F
   How long is EF, to the nearest inch? 5 inches
   To the nearest $\frac{1}{2}$ inch? $\frac{10}{2}$ or $5\frac{0}{2}$ inches
   To the nearest $\frac{1}{4}$ inch? $\frac{20}{4}$ or $5\frac{0}{4}$ inches
   To the nearest $\frac{1}{8}$ inch? $\frac{39}{8}$ or $4\frac{7}{8}$ inches

4. How wide is this book, to the nearest $\frac{1}{8}$ inch? $\frac{66}{8}$ or $8\frac{2}{8}$ inches*

5. How long is this book, to the nearest $\frac{1}{8}$ inch? $\frac{80}{8}$ or $10\frac{0}{8}$ inches*

reference page

* Answers may vary.

H-337

Developmental Experiences

► Draw a 7-inch line segment on the chalkboard. Call on four pupils, in turn, to measure the length of the segment to the nearest 1 inch, $\frac{1}{2}$ inch, $\frac{1}{4}$ inch, and $\frac{1}{8}$ inch. Let each child write the result of his measurement on the chalkboard (7 inches, $\frac{14}{2}$ inches, $\frac{28}{4}$ inches, and $\frac{56}{8}$ inches). Ask the class what they notice about the measures. They should see that there are 2 times as many $\frac{1}{2}$ inches, 4 times as many $\frac{1}{4}$ inches, and 8 times as many $\frac{1}{8}$ inches as there are whole inches needed to measure the same length.

Now place the following table on the chalkboard, and ask the pupils to complete it.

1 inch = 2 inches	$\frac{1}{2}$ inch = 4 inches
1 inch = 4 inches	$\frac{1}{2}$ inch = 8 inches
1 inch = 8 inches	$\frac{1}{4}$ inch = 8 inches

Pages 338 through 340

● Page 338 provides practice in interpreting measurements indicated by points on a ruler.

Be sure each pupil has a 12-inch ruler that is graduated to eighths of an inch. Then ask the class how long each part is for the following:

1) An inch segment divided into two parts of the same length ($\frac{1}{2}$ inch).
2) An inch segment divided into four parts of the same length ($\frac{1}{4}$ inch).
3) An inch segment divided into eight parts of the same length ($\frac{1}{8}$ inch).

Have the class locate the $\frac{1}{2}$-inch, $\frac{1}{4}$-inch, and $\frac{1}{8}$-inch marks on their rulers. Also ask, in turn, how many $\frac{1}{2}$-inch, $\frac{1}{4}$-inch, and $\frac{1}{8}$-inch segments there are in a 1-inch segment.

Ask the class how the inch is partitioned, or divided, in the example at the top of page 338. Discuss the statements and questions pertaining to this part of the ruler.

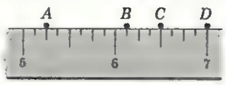
Work the exercises with the class, letting pupils explain why the given points represent the measurements they do.

● Page 339 requires the children to measure distances between pairs of points.

Ask the pupils to locate the 0-inch mark on their rulers (some may find that the 0-inch mark is not indicated, which means that the left end of the ruler may be the 0-mark). Be sure the class realizes that any point on the ruler, not necessarily the 0-inch mark, can be used as a starting point when measuring lengths.

Let the class discuss the example and the first two exercises. When the children agree on the answers, assign the rest of the exercises on page 339 for independent work. Use the results for a class discussion.

For Class Discussion



Part of a ruler is pictured here.

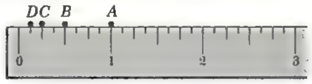
A shows a measurement of $5\frac{1}{4}$ inches from the 0 point of the ruler.

B shows a measurement of $6\frac{1}{8}$ inches from the 0 point. Why? *Because it is located between 6 and 7 at the $\frac{1}{8}$ mark after 6. Answers may vary.*

What measurement is shown by point C? *$6\frac{1}{2}$ inches** point D? *7 inches **

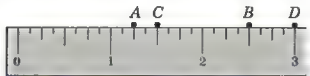
In each exercise, name the measurement from—

1.



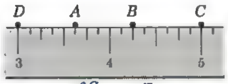
0 to A. *1 inch** 0 to B. *$\frac{1}{2}$ inch**
0 to C. *$\frac{1}{4}$ inch** 0 to D. *$\frac{1}{8}$ inch*

2.



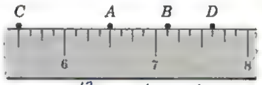
0 to A. *$\frac{5}{4}$ or $1\frac{1}{4}$ inches** 0 to B. *$\frac{5}{2}$ or $2\frac{1}{2}$ inches**
0 to C. *$\frac{3}{2}$ or $1\frac{1}{2}$ inches** 0 to D. *3 inches **

3.



0 to A. *$\frac{29}{8}$ or $3\frac{5}{8}$ inches **
0 to B. *$\frac{17}{4}$ or $4\frac{1}{4}$ inches **
0 to C. *5 inches **
0 to D. *3 inches **

4.



0 to A. *$\frac{13}{2}$ or $6\frac{1}{2}$ inches **
0 to B. *$\frac{57}{8}$ or $7\frac{1}{8}$ inches **
0 to C. *$\frac{11}{2}$ or $5\frac{1}{2}$ inches **
0 to D. *$\frac{61}{8}$ or $7\frac{5}{8}$ inches **

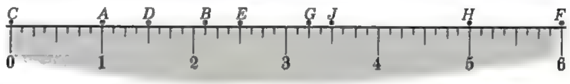
reference page

H-338

** Form of answers may vary.*

Name

Measure the distance between two points.



Example: The distance between points A and B is $\frac{9}{8}$ inches or $1\frac{1}{8}$ inches.

How far is it from—

1. C to point F in inches? *6 inches* in $\frac{1}{2}$ inches? *$\frac{12}{2}$ inches*
2. C to point A in $\frac{1}{4}$ inches? *$\frac{4}{4}$ inch* in $\frac{1}{2}$ inches? *$\frac{2}{2}$ inch*
3. D to point E in inches? *1 inch* in $\frac{1}{2}$ inches? *$\frac{2}{2}$ inch*
4. A to point D in $\frac{1}{2}$ inches? *$\frac{1}{2}$ inch* in $\frac{1}{4}$ inches? *$\frac{2}{4}$ inch*
5. B to point J in $\frac{1}{8}$ inches? *$\frac{11}{8}$ inches or $1\frac{3}{8}$ inches*
6. A to point E in $\frac{1}{2}$ inches? *$\frac{3}{2}$ or $1\frac{1}{2}$ inches* in $\frac{1}{4}$ inches? *$\frac{6}{4}$ or $1\frac{2}{4}$ inches*
7. J to point F in $\frac{1}{2}$ inches? *$\frac{5}{2}$ or $2\frac{1}{2}$ inches* in $\frac{1}{4}$ inches? *$\frac{10}{4}$ or $2\frac{2}{4}$ inches*
8. A to point F in $\frac{1}{2}$ inches? *$\frac{10}{2}$ or $5\frac{0}{2}$ inches* in inches? *5 inches*
9. G to point H in $\frac{1}{4}$ inches? *$\frac{7}{4}$ or $1\frac{3}{4}$ inches* in $\frac{1}{8}$ inches? *$\frac{14}{8}$ or $1\frac{6}{8}$ inches*
10. D to point H in $\frac{1}{4}$ inches? *$\frac{14}{4}$ or $3\frac{2}{4}$ inches* in $\frac{1}{2}$ inches? *$\frac{7}{2}$ or $3\frac{1}{2}$ inches*

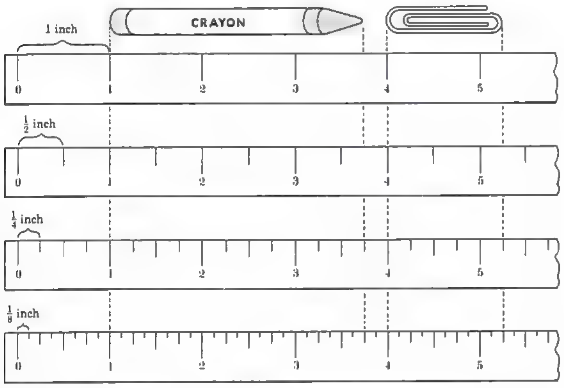
H-339

● Page 340 provides practice in measuring lengths to the nearest whole inch and to the nearest part of an inch. Discuss the idea that a length can be measured better (with less error) when smaller units of measure are used. Ask the class to tell which measurement in each of the following pairs of measurements has the smallest error and why: 3 inches, or $3\frac{1}{2}$ inches; 10 inches, or $9\frac{5}{8}$ inches. The pupils should conclude that whenever a smaller unit of measure is used, a smaller error in the measurement results.

On the chalkboard, draw line segments of the following lengths: $7\frac{1}{2}$ inches, $9\frac{1}{4}$ inches, $11\frac{1}{8}$ inches. Let the pupils measure the lengths of those segments in inches, $\frac{1}{2}$ inches, $\frac{1}{4}$ inches, and $\frac{1}{8}$ inches.

Work and discuss exercises 1 through 4 as a class activity. Then assign the remaining exercises to be completed independently. Discuss the last four exercises when they have been completed.

Use the diagram to answer each question.



1. To the nearest inch, how long is the crayon? 3 inches

2. To the nearest inch, how long is the paper clip? 1 inch

3. To the nearest $\frac{1}{2}$ inch, how long is the crayon? $\frac{5}{2}$ or $2\frac{1}{2}$ inches or $\frac{6}{2}$ or $3\frac{0}{2}$ inches

4. To the nearest $\frac{1}{2}$ inch, how long is the paper clip? $\frac{2}{2}$ or $1\frac{0}{2}$ inch or $\frac{3}{2}$ or $1\frac{1}{2}$ inches

5. To the nearest $\frac{1}{4}$ inch, how long is the crayon? $\frac{11}{4}$ or $2\frac{3}{4}$ inches

6. To the nearest $\frac{1}{4}$ inch, how long is the paper clip? $\frac{5}{4}$ or $1\frac{1}{4}$ inches

7. To the nearest $\frac{1}{8}$ inch, how long is the crayon? $\frac{22}{8}$ or $2\frac{6}{8}$ inches

8. To the nearest $\frac{1}{8}$ inch, how long is the paper clip? $\frac{10}{8}$ or $1\frac{2}{8}$ inches

H-340

Supplemental Experience

■ Show the class six envelopes in which pieces of tagboard measuring between 4 and 12 inches have been placed. Have each envelope marked to indicate the length of the tagboard inside. Call on a pupil to choose one of the envelopes; to note the length indicated; and without using a ruler, to estimate and draw a line segment of that length on the chalkboard. Then he should take the tagboard strip from the envelope and compare it with the line segment.

Continue the activity in this manner. If several children work at the chalkboard at the same time, the activity becomes more challenging and exciting. As the pupils continue with this practice in estimating lengths, they will enjoy seeing how they improve.

UNIT 24
DIVISION REVIEWS COMPUTATION
Pages 341 Through 368

OBJECTIVE

To improve computational skills.

The pupil practices the computation of sums, quotients, differences, and products by using the division algorism and the multiplication check. He is given the opportunity to learn to use the division algorism more efficiently by using fewer partial quotients. He also decides whether one quotient is greater or less than another quotient.

KEY IDEA

Practice.

KEY IDEA

Practice.

Scope

To give pupils more practice in computing quotients.

Fundamentals

Practice exercises should be assigned only as needed. These exercises give the teacher an opportunity to discover the types of errors a pupil makes. After discovering these errors, an effort should be made to provide help for each pupil's particular difficulties. There should be a continued effort to allow for individual differences in computational skill.

The pupils should be allowed to use a variety of partial quotients when they compute quotients using the division algorism. It is enough to expect from most pupils that they be able to successfully use partial quotients. The more capable pupils should be allowed to use shortcuts.

Readiness for Understanding

Knowledge of multiplication.
Understanding of quotient.

Developmental Experiences

► Computation of differences is one of the steps used in the division algorism. This activity provides the pupils with an opportunity to review the computation of differences.

Separate the class into two teams, and write two differences of whole numbers on the chalkboard.

$$\begin{array}{r} 5432 \\ -\ 2465 \\ \hline \end{array} \qquad \begin{array}{r} 7102 \\ -\ 3896 \\ \hline \end{array}$$

Assign one of these differences to each team. Have the first player from each team come to the chalkboard, compute the difference of the ones in his team's exercise, and write the result of his computation in the appropriate place.

$$\begin{array}{r} 2 \\ 54\cancel{3}2 \\ -\ 2465 \\ \hline 7 \end{array} \qquad \begin{array}{r} 9 \\ 7\cancel{10}2 \\ -\ 3896 \\ \hline 6 \end{array}$$

As soon as the first player completes his assignment, he is to give his piece of chalk to the second player on his team. The second player then computes the difference of tens and writes the result in the appropriate place.

$$\begin{array}{r} 32 \\ 5\cancel{4}\cancel{3}2 \\ -\ 2465 \\ \hline 67 \end{array} \qquad \begin{array}{r} 9 \\ 7\cancel{10}2 \\ -\ 3896 \\ \hline 06 \end{array}$$

The third player is to compute and write the difference of hundreds and the fourth player the difference of thousands.

$$\begin{array}{r} 432 \\ \cancel{5}\cancel{4}\cancel{3}2 \\ -\ 2465 \\ \hline 2967 \end{array} \qquad \begin{array}{r} 6\ 9 \\ \cancel{7}\cancel{10}2 \\ -\ 3896 \\ \hline 3206 \end{array}$$

Have the fifth member of each team explain his team's computation step by step. For example, a pupil may explain the computation of $7102 - 3896$ in the following way.

The 6 in 3206 refers to the difference of the ones; this is $(10 - 6) + 2$.

The 0 in 3206 refers to the difference of the tens; this is 9 tens − 9 tens, or 0 tens.

The 2 in 3206 refers to the difference of the hundreds; this is 10 hundreds − 8 hundreds, or 2 hundreds.

The 3 in 3206 refers to the difference of the thousands; this is 6 thousands − 3 thousands, or 3 thousands.

The players who have participated thus far in the activity may earn points for their team in the following ways: one point is earned by each team member who correctly computes his part of the difference; one point is earned by the first team to complete its computation correctly; and one point is earned by the team member who correctly describes the computation.

Continue the activity, starting with the sixth player on each team. The following differences are suggested:

$$\begin{array}{r} 682 \\ -\ 459 \\ \hline \end{array} \qquad \begin{array}{r} 905 \\ -\ 672 \\ \hline \end{array} \qquad \begin{array}{r} 1714 \\ -\ \ 962 \\ \hline \end{array}$$

$$\begin{array}{r} 5620 \\ -\ 3148 \\ \hline \end{array} \qquad \begin{array}{r} 1193 \\ -\ \ 628 \\ \hline \end{array} \qquad \begin{array}{r} 4009 \\ -\ 1234 \\ \hline \end{array}$$

At the end of the activity, let each team total its points.

The one with more points is the winner.

▶ In computing a product, the pupils must often compute the sum of partial products. In computing a quotient, they must often compute the sum of partial quotients. The following activity has been included to give practice in computing sums of whole numbers.

Separate the class into two teams. Then write on the chalkboard the following computation:

$$\begin{array}{r} 4679 \\ +\,2583 \\ \hline 6000 \\ 110 \\ 160 \\ \underline{12} \end{array}$$

Explain that there are errors in the computation of some of the partial sums. Call on a pupil from one of the teams to point out and correct any errors that he sees and to add the correct partial sums.

$$\begin{array}{r} 4679 \\ +\,2583 \\ \hline 6000 \\ 1100 \\ 150 \\ 12 \\ \hline 7262 \end{array}$$

Continue the game with other examples having incorrect partial sums. Ask different pupils, alternately from one team and then the other, to correct the errors and then add. Continue until everyone has participated in the game.

The players earn points for their team as follows: one point for finding all errors and correcting them, and another point for correctly adding the partial sums. At the conclusion of the activity have each team total its points. The one with the greater number wins.

▶ Review multiplication of multiples of 10 and 100. Write the following exercises on the chalkboard:

$50 \times 40 = \_\_\_$
$2 \times 90 = \_\_\_$
$70 \times 600 = \_\_\_$

Ask three children to compute these products at the board; then ask each child to explain his computation. The following explanations may be given:

5 × 4 is 20, and tens × tens is ten tens, or hundreds. Therefore 50 × 40 is 20 hundreds, or 2 thousands (2000).

2 × 9 tens is 18 tens, so 2 × 90 is 180.

7 × 6 is 42, and tens × hundreds is ten hundreds, or thousands. Therefore 70 × 600 is 42 thousands (42,000).

Continue the activity with other products involving multiples of 10. The following exercises are some of the many that could be used.

$20 \times 20 = \_\_\_$	$\_\_\_ \times 20 = 4000$
$50 \times 90 = \_\_\_$	$\_\_\_ \times 3 = 900$
$\_\_\_ \times 40 = 80$	$\_\_\_ \times 40 = 8000$
$60 \times \_\_\_ = 600$	$6 \times \_\_\_ = 600$
$30 \times 300 = \_\_\_$	$30 \times \_\_\_ = 600$
$70 \times 80 = \_\_\_$	$40 \times \_\_\_ = 4000$
$\_\_\_ \times 7 = 700$	$\_\_\_ \times 70 = 700$

▶ Write the exercise 397 × 60 on the chalkboard. Call on a pupil to describe orally how he would compute this product. As he does this, let another pupil write the results on the chalkboard. For example, the children may work together to do the computation in the following way:

6 tens × 7 is 42 tens. Write 2 tens and remember 4 hundreds.

$397 \times 60 = \underline{\quad 20}$

6 tens × 9 tens is 54 ten tens, or 54 hundreds. 54 hundreds + 4 hundreds is 58 hundreds. Write 8 hundreds and remember 5 thousands.

$397 \times 60 = \underline{\quad 820}$

6 tens × 3 hundreds is 18 ten hundreds, or 18 thousands. 18 thousands + 5 thousands is 23 thousands. Write 23 thousands.

$397 \times 60 = \underline{23{,}820}$

Continue this way, letting pairs of children describe and write the computation of such products as the following:

$734 \times 500 = \_\_\_$	$358 \times 70 = \_\_\_$
$67 \times 400 = \_\_\_$	$96 \times 900 = \_\_\_$
$298 \times 60 = \_\_\_$	$24 \times 80 = \_\_\_$

▶ Write a division exercise on the chalkboard. Direct the pupils to compute the quotient at their seats. After everyone has completed the work, ask several pupils in turn to come to the chalkboard and show how they computed. If the exercise was 688 ÷ 16, the pupils may have computed in the following ways:

$$\begin{array}{r|l} 16\overline{)\,688} & \\ \underline{160} & 10 \\ 528 & \\ \underline{160} & 10 \\ 368 & \\ \underline{160} & 10 \\ 208 & \\ \underline{160} & 10 \\ 48 & \\ \underline{48} & \underline{\ 3} \\ 0 & 43 \end{array} \qquad \begin{array}{r|l} 16\overline{)\,688} & \\ \underline{320} & 20 \\ 368 & \\ \underline{320} & 20 \\ 48 & \\ \underline{48} & \underline{\ 3} \\ 0 & 43 \end{array}$$

Discuss the work that is done on the chalkboard. Then ask for volunteers to suggest and demonstrate ways of checking the computation. Some possible checks are shown below.

1.
$$\begin{array}{r} 16 \\ \times\,43 \\ \hline 18 \\ 30 \\ 240 \\ 400 \\ \hline 688 \end{array}$$

2.
$$\begin{array}{r} 16 \\ \times\,40 \\ \hline 640 \end{array} \qquad \begin{array}{r} 16 \\ \times\ \ 3 \\ \hline 48 \end{array} \qquad \begin{array}{r} 640 \\ +\ 48 \\ \hline 688 \end{array}$$

3.

$$\begin{array}{r} 20 \\ \times 16 \\ \hline 120 \\ 200 \\ \hline 320 \end{array} \qquad \begin{array}{r} 20 \\ \times\ 16 \\ \hline 120 \\ 200 \\ \hline 320 \end{array} \qquad \begin{array}{r} 16 \\ \times\ 3 \\ \hline 48 \end{array}$$

$$\begin{array}{r} 320 \\ 320 \\ +\ 48 \\ \hline 688 \end{array}$$

4.

$$\begin{array}{r} 16 \\ \times\ 30 \\ \hline 480 \end{array} \qquad \begin{array}{r} 16 \\ \times\ 10 \\ \hline 160 \end{array} \qquad \begin{array}{r} 16 \\ \times\ 3 \\ \hline 48 \end{array}$$

$$\begin{array}{r} 480 \\ 160 \\ +\ 48 \\ \hline 688 \end{array}$$

Since multiplication is distributive over addition, all these checks are acceptable. And since multiplication is commutative, it makes no difference which factor is written first.

In checking division, the pupils should be allowed to employ any shortcut they can use successfully. The following is another shortcut:

$$\begin{array}{r} 16 \\ \times\ 43 \\ \hline 688 \end{array}$$

3 × 6 is 18. Write the 8 and remember 1 ten.

3 × 1 ten is 3 tens.
4 tens × 6 is 24 tens.
1 ten (remembered) + 3 tens + 24 tens is 28 tens. Write 8 tens and remember 2 hundreds.

4 tens × 1 ten is 4 ten tens, or 4 hundreds.
2 hundreds (remembered) + 4 hundreds is 6 hundreds. Write 6 hundreds.

Pages 341 through 368

● The following general procedure is suggested for the pages in this unit.

With the class, work and discuss the example at the top of the page, if there is one. Also discuss several of the exercises so that the pupils understand completely what is to be done.

Assign the computational exercises as independent work. Adjust the length of the assignments to the pupils' abilities. Do not insist that the pupils use a shortcut method in computing products or that everyone use the same number of steps in computing quotients.

When a given assignment has been completed, ask individuals to describe how they computed specific exercises. This procedure can be varied by dividing the class into three or more teams to compete for accuracy. Let the teams take turns choosing a row of exercises and computing them. When a team has finished its row of exercises, discuss the results. Then let one member of the team total the team's score, 1 point for each correct computation. The team with the most points wins.

● After the pupils have completed pages 341 through 343, it is possible that several of them will see a connection between the addition and subtraction exercises on page 343. If someone does see a connection, let him explain to the class what he sees. If none of the pupils sees a connection, the matter should not be discussed.

Name

UNIT 24 DIVISION REVIEWS COMPUTATION

For Class Discussion

What is a mixed fraction for 267 ÷ 15?

$$\begin{array}{r|l} 15\,\overline{)\,267} & \\ 150 & 10 \\ \hline 117 & \\ 105 & 7 \\ \hline 12 & \\ 12 & \frac{12}{15} \\ \hline 0 & 17 + \frac{12}{15} \end{array}$$

$\frac{267}{15} = 17\frac{12}{15}$

Check: Does $15 \times 17\frac{12}{15} = 267$?

$$\begin{array}{r} 17\frac{12}{15} \\ \times\ 15 \\ \hline 12 \\ 35 \\ 50 \\ 70 \\ 100 \\ \hline 267 \end{array}$$

Compute a mixed fraction. Check. Partial quotients may vary.

1. 86 ÷ 15 $5\frac{11}{15}$
2. 102 ÷ 9 $11\frac{3}{9}$
3. 186 ÷ 11 $16\frac{10}{11}$
4. 256 ÷ 19 $13\frac{9}{19}$
5. 278 ÷ 12 $23\frac{2}{12}$
6. 341 ÷ 23 $14\frac{19}{23}$

H-341

Compute a mixed fraction. Check. *Partial quotients may vary*

1. 372 ÷ 12 $31\frac{0}{12}$

2. 375 ÷ 41 $9\frac{6}{41}$

3. 462 ÷ 37 $12\frac{18}{37}$

4. 489 ÷ 15 $32\frac{9}{15}$

5. 973 ÷ 24 $40\frac{13}{24}$

6. 878 ÷ 26 $33\frac{20}{26}$

7. 177 ÷ 32 $5\frac{17}{32}$

8. 677 ÷ 32 $21\frac{5}{32}$

H-342

Name

Compute. Write the answer as a mixed fraction. *Partial quotients may vary.*

1. $\frac{677}{32} + \frac{177}{32}$ $\frac{854}{32} = 26\frac{22}{32}$

2. $\frac{854}{32} - \frac{177}{32}$ $\frac{677}{32} = 21\frac{5}{32}$

3. $\frac{854}{32} - \frac{677}{32}$ $\frac{177}{32} = 5\frac{17}{32}$

4. $\frac{458}{27} - \frac{316}{27}$ $\frac{142}{27} = 5\frac{7}{27}$

5. $\frac{458}{27} - \frac{142}{27}$ $\frac{316}{27} = 11\frac{19}{27}$

6. $\frac{61}{18} + \frac{495}{18}$ $\frac{556}{18} = 30\frac{16}{18}$

7. $\frac{212}{106} + \frac{503}{106}$ $\frac{715}{106} = 6\frac{76}{106}$

8. $\frac{715}{106} - \frac{503}{106}$ $\frac{212}{106} = 2\frac{0}{106}$

9. $\frac{104}{95} + \frac{787}{95}$ $\frac{891}{95} = 9\frac{36}{95}$

H-343

● Although it is not the purpose of this unit to have the children look for patterns, such an awareness may be a by-product of their work. For example, a child may observe that in each row of exercises on pages 344 and 345, the divisors are the same and the dividends become greater from left to right. Each quotient may thus be greater than the one before it. If the pupils do not see patterns on their own, the teacher should not call attention to them.

● In checking their mixed fractions on pages 346 through 350, the pupils should be allowed to use any shortcuts they prefer.

● The story questions in this unit, such as those on pages 348 through 352, should be used as a class activity. The pupils have worked with enough story exercises to be able to see the number structure in each of these stories. The computation of mixed fractions on these pages should be assigned according to the pupils' abilities.

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. 14)253 $18\frac{1}{14}$

2. 14)462 $33\frac{0}{14}$

3. 14)745 $53\frac{3}{14}$

4. 27)142 $5\frac{7}{27}$

5. 27)316 $11\frac{19}{27}$

6. 27)458 $16\frac{26}{27}$

7. 54)578 $10\frac{38}{54}$

8. 54)690 $12\frac{42}{54}$

9. 54)829 $15\frac{19}{54}$

H-344

Name

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. $76\overline{)632}$ $8\frac{24}{76}$
2. $76\overline{)758}$ $9\frac{74}{76}$
3. $76\overline{)930}$ $12\frac{18}{76}$
4. $95\overline{)104}$ $1\frac{9}{95}$
5. $95\overline{)787}$ $8\frac{27}{95}$
6. $95\overline{)891}$ $9\frac{36}{95}$
7. $106\overline{)212}$ $2\frac{0}{106}$
8. $106\overline{)503}$ $4\frac{79}{106}$
9. $106\overline{)715}$ $6\frac{79}{106}$

H-345

Name

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. $\frac{6284}{17}$ $369\frac{11}{17}$
2. $\frac{4175}{13}$ $321\frac{2}{13}$
3. $\frac{2445}{29}$ $84\frac{9}{29}$
4. $\frac{4976}{16}$ $311\frac{0}{16}$
5. $\frac{5575}{31}$ $179\frac{26}{31}$
6. $\frac{3146}{29}$ $108\frac{14}{29}$
7. $\frac{5384}{37}$ $145\frac{19}{37}$
8. $\frac{3860}{42}$ $91\frac{38}{42}$

H-347

What is a mixed fraction for $\frac{6269}{25}$?

$$\begin{array}{r|l} 25\overline{)6269} & \\ \underline{5000} & 200 \\ 1269 & \\ \underline{1250} & 50 \\ 19 & \\ 19 & \frac{19}{25} \\ \hline 0 & 250 + \frac{19}{25} \end{array}$$

$\frac{6269}{25} = 250\frac{19}{25}$

Check:
Does $25 \times 250\frac{19}{25} = 6269$?

$$\begin{array}{r} 250\frac{19}{25} \\ \times\ \ 25 \\ \hline 19 \\ 250 \\ 1000 \\ 1000 \\ \underline{4000} \\ 6269 \end{array}$$

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. $\frac{9801}{48}$ $204\frac{9}{48}$
2. $\frac{7361}{24}$ $306\frac{17}{24}$
3. $\frac{3724}{18}$ $206\frac{16}{18}$
4. $\frac{5147}{29}$ $177\frac{14}{29}$
5. $\frac{6179}{37}$ $167\frac{0}{37}$
6. $\frac{3560}{21}$ $169\frac{11}{21}$

H-346

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. $\frac{6806}{41}$ $166\frac{0}{41}$
2. $\frac{3749}{32}$ $117\frac{5}{32}$
3. $\frac{2897}{704}$ $4\frac{81}{704}$
4. $\frac{3586}{290}$ $12\frac{106}{290}$
5. $\frac{8990}{555}$ $16\frac{110}{555}$
6. $\frac{7500}{83}$ $90\frac{30}{83}$
7. A newspaper boy delivers the same number of papers each day. He delivered 420 papers in 7 days. How many did he deliver each day? 60

H-348

Name

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. $\frac{6040}{7}$ $862\frac{6}{7}$

2. $\frac{8417}{9}$ $935\frac{2}{9}$

3. $\frac{9741}{17}$ $573\frac{0}{17}$

4. $\frac{6535}{64}$ $102\frac{7}{64}$

5. $\frac{999}{8}$ $124\frac{7}{8}$

6. $\frac{7995}{57}$ $140\frac{15}{57}$

7. $\frac{3726}{39}$ $95\frac{21}{39}$

8. $\frac{6300}{77}$ $81\frac{63}{77}$

9. Dave cut an 8-foot board into 3 equal lengths. How long is each piece? *$2\frac{2}{3}$ feet*

H-349

Compute a mixed fraction. Check.

1. $\frac{9580}{11}$ $870\frac{10}{11}$

2. $\frac{5224}{8}$ $653\frac{0}{8}$

3. $\frac{2450}{348}$ $7\frac{14}{348}$

4. $\frac{8600}{537}$ $16\frac{8}{537}$

5. $\frac{7439}{179}$ $41\frac{100}{179}$

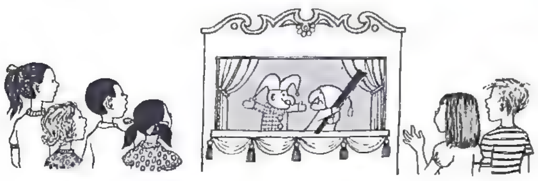
6. $\frac{2987}{490}$ $6\frac{47}{490}$

7. $\frac{9870}{318}$ $36\frac{12}{318}$

8. $\frac{12420}{414}$ $30\frac{0}{414}$

9. There are 840 trees in rows of 28. How many rows are there? *30 rows*

10. Kim paid 65¢ for 5 tickets to a puppet show. How much did she pay for one ticket? *\$.13 for one ticket*



H-350

● Another opportunity to see patterns is provided on pages 351 and 352. Those pupils who are alert to patterns may observe that in each row of exercises on page 351 the dividends are the same and the divisor becomes greater from left to right. The same pattern occurs in exercises 1 through 3, 4 through 6, and 7 through 9 on page 352. The result may be that each quotient is less than the one before it. If some child comments about this pattern, have him explain his findings to the class. Otherwise make no comment on the subject.

● After pages 353 through 356 have been completed, some of the pupils may see a connection between the addition and subtraction exercises and some of the division exercises. Ask the pupils who see a connection to tell the class about it.

Name

Compute a mixed fraction. *Partial quotients may vary.*

1. $\frac{652}{27}$ $24\frac{4}{27}$ 2. $\frac{652}{53}$ $12\frac{16}{53}$ 3. $\frac{652}{71}$ $9\frac{13}{71}$

4. $\frac{804}{17}$ $47\frac{5}{17}$ 5. $\frac{804}{36}$ $22\frac{12}{36}$ 6. $\frac{804}{67}$ $12\frac{0}{67}$

7. $\frac{739}{9}$ $82\frac{1}{9}$ 8. $\frac{739}{47}$ $15\frac{34}{47}$ 9. $\frac{739}{102}$ $7\frac{25}{102}$

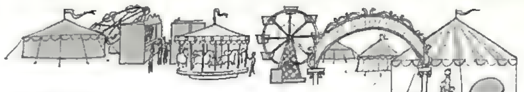
10. Mary has 816 books to pack. If she can pack 24 books in a box, how many boxes are needed? *34 boxes*

H-351

Compute a mixed fraction. *Partial quotients may vary.*

1. $\frac{5830}{231}$ $25\frac{55}{231}$
2. $\frac{5830}{376}$ $15\frac{190}{376}$
3. $\frac{5830}{576}$ $10\frac{70}{576}$
4. $\frac{8476}{99}$ $85\frac{61}{99}$
5. $\frac{8476}{213}$ $39\frac{169}{213}$
6. $\frac{8476}{447}$ $18\frac{430}{447}$
7. $\frac{7777}{103}$ $75\frac{52}{103}$
8. $\frac{7777}{111}$ $70\frac{7}{111}$
9. $\frac{7777}{152}$ $51\frac{25}{152}$
10. A library has 738 books. Each shelf holds 23 books. How many shelves are needed to hold these books? *33 shelves*

H-352



Compute a mixed fraction.

1. $\frac{4908}{36}$ $136\frac{12}{36}$
2. $\frac{9909}{90}$ $110\frac{9}{90}$
3. $\frac{2619}{307}$ $8\frac{163}{307}$
4. $\frac{7437}{37}$ $201\frac{0}{37}$
5. $\frac{8437}{3}$ $2812\frac{1}{3}$
6. $\frac{6804}{45}$ $151\frac{9}{45}$
7. $\frac{2005}{6}$ $334\frac{1}{6}$
8. $\frac{3842}{241}$ $15\frac{227}{241}$
9. $\frac{7962}{4}$ $1990\frac{2}{4}$
10. $\frac{9081}{70}$ $129\frac{51}{70}$
11. Tom paid \$5.76 for 48 carnival tickets. How much did each ticket cost? *\$.12*

H-354

Name

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. $\frac{9801}{9}$ $1089\frac{0}{9}$
2. $\frac{7654}{20}$ $382\frac{14}{20}$
3. $\frac{8341}{6}$ $1390\frac{1}{6}$
4. $\frac{5002}{23}$ $217\frac{11}{23}$
5. $\frac{6781}{7}$ $968\frac{5}{7}$
6. $\frac{4802}{70}$ $68\frac{42}{70}$
7. $\frac{3560}{5}$ $712\frac{0}{5}$
8. $\frac{8650}{600}$ $14\frac{250}{600}$
9. $\frac{9038}{400}$ $22\frac{238}{400}$
10. $\frac{1798}{203}$ $8\frac{174}{203}$

H-353

Name

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. $643 \div 48$ $13\frac{19}{48}$
2. $472 \div 48$ $9\frac{40}{48}$
3. $268 \div 48$ $5\frac{28}{48}$
4. $183 \div 71$ $2\frac{41}{71}$
5. $375 \div 71$ $5\frac{20}{71}$
6. $558 \div 71$ $7\frac{61}{71}$
7. $382 \div 18$ $21\frac{4}{18}$
8. $382 \div 15$ $25\frac{7}{15}$
9. $382 \div 7$ $54\frac{4}{7}$

H-355

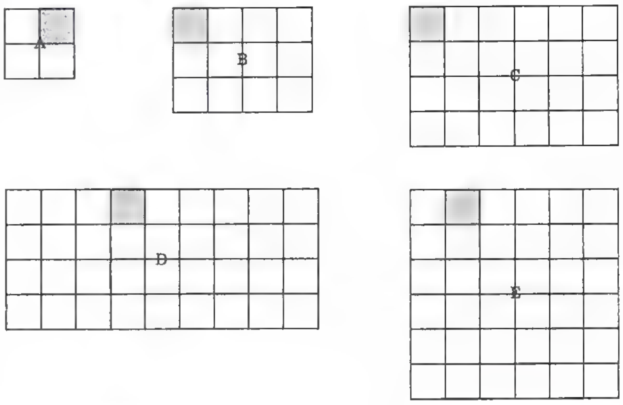
Compute a mixed fraction. *Partial quotients may vary.*

1. $27\overline{)4563}$ $169\frac{0}{27}$
2. $41\overline{)4563}$ $111\frac{12}{41}$
3. $58\overline{)4563}$ $78\frac{39}{58}$
4. $305\overline{)386}$ $1\frac{81}{305}$
5. $305\overline{)8894}$ $29\frac{49}{305}$
6. $305\overline{)9280}$ $30\frac{130}{305}$
7. $27\overline{)5678}$ $210\frac{8}{27}$
8. $27\overline{)2437}$ $90\frac{7}{27}$
9. $27\overline{)3241}$ $120\frac{1}{27}$
10. $\frac{183}{71}+\frac{375}{71}$ $7\frac{61}{71}$
11. $\frac{5678}{27}-\frac{2437}{27}$ $120\frac{1}{27}$
12. $\frac{9280}{305}-\frac{8894}{305}$ $1\frac{81}{305}$

H-356

Name

Compare the areas.



If the area of  is 3 square feet, then what is the area of—

A? *12 square feet*
B? *36 square feet*
C? *72 square feet*
D? *108 square feet*
E? *108 square feet*

If the area of E is 25 square feet, then what is the area of—

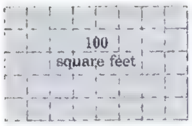
A? *$2\frac{7}{9}$ square feet*
B? *$8\frac{1}{3}$ square feet*
C? *$16\frac{2}{3}$ square feet*
D? *25 square feet*

H-357

● Pages 357 and 358 provide an opportunity for pupils to compare areas and use their knowledge of multiplication and division.

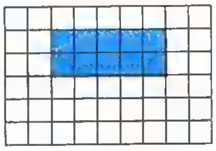
In the first exercise on page 357, the square unit shown represents 3 square feet. Let pupils suggest ways to compute the area of rectangle A. Then let them work independently to compute the area of each of the other rectangles. As the second exercise is discussed, pupils may suggest that E is 9 times as large as A, so the area of A is $\frac{25}{9}$ or $2\frac{7}{9}$ square feet. Let pupils complete the exercise independently and then discuss their results.

Assign page 358 for independent work. Let pupils explain how they showed areas of the requested sizes.

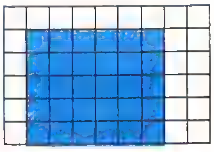


Show these areas: *The shape of the pictures may vary.*

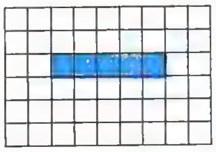
1. 25 square feet



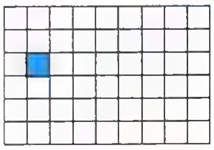
2. 75 square feet



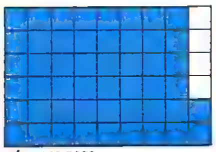
3. $\frac{100}{8}$ square feet



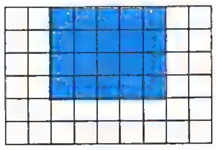
4. $\frac{100}{40}$ square feet



5. 125 square feet



6. $\frac{1000}{20}$ square feet



reference page

H-358

● A discussion of the exercises at the top of page 359 may help some of the pupils to decrease the number of steps they use in working with the division algorithm. After pages 359 and 360 have been completed, you may decide to have some of the pupils work specific exercises a second time so you can encourage them to combine some of their steps.

● The exercises at the top of page 361 require only that the pupils check the given computation to decide whether they are correct. They need not compute the correct mixed fraction when they find an error. It is hoped, however, that in checking these computations, the pupils will learn to uncover and correct their own errors. Additional practice is provided in exercises 4 through 9.

Name

Complete.

1. $16\overline{)742}$

742	
160	10
582	
160	10
422	
160	10
262	
160	10
102	
32	2
70	
32	2
38	
32	2
6	
6	$\frac{6}{16}$
0	$46\frac{6}{16}$

2. $16\overline{)742}$

742	
320	20
422	
320	20
102	
80	5
22	
16	1
6	
6	$\frac{6}{16}$
0	$46\frac{6}{16}$

3. $16\overline{)742}$

742	
640	40
102	
96	6
6	
6	$\frac{6}{16}$
0	$46\frac{6}{16}$

Compute a mixed fraction. *Partial quotients may vary.*

4. $378 \div 7$ $54\frac{0}{7}$

5. $945 \div 30$ $31\frac{15}{30}$

6. $472 \div 9$ $52\frac{4}{9}$

7. $4720 \div 90$ $52\frac{40}{90}$

8. $513 \div 42$ $12\frac{9}{42}$

9. $5130 \div 4200$ $1\frac{930}{4200}$

H-359

Compute a mixed fraction. *Partial quotients may vary.*

1. $3625 \div 711$ $5\frac{70}{711}$

2. $1876 \div 18$ $104\frac{4}{18}$

3. $692 \div 21$ $32\frac{20}{21}$

4. $3501 \div 901$ $3\frac{798}{901}$

5. $8943 \div 41$ $218\frac{5}{41}$

6. $23418 \div 7$ $3345\frac{3}{7}$

7. $82973 \div 82$ $1011\frac{71}{82}$

8. $34999 \div 341$ $102\frac{217}{341}$

9. $47803 \div 20$ $2390\frac{3}{20}$

10. $60000 \div 205$ $292\frac{140}{205}$

11. $88888 \div 7$ $12698\frac{2}{7}$

H-360

Name

Is the computation correct? Check by multiplication.

1. $83\overline{)7891}$

7891	
7470	90
421	
415	5
6	
6	$\frac{6}{83}$
0	$95 + \frac{6}{83}$

$95\frac{6}{83}$
$\times 83$
6
285
7600
7891

2. $98\overline{)88453}$

88453	
88200	900
253	
196	2
57	
57	$\frac{57}{98}$
0	$902 + \frac{57}{98}$

$902\frac{57}{98}$
$\times 98$
57
7216
81180
88453

3. $17\overline{)4685}$

4685	
2900	200
1785	
1700	100
85	
85	5
0	305

305
$\times 17$
2135
3050
5185

Compute a mixed fraction. Check. *Partial quotients may vary.*

4. $\frac{3475}{34}$ $102\frac{7}{34}$

5. $\frac{1010}{72}$ $14\frac{2}{72}$

6. $\frac{2397}{82}$ $29\frac{19}{82}$

7. $\frac{3547}{34}$ $104\frac{11}{34}$

8. $\frac{1050}{72}$ $14\frac{42}{72}$

9. $\frac{1902}{82}$ $23\frac{16}{86}$

H-361

● Page 362 may be assigned for independent work. You may want to vary this assignment by having the pupils work in pairs, letting each pupil check his partner's computation.

● As they work the exercises on page 363, some of the pupils may observe these patterns: in exercises 1 through 6 the divisor is 37; in exercises 7 through 12 the divisor is 18; in exercises 1 through 6, and again in exercises 7 through 12, the dividends (and possibly the quotients) increase from problem to problem. It is important that any comment about these patterns come from the pupils rather than the teacher.

Discuss the pupils' answers to exercises 13 and 14.

● The exercises at the top of page 364 require only that the pupils check the given computation of each exercise to decide whether it is correct.

If a child comments about a pattern in exercises 4 through 6, have him explain his findings to the class.

● In their work on pages 365 and 366, some pupils may observe that when the dividend and divisor are both multiplied by the same number, the quotient remains the same. On page 365, this pattern can be seen in exercises 6, 7, and 8, but it should not be discussed unless the pupils make the observation themselves.

Name

Compute a mixed fraction. *Partial quotients may vary.*

1. $926 \div 37$ $25\frac{1}{37}$
2. $1042 \div 37$ $28\frac{6}{37}$
3. $1067 \div 37$ $28\frac{31}{37}$
4. $1383 \div 37$ $37\frac{14}{37}$
5. $2579 \div 37$ $69\frac{26}{37}$
6. $25790 \div 37$ $697\frac{1}{37}$
7. $342 \div 18$ $19\frac{0}{18}$
8. $789 \div 18$ $43\frac{15}{18}$
9. $1254 \div 18$ $69\frac{12}{18}$
10. $8976 \div 18$ $498\frac{12}{18}$
11. $14000 \div 18$ $777\frac{14}{18}$
12. $18400 \div 18$ $1022\frac{4}{18}$

13. Ann has 70 photos to put into her album. She can put 16 photos on each page. How many pages can she fill? *4 pages*

14. There are 100 pins in a box. Mrs. Bell needs 250 pins for a class project. How many boxes of pins does she need? *3 boxes*

H-363

Compute a mixed fraction. *Partial quotients may vary.*

1. $\frac{112}{34}$ $3\frac{10}{34}$
2. $\frac{3503}{72}$ $48\frac{47}{72}$
3. $\frac{495}{82}$ $6\frac{3}{82}$
4. $\frac{7134}{34}$ $209\frac{28}{34}$
5. $\frac{1443}{72}$ $20\frac{3}{72}$
6. $\frac{4299}{82}$ $52\frac{35}{82}$
7. $\frac{13605}{32}$ $425\frac{5}{32}$
8. $\frac{42108}{58}$ $726\frac{0}{58}$
9. $\frac{17369}{40}$ $434\frac{9}{40}$

10. There are 938 pupils in Central School. How many classes of 29 pupils can be formed? *32 classes*

H-362

Is the computation correct? Check by multiplication.

1. $57\overline{)3975}$
   3420 | 60
   555
   399 | 7
   56
   56
   0 | $67 + \frac{56}{57}$

   $67\frac{56}{57}$; $\times 57$; 56; 469; 3350; 3875; $\frac{56}{57}$

2. $84\overline{)2974}$
   1680 | 20
   1294
   1252 | 3
   42
   42
   0 | $23 + \frac{42}{84}$

   $23\frac{42}{84}$; $\times 84$; 42; 92; 1840; 1974; $\frac{42}{84}$

3. $362\overline{)17143}$
   14480 | 40
   2763
   2534 | 7
   229
   229
   0 | $47 + \frac{229}{362}$

   $47\frac{229}{362}$; $\times 362$; 229; 94; 2820; 14100; 17243; $\frac{229}{362}$

Compute a mixed fraction. *Partial quotients may vary.*

4. $18\overline{)3890}$
   3600 | 200
   290
   180 | 10
   110
   108 | 6
   2
   2 | $\frac{2}{18}$
   0 | $216\frac{2}{18}$

5. $18\overline{)5006}$
   3600 | 200
   1406
   1260 | 70
   146
   144 | 8
   2
   2 | $\frac{2}{18}$
   0 | $278\frac{2}{18}$

6. $18\overline{)6809}$
   5400 | 300
   1409
   1260 | 70
   149
   126 | 7
   23
   18 | 1
   5
   5 | $\frac{5}{18}$
   0 | $378\frac{5}{18}$

H-364

● Those pupils who have developed an awareness of patterns will need to do very little computation in working the exercises on page 367. Make sure the pupils understand the procedure to be followed; then assign the exercises for independent work. Point out that ≠ means "does not equal." When everyone has completed the assignment, ask those who finished first to explain how they were able to complete the assignment so quickly. This will give the pupils who have learned to look for patterns an opportunity to share their findings with the class.

Name

Complete. Check.

1. 204) 69453

69453		
40800	200	
28653		
20400	100	
8253		
8160	40	
93		
93	$\frac{93}{204}$	
0	$340\frac{93}{204}$	

Check: $340\frac{93}{204}$; 340 × 204 = 1360 + 68000 = 68360; + 93 = 69453

2. 315) 84536

84536		
63000	200	
21536		
18900	60	
2636		
2520	8	
116		
116	$\frac{116}{315}$	
0	$268\frac{116}{315}$	

Check: $268\frac{116}{315}$; 268 × 315 = 1340 + 2680 + 80400; + 116 = 84536

Compute a mixed fraction. *Partial quotients may vary.*

3. $\frac{384}{17}$ $22\frac{10}{17}$

4. $\frac{628}{25}$ $25\frac{3}{25}$

5. $\frac{628}{225}$ $2\frac{178}{225}$

6. $\frac{3900}{50}$ $78\frac{0}{50}$

7. $\frac{39000}{500}$ $78\frac{0}{500}$

8. $\frac{390}{5}$ $78\frac{0}{5}$

H-365

Compute a mixed fraction. Check. *Partial quotients may vary.*

1. $\frac{1616}{404}$ $4\frac{0}{404}$

2. $\frac{16160}{4040}$ $4\frac{0}{4040}$

3. $\frac{7557}{25}$ $302\frac{7}{25}$

4. $\frac{808}{202}$ $4\frac{0}{202}$

5. $\frac{36515}{72}$ $507\frac{11}{72}$

6. $\frac{84953}{63}$ $1348\frac{29}{63}$

H-366

Name

Complete each sentence using = or ≠. If ≠ is used, write "greater" or "less" after the sentence to tell whether the first number is greater or less than the second.

Example: 16 ÷ 8 = 8 ÷ 4
16 ÷ 14 ≠ 28 ÷ 14 (less)
16 ÷ 4 ≠ 12 ÷ 4 (greater)

1. 93 ÷ 2 ≠ 93 ÷ 3 *greater*
2. 176 ÷ 27 ≠ 203 ÷ 27 *less*
3. 168 ÷ 20 = 1680 ÷ 200
4. 27 ÷ 27 = 127 ÷ 127
5. 2891 ÷ 95 ≠ 2897 ÷ 95 *less*
6. 3210 ÷ 80 = 321 ÷ 8
7. 2891 ÷ 95 ≠ 2800 ÷ 95 *greater*
8. 267 ÷ 103 ≠ 267 ÷ 82 *less*
9. 93 ÷ 1 ≠ 93 ÷ 93 *greater*
10. 5843 ÷ 43 ≠ 5843 ÷ 86 *greater*
11. 0 ÷ 384 = 0 ÷ 21
12. 54 ÷ 2 = 108 ÷ 4
13. 50 ÷ 3 ≠ 100 ÷ 12 *greater*
14. 484 ÷ 4 ≠ 968 ÷ 20 *greater*

Compute and check.

15. In the supermarket, Ken filled bags with 12 onions in each bag. How many bags did he fill from a pile of 996 onions? *83 bags*

reference page

H-367

● Additional experience in seeing the number structure in story exercises is provided on page 368. Assign the exercises to be completed independently. Upon completion of the assignment, discuss the pupils' answers.

Answer each question.

1. Dave has 773 bottles of root beer to put into cases. Each case holds 24 bottles. How many cases can he fill? 32 cases

2. Dave made a display with 145 cans of peas. Each can weighed 2 pounds. How much did the display weigh? 290 pounds

3. On a shelf there are 24 cans of peas, 15 cans of carrots, 17 cans of beans, and 11 cans of beets. How many cans are on the shelf? 67 cans

4. A clerk at a bank has 928 pennies to put into rolls. A roll is 50 pennies. How many rolls can he make? 18 rolls

5. The area of a field is 52,834 square feet. When the field is divided into 4 equal parts, what is the area of each part? $13{,}208\frac{2}{4}$ sq. ft.

H-368

Supplemental Experience

■ The pupils may enjoy using an interesting shortcut to find the remainder when the divisor is 9. Consider the following example.

9) 1234	
900	100
334	
270	30
64	
63	7
1	137

The remainder (1) can be found by adding the digits of the dividend $(1 + 2 + 3 + 4 = 10)$ and then adding the digits of the computed sum $(1 + 0 = 1)$. Sometimes this process must be continued in order to reach a sum of 9 or less.

Tell the pupils to use this shortcut to find the remainders in the following exercises. Then have them check their answers by using the division algorism.

9) 3475 The remainder is 1.
$(3 + 4 + 7 + 5 = 19; 1 + 9 = 10; 1 + 0 = 1)$

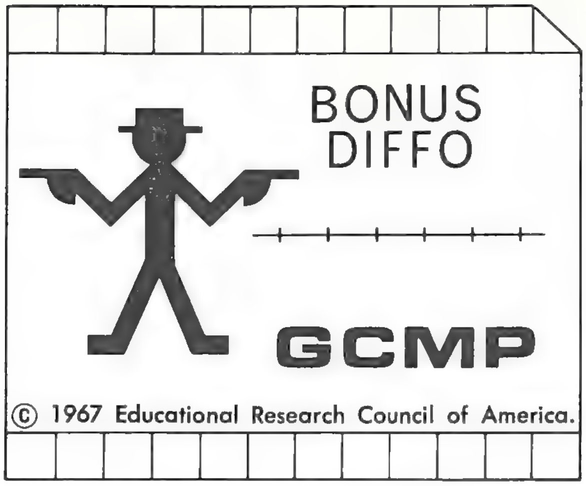
9) 2579 The remainder is 5.
$(2 + 5 + 7 + 9 = 23; 2 + 3 = 5)$

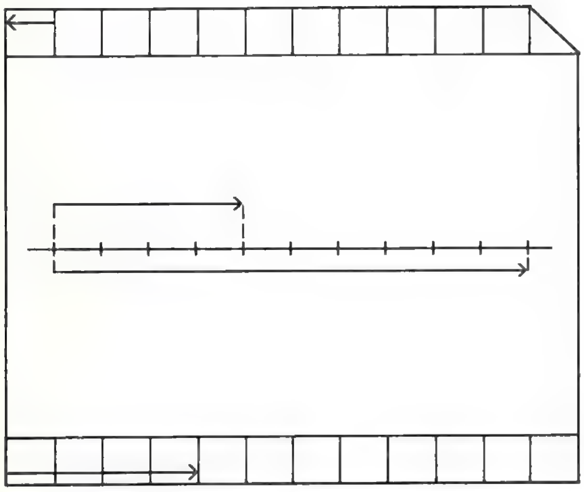
(Notice that when the remainder is 0, this method gives not 0, but 9. For example, $747 \div 9$: $7 + 4 + 7 = 18$, $1 + 8 = 9$.)

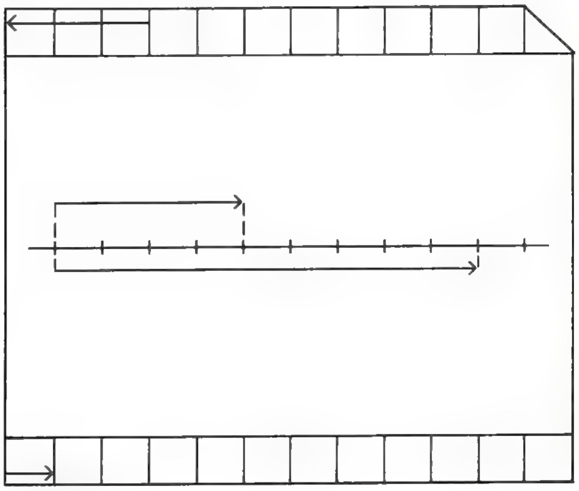
The more able pupils may want to try to find out why this shortcut works. Their investigation will lead them into remainder arithmetic, otherwise known as modulo systems. Encourage them to try dividing round hundreds by 9.

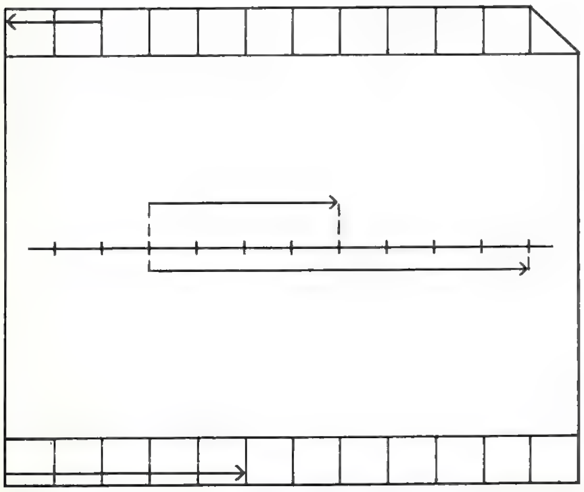
$100 \div 9$ has a remainder of 1. $(100 = 99 + 1)$
$200 \div 9$ has a remainder of 2. $(200 = 2 \times 99 + 2)$
$300 \div 9$ has a remainder of 3. $(300 = 3 \times 99 + 3)$

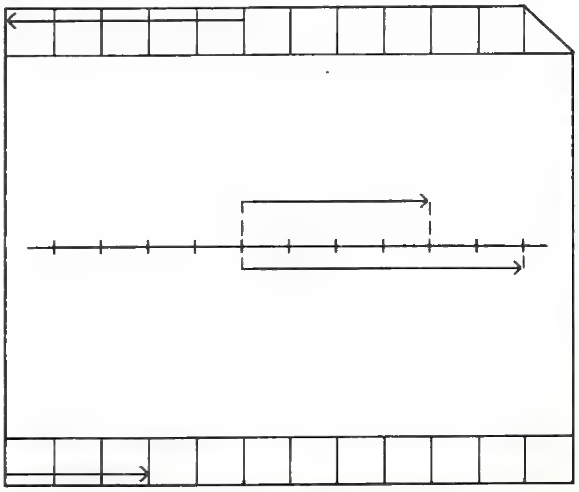
The pupils may also try dividing round thousands by 9 and round tens by 9.

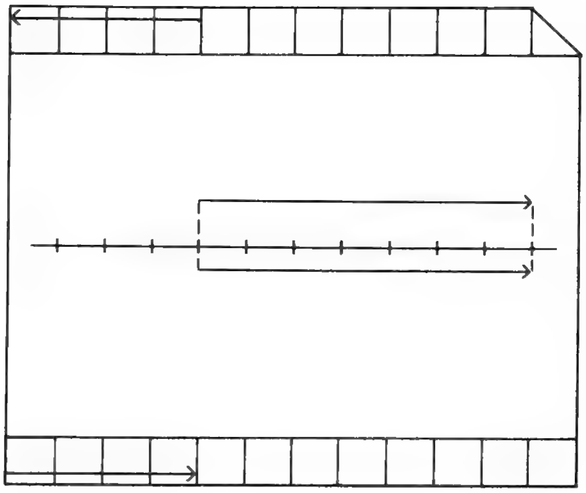

BONUS
DIFFO
GCMP
© 1967 Educational Research Council of America.

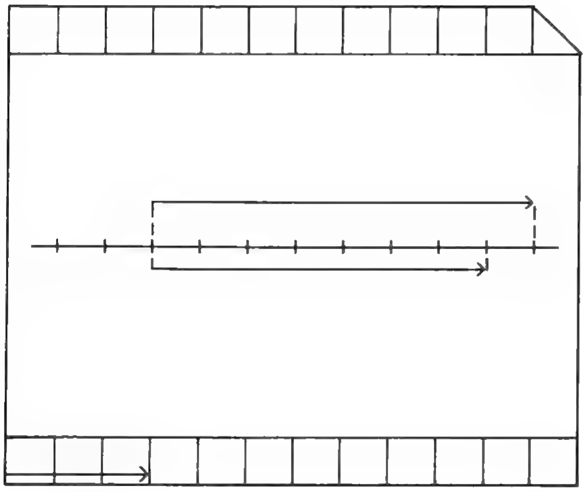


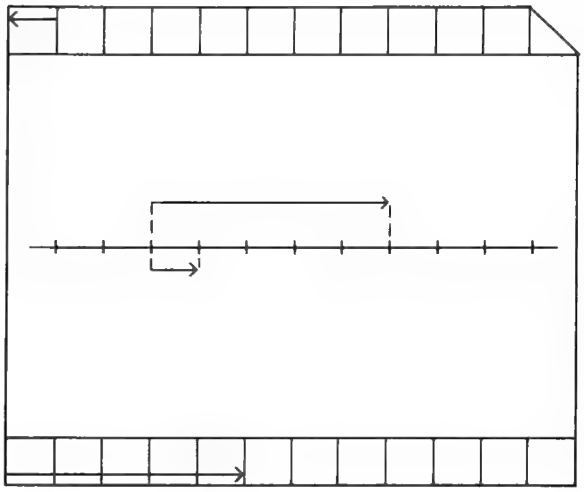


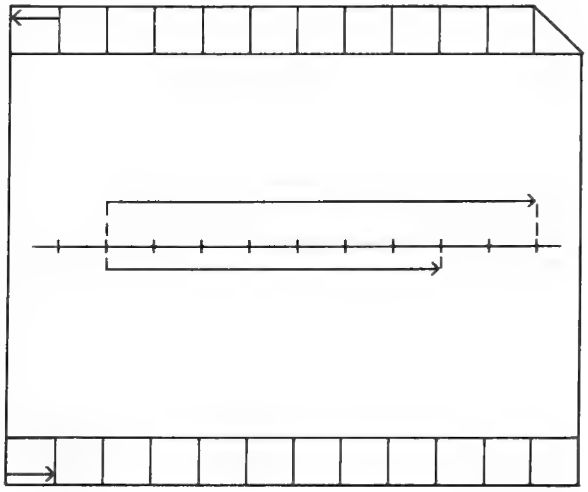


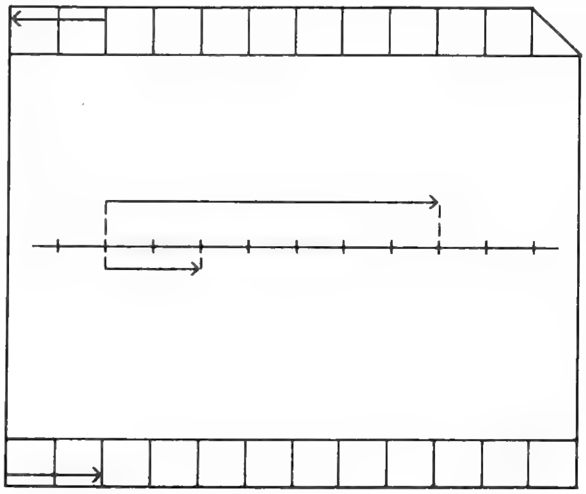


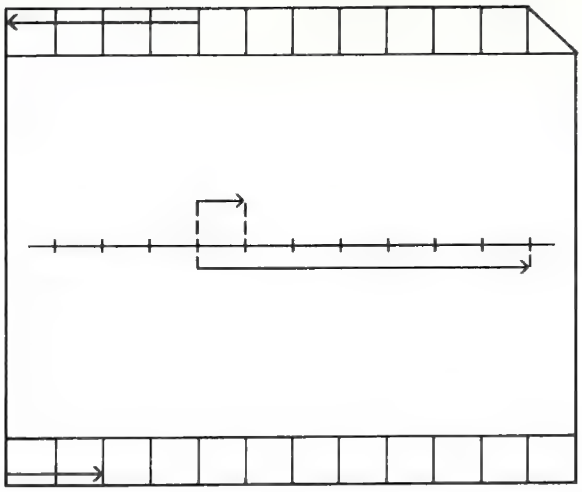


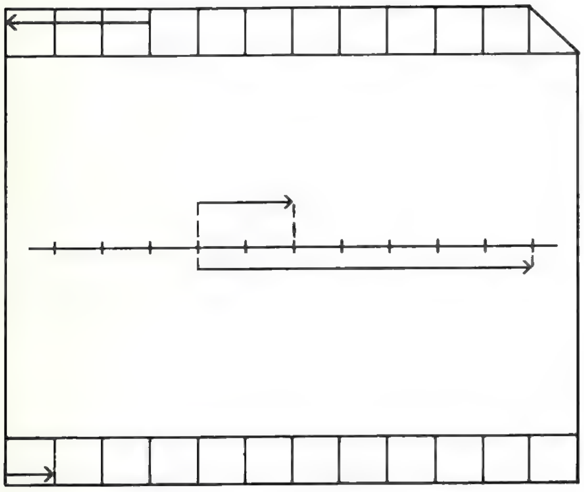


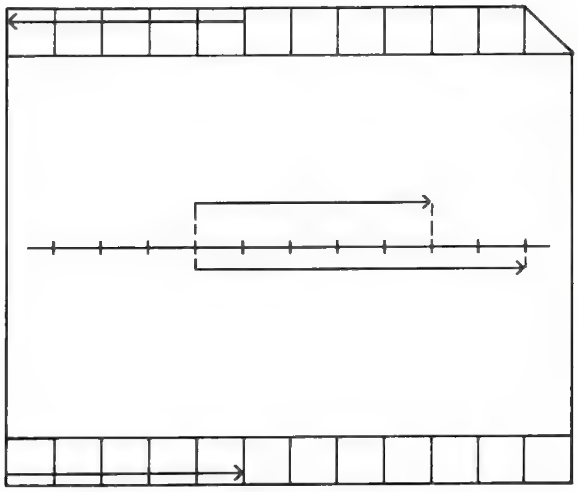


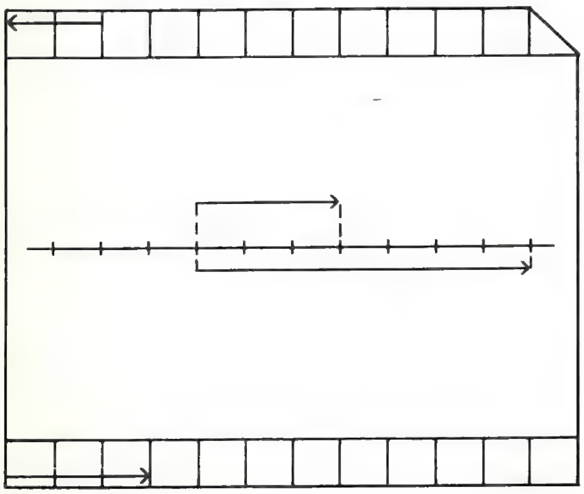


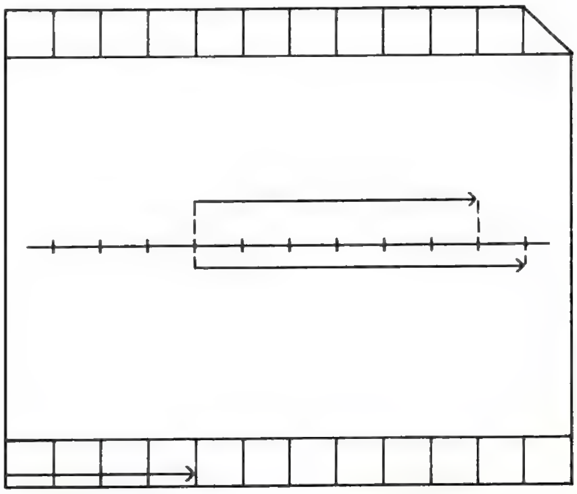


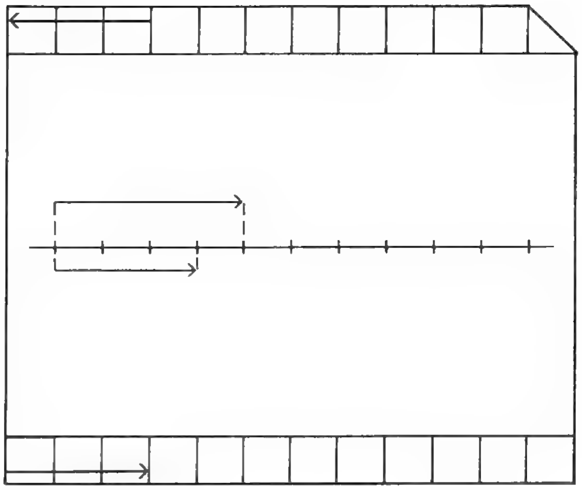


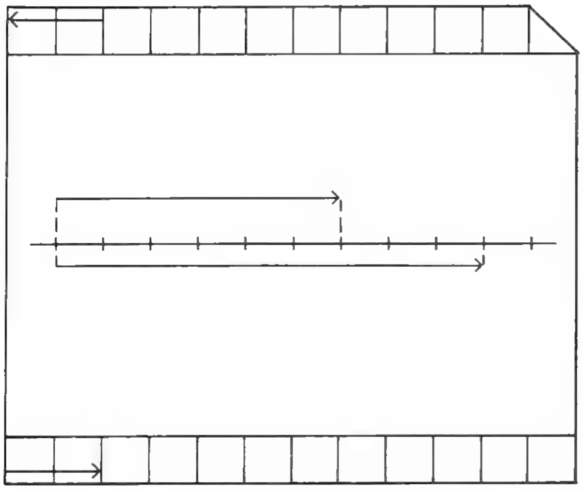


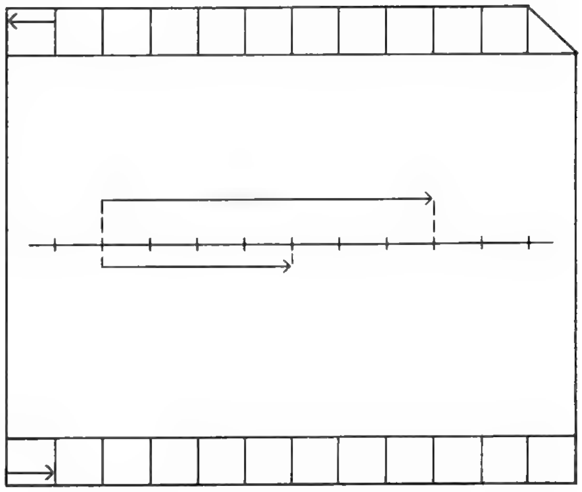


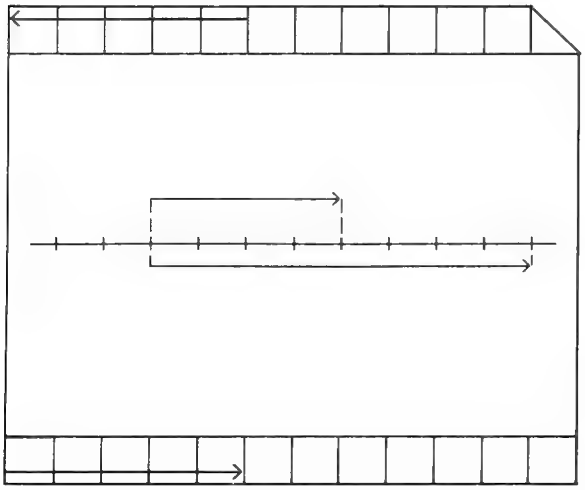


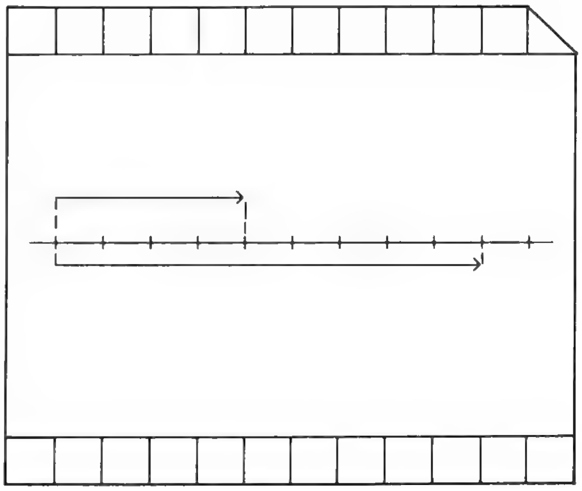


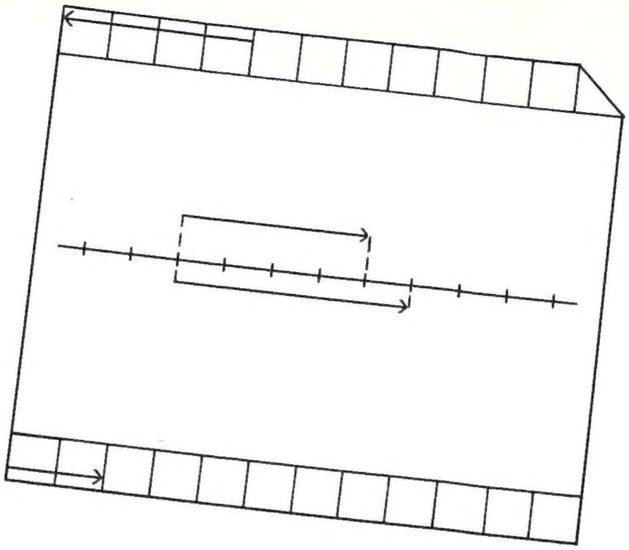


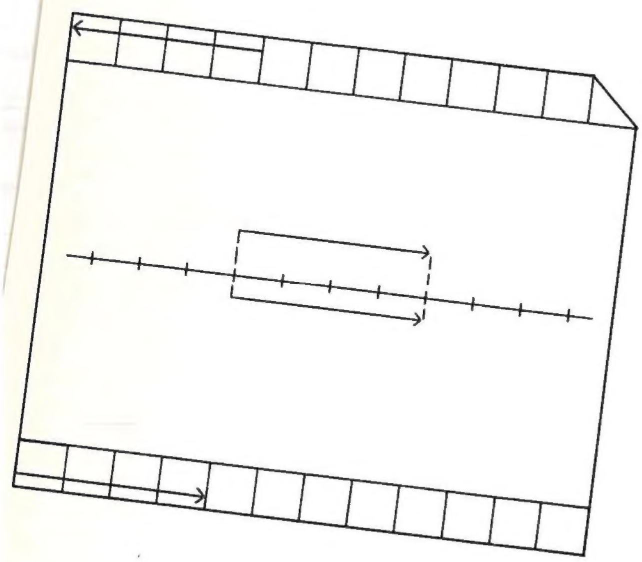


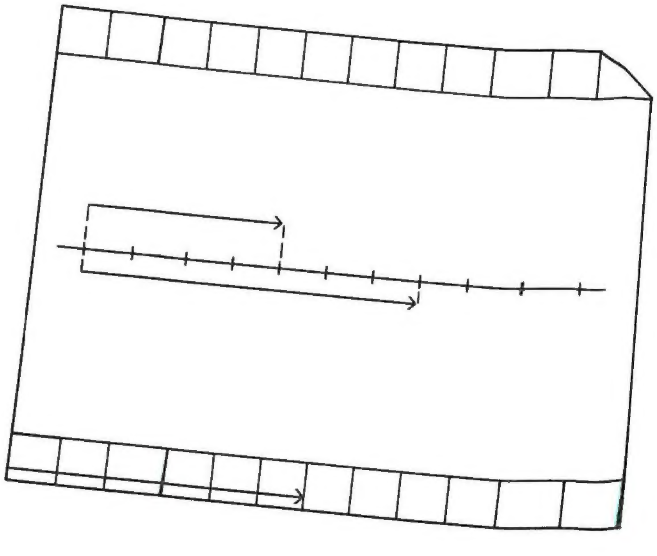


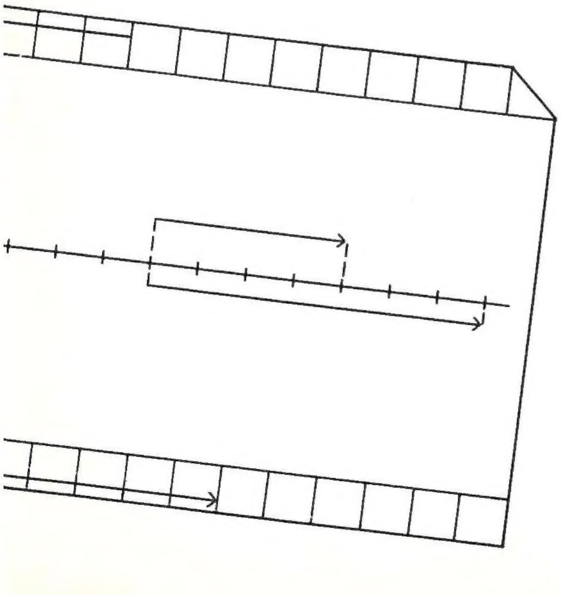


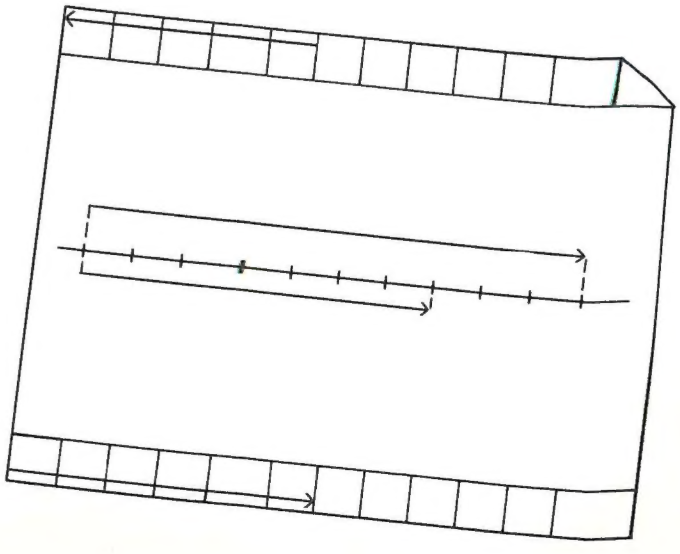




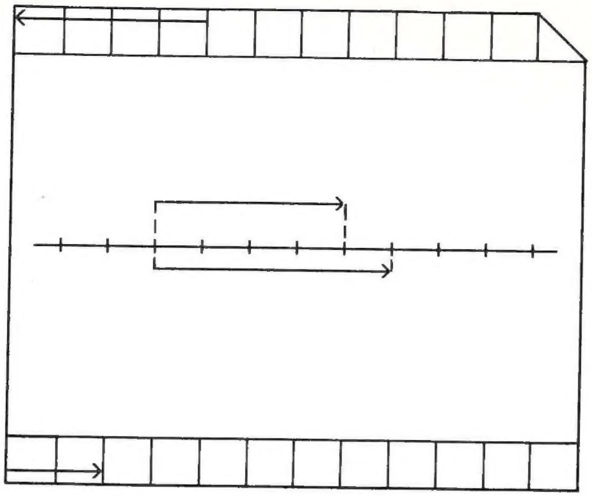


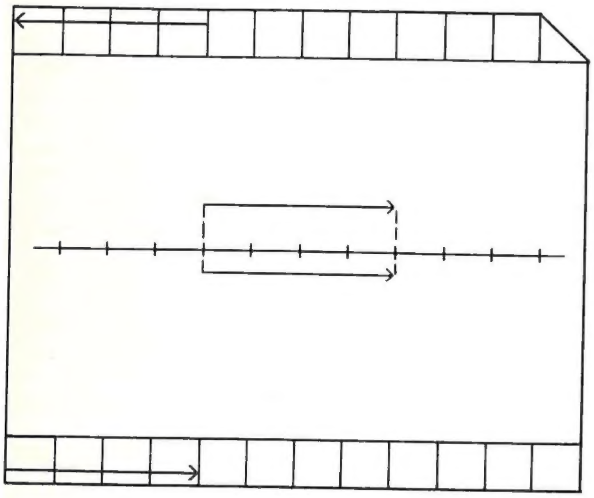


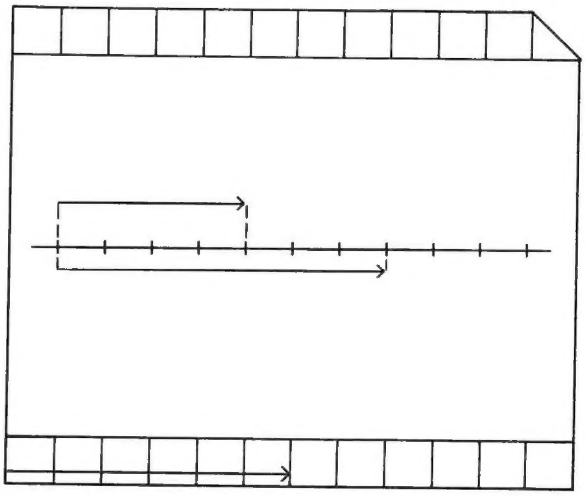


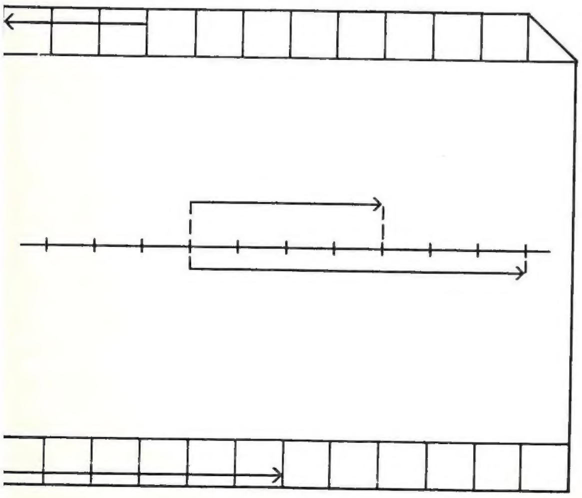


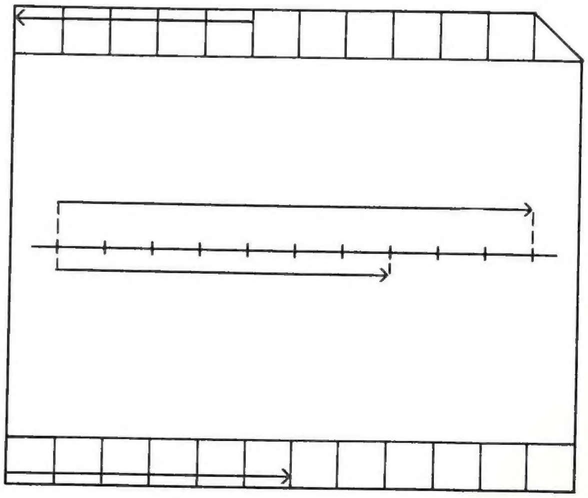
This is the last page (300) required)r
this book. The two pages that foll(
(contibutors, and the back page) not
necessary.











This is the last page (300) required for this book. The two pages that follow (contibutors, and the back page) are not necessary.